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ON COMPLETENESS AND MINIMALITY OF A SYSTEM OF SINES AND COSINES IN THE SPACE OF CONTINUOUSLY-DIFFERENTIABLE FUNCTIONS

Abstract

In the paper we obtain necessary and sufficient conditions of completeness and minimality of a system of sines and cosines in the space of continuously-differentiable functions.

Let $C[0,\pi]$ and $C^1[0,\pi]$ be Banach spaces of continuous and continuously-differentiable functions, respectively, with ordinary sup-norms.

Denote:

$$C_{x_1,x_2,\ldots,x_n}[a,b]=\left\{f\in C[a,b]: f(x_i)=0,\quad i=\overline{1,n}\right\},$$

$$C^1_{x_1,x_2,\ldots,x_n}[a,b]=\left\{f\in C^1[a,b]: f(x_i)=0,\quad i=\overline{1,n}\right\},$$

$$C^{1;y_1,y_2,\ldots,y_n}[a,b]=\left\{f\in C^1[a,b]: f'(y_i)=0,\quad i=\overline{1,n}\right\},$$

$$C^{1;y_1,y_2,\ldots,y_m}_{x_1,x_2,\ldots,x_n}[a,b]=C^1_{x_1,x_2,\ldots x_n}[a,b]\cap C^{1;y_1,y_2,\ldots,y_m}[a,b],$$

$$C_0^{\alpha}[0,\pi]=\{f\in C^{\alpha}[0,\pi]: f(0)=0\},$$

$$C_{\pi}^{\alpha}[0,\pi]=\{f\in C^{\alpha}[0,\pi]: f(\pi)=0\},$$

$$C_{0,\pi}^{\alpha}[0,\pi]=C_0^{\alpha}[0,\pi]\cap C_{\pi}^{\alpha}[0,\pi],$$

$$C^{1+\alpha}=\left\{f: f'\in C^{\alpha}[0,\pi]\right\},$$

where $C_0^{\alpha}[0,\pi]$ is a Banach space of Holder functions with appropriate norm.

Let's consider the systems

$$\left\{\sin\left(n-\frac{\beta}{2}\right)\theta\right\}_{n=1}^{\infty}, \tag{1}$$

and

$$\left\{\cos\left(n-\frac{\beta}{2}\right)\theta\right\}_{n=1}^{\infty}, \tag{2}$$

where $\theta\in[0,\pi]$, β is a real parameter.

First of all revise the results of the paper [1] for the completeness and minimality of systems (1) and (2) in $C[0,\pi]$.

Lemma 1. *For $0<\beta<2$, for any function $\psi(\theta)\in C_0^{\alpha}[0,\pi]$, the biorthogonal series*

$$\sum_{n=1}^{\infty}A_n\sin\left(n-\frac{\beta}{2}\right)\theta \tag{3}$$

uniformly converges on $[0,\pi]$ to the function $\psi(\theta)$; for $\beta=0$, if $\psi(\theta)\in C_{0,\pi}^{\alpha}[0,\pi]$, series (3) uniformly converges on $[0,\pi]$ to the function $\psi(\theta)$.

Lemma 2. *For $1<\beta<3$, for any function $\psi(\theta)\in C^{\alpha}[0,\pi]$, the biorthogonal series*

$$\sum_{n=1}^{\infty} B_n \cos\left(n-\frac{\beta}{2}\right)\theta \tag{4}$$

uniformly converges on $[0,\pi]$ to the function $\psi(\theta)$; for $\beta=1$, if $\psi(\theta)\in C_{\pi}^{\alpha}[0,\pi]$, series (3) uniformly converges on $[0,\pi]$ to the function $\psi(\theta)$.

Theorem 1. *For the system (1) the following statements are valid:*

1) for $0<\beta<2$ system (1) is complete and minimal in $C_0[0,\pi]$;

2) for $\beta\le 0$ system (1) is minimal, but not complete in $C_0[0,\pi]$;

3) for $\beta\in(2k,2k+2)$, $k=1,2,...$, system (1) is complete, but not minimal in $C_0[0,\pi]$;

4) for $\beta=0$ system (1) is complete and minimal in $C_{0,\pi}[0,\pi]$;

5) for $\beta=2k$, $k=1,2,...$,system (1) is complete, but not minimal in $C_{0,\pi}[0,\pi]$;

6) for $\beta=-2k$, $k=1,2,...$,system (1) is minimal, but not complete in $C_{0,\pi}[0,\pi]$;

7) in the case of minimality the biorthogonal system $\{h_n^s(\theta)\}_{n=1}^{\infty}$ is of the form:

$$h_n^s(\theta)=\frac{2}{\pi}\sum_{k=0}^{n-1} C_{\beta}^k \sin(n-k)\theta\left(2\cos\frac{\theta}{2}\right)^{-\beta},$$

where C_{β}^k are binomial coefficients.

The similar theorem is true for system (2) as well.

Theorem 2. *The following statements hold:*

1) for $1<\beta<3$ system (2) is complete and minimal in $C[0,\pi]$;

2) for $\beta\le 1$ system (2) is minimal, but not complete in $C[0,\pi]$;

3) for $\beta\in(1+2k,3+2k)$, $k=1,2,...$, system (2) is complete, but not minimal in $C[0,\pi]$;

4) for $\beta=1$ system (2) is complete and minimal in $C_{\pi}[0,\pi]$;

5) for $\beta=1{+}2k$, $k=1,2,...$,system (2) is complete, but not minimal in $C_{\pi}[0,\pi]$;

6) for $\beta=1{-}2k$, $k=1,2,...$,system (2) is minimal, but not complete in $C_{\pi}[0,\pi]$;

7) in the case of minimality the system $\left\{\hat{h}_{n,\beta}^c(\theta)\right\}_{n=1}^{\infty}$ biorthogonal to system (2) is of the form:

$$\hat{h}_{n,\beta}^c(\theta)=h_{n-1,\beta-2}^c(\theta),$$

where

$$h_{n,\beta}^c(\theta)=\frac{2}{\pi\left(2\cos\frac{\theta}{2}\right)^{\beta}}\left[\sum_{k=0}^{n} C_{\beta}^k \sin(n-k)\theta-\frac{C_{\beta}^n}{2}\right],$$

C_{β}^k are binomial coefficients.

State the main results of the paper.

Theorem 3. *The following statements are true:*

1) for $1<\beta<3$, $\beta\ne 2$ system (1) is complete and minimal in $C_0^1[0,\pi]$;

2) for $1+2k<\beta<3+2k$, $\beta\ne 2k$, $k=1,2,...$, system (1) is complete, but not minimal in $C_0^1[0,\pi]$;

3) for $1-2k<\beta<3-2k$, $k=1,2,...,$, system (1) is minimal, but not complete in $C_0^1[0,\pi]$;

4) for $\beta=0$ system (1) is complete and minimal in $C_{0,\pi}^1[0,\pi]$;

5) for $\beta = 2k$, $k = 1, 2, ...$, system (1) is complete, but not minimal in $C^1_{0,\pi}[0, \pi]$;

6) for $\beta = -2k$, $k = 1, 2, ...$, system (1) is minimal, but not complete in $C^1_{0,\pi}[0, \pi]$;

7) for $\beta = 1$ system (1) is complete and minimal in $C_0^{1;\pi}[0, \pi]$;

8) *for $\beta = 1 + 2k$, $k = 1, 2, ...$, system (1) is complete, but not minimal in $C_0^{1;\pi}[0, \pi]$;*

9) *for $\beta = 1 - 2k$, $k = 1, 2, ...$, system (1) is minimal, but not complete in* $C_0^{1;\pi}[0, \pi]$;

We'll need the following lemma that may be easily proved.

Lemma 3. *A space of functions $C^{1+\alpha}[0, \pi]$ is dense in $C^1[0, \pi]$ with respect to the norm $\|\cdot\|_1$.*

Proof. Since the Holder space $C^\alpha[0, \pi]$ is dense in the space $C[0, \pi]$, then

$$\forall f(t) \in C^1[0, \pi], \ \forall \varepsilon > 0, \ \exists \varphi(t) \in C^\alpha[0, \pi], \ \left\| f'(t) - \varphi(t) \right\|_C < \frac{\varepsilon}{\pi + 1}. \tag{5}$$

Let $\psi(t) = \int_0^t \varphi(\theta) d\theta - f(0)$. Obviously, $\psi(t) \in C^\alpha[0, \pi]$ and $\psi'(t) = \varphi(t)$, $\forall t \in [0, \pi]$. Further

$$f(t) - \psi(t) = \int_0^t f'(\theta) d\theta - f(0) - \int_0^t \varphi(\theta) d\theta + f(0) = \int_0^t \left(f'(\theta) - \varphi(\theta) \right) d\theta.$$

Then, allowing for (5) we get:

$$|f(t) - \psi(t)| = \left| \int_0^t \left(f'(\theta) - \varphi(\theta) \right) d\theta \right| \le \int_0^t \left| f'(\theta) - \varphi(\theta) \right| d\theta \le$$

$$\le \max_{[0,\pi]} \left| f'(\theta) - \varphi(\theta) \right| \cdot \pi = \left\| f'(t) - \psi(t) \right\|_C \cdot \pi < \frac{\varepsilon \pi}{\pi + 1}$$

or

$$\|f(t) - \psi(t)\|_C < \frac{\varepsilon \pi}{\pi + 1}$$

From relations (5) and (6) it directly follows that $\forall f(t) \in C^1[0, \pi]$, $\forall \varepsilon > 0$, $\exists \psi(t) \in C^{1+\alpha}[0, \pi]$,

$$\|f(t) - \psi(t)\|_1 = \|f(t) - \psi(t)\|_C + \left\| f'(t) - \psi'(t) \right\|_C < \varepsilon.$$

The lemma is proved.

Proof of theorem 3. By proving the theorem we'll use the method suggested in the paper [2]. Let

$$f(\theta) \in C^{1+\alpha}[0, \pi], \quad f(0) = 0.$$

Consider the series

$$\sum_{n=1}^{\infty} A_n \sin\left(n - \frac{\beta}{2}\right)\theta, \tag{7}$$

where the coefficients A_n are determined by the relations

$$A_n = \int_0^\pi f'(\theta) \hat{h}^c_{n,\beta}(\theta) d\theta \left(n - \frac{\beta}{2}\right)^{-1}. \tag{8}$$

After formal differentation of series (7) we get:

$$\sum_{n=1}^{\infty}\left(n-\frac{\beta}{2}\right)A_n\cos\left(n-\frac{\beta}{2}\right)\theta. \tag{9}$$

Then, it follows from lemma 2 that for $1<\beta<3$, series (9) uniformly converges to $f'(\theta)$ on the segment $[0,\pi]$, i.e.

$$f'(\theta)=\sum_{n=1}^{\infty}\left(n-\frac{\beta}{2}\right)A_n\cos\left(n-\frac{\beta}{2}\right)\theta. \tag{10}$$

Integrate series (10) within 0 and θ and get the relation

$$f(\theta)=\sum_{n=1}^{\infty}A_n\sin\left(n-\frac{\beta}{2}\right)\theta. \tag{11}$$

Obviously, series (11) uniformly converges to $f(\theta)$ on $[0,\pi]$. Since series (10) and (11) uniformly converge, then it follows from lemma 3 that system $\left\{\sin\left(n-\frac{\beta}{2}\right)\theta\right\}_{n=1}^{\infty}$ for $1<\beta<3$, $\beta\neq 2$ is complete in $C_0^1[0,\pi]$. For $\beta=2$ we get the system $\{\sin n\theta\}_{n=1}^{\infty}$. All the elements of this system vanish at the point π. Therefore, it may not be complete in $C_0^1[0,\pi]$.

Denote $H_{n,\beta}^{s}(\theta)=\hat{h}_{n,\beta}^{c}(\theta)\left(n-\frac{\beta}{2}\right)^{-1}$.

From the fact the system $\left\{\hat{h}_{n,\beta}^{c}(\theta)\right\}_{n=1}^{\infty}$ is biorthogonal to the system $\left\{\cos\left(n-\frac{\beta}{2}\right)\theta\right\}_{n=1}^{\infty}$ it follows that

$$\int_0^{\pi}H_{n,\beta}^{s}(\theta)\left(\sin\left(n-\frac{\beta}{2}\right)\theta\right)'d\theta=\int_0^{\pi}\hat{h}_{n,\beta}^{c}(\theta)\cos\left(n-\frac{\beta}{2}\right)\theta d\theta=\delta_{nm}.$$

So, system (1) for $1<\beta<3$, $\beta\neq 2$ is minimal in $C_0^1[0,\pi]$ and $\left\{H_{n,\beta}^{s}(\theta)\right\}_{n=1}^{\infty}$ is biorthogonal to it.

Show that for $\beta\in(3,5)$, $\beta\neq 4$ system (1) is complete in $C_0^1[0,\pi]$. Substitution of $\beta'=\beta-2$ and rejection of the first element leads to the system of sines $\left\{\sin\left(n-\frac{\beta'}{2}\right)\right\}_{n=1}^{\infty}$ that by the above-proved, for $1<\beta'<3$, $\beta'\neq 2$ is complete and minimal in $C_0^1[0,\pi]$. Therefore, initial system is complete, but not minimal in $C_0^1[0,\pi]$. We continue this process and get that for $\beta\in(1+2k,3+2k)$, $\beta\neq 2k+2$, $k=1,2,..$, system (1) is complete, but not not minimal in $C_0^1[0,\pi]$.

Similarly, for $\beta\in(-1,1)$, $\beta'\neq 0$ substitution $\beta'=\beta+2$ leads system (1) to the system $\left\{\sin\left(n-\frac{\beta'}{2}\right)\theta\right\}_{n=2}^{\infty}$ wherein the function $\sin\left(1-\frac{\beta'}{2}\right)\theta$ is absent. Since $\beta'\in(1,3)$, $\beta'\neq 2$ then by the above proved, the system $\left\{\sin\left(n-\frac{\beta'}{2}\right)\theta\right\}_{n=1}^{\infty}$ is complete and minimal in $C_0^1[0,\pi]$. So, the system $\left\{\sin\left(n-\frac{\beta'}{2}\right)\theta\right\}_{n=2}^{\infty}$ is minimal, but not complete. We continue this process and get that for $\beta\in(-2k+1,-2k+3)$, $\beta\neq-2k+2$, $k=1,2,...$, the system is minimal, but not complete in $C_0^1[0,\pi]$.

For $\beta=0$ we get the classic system of sines $\{\sin n\theta\}_{n=1}^{\infty}$.

First we show that the system $\theta\cup\{\sin n\theta\}_{n=1}^{\infty}$ is complete and minimal in $C_0^1[0,\pi]$. After formal integration of the series $A_0\theta+\sum\limits_{n=1}^{\infty}A_n\sin n\theta$ we get the series

$$A_0+\sum_{n=1}^{\infty}nA_n\cos n\theta. \tag{12}$$

Let $f\in C^{1+\alpha}[0,\pi]$, $f(0)=0$ and

$$A_0=\int_0^{\pi}f'(\theta)\,h_{0,0}^c(\theta)\,d\theta,\quad A_n=\frac{1}{n}\int_0^{\pi}f'(\theta)\,h_{n,0}^c(\theta)\,d\theta,\ n\geq1.$$

It again follows from lemma 2 that series (12) uniformly converges to $f'(\theta)$ on the segment $[0,\pi]$, i.e.

$$f'(\theta)=A_0+\sum_{n=1}^{\infty}nA_n\cos n\theta. \tag{13}$$

After integration of series (13) within 0 and θ we get a uniformly convergent series:

$$f(\theta)=A_0\theta+\sum_{n=1}^{\infty}A_n\sin n\theta. \tag{14}$$

So, series (12) converges to $f(\theta)$ by the norm $\|\cdot\|_1$. It follows from the proved lemma that the system $\theta\cup\{\sin n\theta\}_{n=1}^{\infty}$ is complete in $C_0^1[0,\pi]$.

Denote by

$$H_0^s(\theta)=h_{0,0}^c(\theta),\quad H_n^s(\theta)=\frac{1}{n}h_{n,0}^c(\theta),\ n\geq1.$$

From the biorthogonality of the system $\left\{h_{n,0}^c\right\}_{n=0}^{\infty}$ to the system $1\cup\{\cos nt\}_{n=1}^{\infty}$ (Theorem 2) it follows that

$$\int_0^{\pi}H_0^s(\theta)\,\theta'd\theta=\int_0^{\pi}h_{0,0}^c(\theta)\cdot1d\theta=1,$$

$$\int_0^{\pi}H_n^s(\theta)\,\theta'd\theta=\frac{1}{n}\int_0^{\pi}h_{n,0}^c(\theta)\cdot1d\theta=0,\quad n\geq1,$$

$$\int_0^{\pi}H_0^s(\theta)\,(\sin n\theta)'\,d\theta=n\int_0^{\pi}h_{0,0}^c(\theta)\cos n\theta d\theta=0,$$

$$\int_0^\pi H_n^s(\theta)(\sin m\theta)' d\theta = \frac{m}{n}\int_0^\pi h_{n,0}^c(\theta)\cos m\theta d\theta = \delta_{nm},\ n \geq 1, m \geq 1.$$

So, the system $\theta \cup \{\sin n\theta\}_{n=1}^\infty$ is minimal in $C_0^1[0,\pi]$ and $\{H_n^s(\theta)\}_{n=1}^\infty$ is a system biorthogonal to it.

By B we denote a closure of linear shell of the system $\{\sin n\theta\}_{n=1}^\infty$ with respect to the norm $\|\cdot\|_1$. Obviously, $B \subset C_{0,\pi}^1[0,\pi]$. Since the system $\theta \cup \{\sin n\theta\}_{n=1}^\infty$ is complete in $C_0^1[0,\pi]$, then

$$\forall f(t) \in C_{0,\pi}^1[0,\pi], \quad \forall\varepsilon > 0, \quad \exists\lambda_\varepsilon \in \mathbb{C}, \quad \exists b_\varepsilon(t) \in B,$$

$$\|f(t) - \lambda_\varepsilon t - b_\varepsilon(t)\|_1 < \frac{\varepsilon}{2}. \tag{15}$$

Then$|f(t) - \lambda_\varepsilon t - b_\varepsilon(t)| < \frac{\varepsilon}{2}$ or $|\lambda_\varepsilon| < \frac{\varepsilon}{2\pi}$. We get from relation (15), that $\forall t \in [0,\pi]$,

$$-\frac{\varepsilon}{2} < f(t) - \lambda_\varepsilon t - b_\varepsilon(t) < \frac{\varepsilon}{2}, \quad -\frac{\varepsilon}{2} + \lambda_\varepsilon t < f(t) - b_\varepsilon(t) < \frac{\varepsilon}{2} + \lambda_\varepsilon t$$

$$-\frac{\varepsilon}{2} - \frac{\varepsilon}{2} < f(t) - b_\varepsilon(t) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}, \quad -\varepsilon < f(t) - b_\varepsilon(t) < \varepsilon.$$

So,

$$\|f(t) - b_\varepsilon(t)\|_C < \varepsilon. \tag{16}$$

We get from relation (15) that $\forall t \in [0,\pi]$

$$-\frac{\varepsilon}{2} < f'(t) - \lambda_\varepsilon - b'_\varepsilon(t) < \frac{\varepsilon}{2}, \quad -\frac{\varepsilon}{2} + \lambda_\varepsilon < f'(t) - b'_\varepsilon(t) < \frac{\varepsilon}{2} + \lambda_\varepsilon$$

$$-\frac{\varepsilon}{2} - \frac{\varepsilon}{2\pi} < f'(t) - b'_\varepsilon(t) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2\pi},$$

$$-\varepsilon < f'(t) - b'_\varepsilon(t) < \varepsilon, \quad \left|f'(t) - b_\varepsilon(t)\right| < \varepsilon,$$

i.e.

$$\left\|f'(t) - b'_\varepsilon(t)\right\|_C < \varepsilon. \tag{17}$$

It follows from (15) and (16)

$$\|f(t) - b_\varepsilon(t)\|_1 < 2\varepsilon.$$

By definition of completeness this means that the system $\{\sin n\theta\}_{n=1}^\infty$ is complete in the space $C_{0,\pi}^1[0,\pi]$.

From the above-mentioned arguments we easily get that for $\beta = 2k$, $k = 1,2,...$, the system $\left\{\sin\left(n - \frac{\beta}{2}\right)\theta\right\}_{n=1}^\infty$ is complete, but not minimal in $C_{0,\pi}^1[0,\pi]$, and for $\beta = -2k$, $k = 1,2,...$ it is minimal, but not complete in $C_{0,\pi}^1[0,\pi]$.

From the minimality of the system $\theta \cup \{\sin n\theta\}_{n=1}^\infty$ in $C_0^1[0,\pi]$ it directly follows that the system $\{\sin n\theta\}_{n=1}^\infty$ is minimal in $C_{0,\pi}^1[0,\pi]$.

For $\beta = 1$ the system $\left\{\sin\left(n-\frac{\beta}{2}\right)\theta\right\}_{n=1}^{\infty}$ is complete and minimal in $C_0^{1;\pi}[0,\pi]$. Indeed, we differentiate series

$$\sum_{n=1}^{\infty} A_n \sin\left(n-\frac{1}{2}\right)\theta \tag{18}$$

and get:

$$\sum_{n=1}^{\infty}\left(n-\frac{1}{2}\right) A_n \cos\left(n-\frac{1}{2}\right)\theta. \tag{19}$$

Let $f(\theta) \in C^{1+\alpha}[0,\pi]$, $f(0)=0$, $f'(\pi)=0$

$$A_n = \int_0^{\pi} f'(\theta)\,\hat{h}^c_{n,\frac{1}{2}}(\theta)\,d\theta\left(n-\frac{1}{2}\right)^{-1}. \tag{20}$$

Then, it follows from lemma 2 that series (19) uniformly converges to $f'(\theta)$ on the segment $[0,\pi]$, i.e.

$$f'(\theta) = \sum_{n=1}^{\infty}\left(n-\frac{1}{2}\right) A_n \cos\left(n-\frac{1}{2}\right)\theta. \tag{21}$$

We integrate series (21) within 0 and θ, get the relation

$$f(\theta) = \sum_{n=1}^{\infty} A_n \sin\left(n-\frac{1}{2}\right)\theta. \tag{22}$$

By uniform convergence of series (21), series (22) also uniformly converges to $f(\theta)$ on $[0,\pi]$.

By lemma 3 it follows from uniform convergence of series (21) and (22) that the system $\left\{\sin\left(n-\frac{1}{2}\right)\theta\right\}_{n=1}^{\infty}$ is complete in the space $C_0^{1;\pi}[0,\pi]$.

Denote $H^s_{n,\frac{1}{2}}(\theta) = \hat{h}^c_{n,\frac{1}{2}}(\theta)\,d\theta\left(n-\frac{1}{2}\right)^{-1}$.

It follows from the biorthogonality of the systems $\left\{\hat{h}^c_{n,\frac{1}{2}}(\theta)\right\}_{n=1}^{\infty}$ and $\left\{\cos\left(n-\frac{1}{2}\right)\theta\right\}_{n=1}^{\infty}$ that

$$\int_0^{\pi} H^s_{m,\frac{1}{2}}(\theta)\left(\sin\left(n-\frac{1}{2}\right)\theta\right)' d\theta = \int_0^{\pi} \hat{h}^c_{m,\frac{1}{2}}(\theta)\,d\theta\cos\left(n-\frac{1}{2}\right)\theta d\theta = \delta_{nm}.$$

So, for $\beta = \frac{1}{2}$ system (1) is minimal in $C_0^1[0,\pi]$, and therefore minimal in $C_0^{1;\pi}[0,\pi]$.

By similar arguments, we can prove that for $\beta = 1+2k$, $k=1,2,...$ the system $\left\{\sin\left(n-\frac{\beta}{2}\right)\theta\right\}_{n=1}^{\infty}$ is complete, but not minimal in $C_0^{1;\pi}[0,\pi]$, and for $\beta = 1-2k$, $k=1,2,...$ it is minimal but not complete in $C_0^{1;\pi}[0,\pi]$.

Theorem 3 is proved.

For the system

$$1 \cup \left\{ \cos\left(n - \frac{\beta}{2}\right)\theta \right\}_{n=1}^{\infty}, \tag{23}$$

we have the following theorem:

Theorem 4. *The following statements, hold:*

1) for $0 < \beta < 2$ *system (23) is complete and minimal in* $C^{1;0}[0,\pi]$*;*

2) for $2k < \beta < 2k+2$, $k = 1,2,...,$ *system (23) is complete, but not minimal in* $C^{1;0}[0,\pi]$*;*

3) for $-2k < \beta < -2k+2$, $k = 1,2,...$ *system (23) is minimal, but not complete in* $C^{1;0}[0,\pi]$*;*

4) for $\beta = 0$ *system (23) is complete and minimal in* $C^{1;0,\pi}[0,\pi]$*;*

5) for $\beta = 2k$, $k = 1,2,...$*system (23) is complete, but not minimal in* $C^{1;0,\pi}[0,\pi]$*;*

6) for $\beta = -2k$, $k = 1,2,...,$*system (23) is minimal, but not complete in* $C^{1;0,\pi}[0,\pi]$*;*

Proof. Let $f(\theta) \in C^{1+\alpha}[0,\pi]$, $f'(0) = 0$ and

$$B_n = -\int_0^{\pi} f'(\theta)\, h_n^s(\theta)\, d\theta \left(n - \frac{\beta}{2}\right)^{-1}, \quad n = 1,2,...$$

Let's consider the series:

$$B_0 + \sum_{n=1}^{\infty} B_n \cos\left(n - \frac{\beta}{2}\right)\theta. \tag{24}$$

We formally differentiate series (24)

$$-\sum_{n=1}^{\infty} B_n \left(n - \frac{\beta}{2}\right) \sin\left(n - \frac{\beta}{2}\right)\theta. \tag{25}$$

Then, it follows from lemma 1 that for $0 < \beta < 2$ series (25) $f'(\theta)$ on $[0,\pi]$, i.e.

$$f'(\theta) = -\sum_{n=1}^{\infty} B_n \left(n - \frac{\beta}{2}\right) \sin\left(n - \frac{\beta}{2}\right)\theta. \tag{26}$$

We integrate series (26) within 0 and θ, get:

$$f(\theta) - f(0) = \sum_{n=1}^{\infty} B_n \cos\left(n - \frac{\beta}{2}\right)\theta - \sum_{n=1}^{\infty} B_n.$$

It is shown in the paper [2] the series $\sum\limits_{n=1}^{\infty} B_n$ absolutely converges. Since, series (26) uniformly converges, the series

$$B_0 + \sum_{n=1}^{\infty} B_n \cos\left(n - \frac{\beta}{2}\right)\theta$$

uniformly converges to $f(\theta)$, where $B_0 = f(0) - \sum\limits_{n=1}^{\infty} B_n$. Then, it follows from lemma 3 that for $0 < \beta < 2$ system (23) is complete in $C^{1;0}[0,\pi]$.

For $-\frac{1}{p} < \beta < 2 - \frac{1}{p}$ system (23) is minimal in $W_p^1(0,\pi)$ [2]. It follows from embedding $W_p^1(0,\pi)$ in $C^1[0,\pi]$ that for $\forall p > 1$, $-\frac{1}{p} < \beta < 2 - \frac{1}{p}$ system (23) is minimal in $C^1[0,\pi]$. So, for $0 < \beta < 2$ it is minimal in $C^1[0,\pi]$.

Similar to the proof scheme of theorem 3 we can show that for $2k < \beta < 2k+2$, $k = 1,2,...$ system (23) is complete, but not minimal in $C^{1;0}[0,\pi]$, and for $-2k < \beta < -2k+2$, $k = 1,2,...$ it is minimal, but not complete in $C^{1;0}[0,\pi]$.

For $\beta = 0$ we get the system

$$1 \cup \{\cos n\theta\}_{n=1}^{\infty} \tag{27}$$

After differentiation of series (24) for $\beta = 0$ we get

$$-\sum_{n=1}^{\infty} nB_n \sin n\theta \tag{28}$$

If $f(\theta) \in C^{1+\alpha}[0,\pi]$, $f'(0) = f'(\pi) = 0$,

$$B_n = -\frac{1}{n}\int_0^{\pi} f'(\theta)\, h_n^s(\theta)\, d\theta, \quad n = 1,2,...,$$

then by lemma 1 series (28) uniformly converges to $f'(\theta)$ on $[0,\pi]$, i.e.

$$f'(\theta) = -\sum_{n=1}^{\infty} nB_n \sin n\theta \tag{29}$$

We integrate series (29) within 0 and θ, get

$$f(\theta) - f(0) = \sum_{n=1}^{\infty} B_n \cos n\theta - \sum_{n=1}^{\infty} B_n.$$

Since the series $\sum\limits_{n=1}^{\infty} B_n$ absolutely converges, we take $B_0 = f(0) - \sum\limits_{n=1}^{\infty} B_n$ and get that the series

$$B_0 + \sum_{n=1}^{\infty} B_n \cos n\theta$$

uniformly converges to $f(\theta)$ on $[0,\pi]$. Hence it follows from lemma 3 that the system $1 \cup \{\cos n\theta\}_{n=1}^{\infty}$ is complete in the space $C^{1;0,\pi}[0,\pi]$. Minimality of the system $1 \cup \{\cos n\theta\}_{n=1}^{\infty}$ in $C^{1;0,\pi}[0,\pi]$ follows from the minimality of system (23) for $-\frac{1}{p} < \beta < 2 - \frac{1}{p}$ in the space $W_p^1(0,\pi)$.

The authors express their deep gratitude to B.T.Bilalov for his constant attention to the paper and discussion of the obtained results.

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Received January 10, 2007; Revised April 19, 2007.