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OPTIMAL CONTROL PROBLEM FOR BAR OSCILLATIONS EQUATION

Abstract

In the optimal problem for oscillation equation of a bar we derive necessary condition of optimality in the form of Pontryagin's maximum principle.

It is known [1] that many problems from practice, e.g. oscillations of a beam and membrane are reduced to fourth order equations. Therefore, optimal control problems for such oscillating systems are of urgent character. In the given paper optimal control problem is studied for the equation of lateral oscillations of a bar and necessary optimality condition is derived in the form of Pontryagin's maximum principle.

Problem statement. Let's consider an optimal control problem for a bar oscillation equation

$$\frac{\partial^2 u(x,t)}{\partial t^2}+\frac{\partial^4 u(x,t)}{\partial x^4}=f(x,t,u(x,t),v(x,t)),$$

$$(x,t)\in Q=\{0<x<l,\ 0<t<T\}, \tag{1}$$

with initial conditions

$$u(x,0)=\varphi_0(x),\ \frac{\partial u(x,0)}{\partial t}=\varphi_1(x),\quad 0<x<l \tag{2}$$

and boundary conditions

$$u(0,t)=u(l,t)=0,\quad \frac{\partial u(0,t)}{\partial x}=\frac{\partial u(l,t)}{\partial x}=0,\quad 0<l<T, \tag{3}$$

where $u(x,t)$ is a function of state, $v(x,t)$ is a control function.

As a class of admissible controls U_{ad} we take a set of measurable bounded functions $v(x,t)$ on Q accepting values from $[\alpha,\beta]$, where α,β are the given numbers.

The following problem is stated: to find such an admissible control from the U_{ad} that delivers minimum to functional

$$J(v)=\iint\limits_Q f_0(x,t,u(x,t),v(x,t))\,dxdt. \tag{4}$$

Assume that the given problems satisfy the following conditions:

1) $\varphi_0\in \overset{0}{W}{}_2^2(0,l)$, $\varphi_1\in L_2(0,l)$;

2) $f(x,t,u,v)$ and $f_0(x,t,u,v)$ are continuous on $\bar{Q}\times R\times V$ and have continuous derivatives $\frac{\partial f}{\partial u}$, $\frac{\partial f_0}{\partial u}$, moreover $\frac{\partial f}{\partial u}$ is bounded, $\frac{\partial f}{\partial u}$ and $\frac{\partial f_0}{\partial u}$ satisfy Lipschitz condition with respect to u.

Definition 1. *Under the generalized solution of problem (1)-(3) corresponding to* $v=v(x,t)\in U_{ad}$ *we'll understand the function* $u(x,t)\in C\left(0,T;W^1,W^0\right)$*, where*

$W^1 = \overset{0}{W}{}_2^2(0,l)$, $W^0 = L_2(0,l)$ and satisfying conditions (2), (3) in the sense of equality of appropriate traces and integral identity

$$\iint_Q \Big(-\frac{\partial u(x,t)}{\partial t}\frac{\partial \Phi(x,t)}{\partial t} + \frac{\partial^2 u(x,t)}{\partial x^2}\frac{\partial^2 \Phi(x,t)}{\partial x^2} +$$

$$+ f(x,t,u(x,t),v(x,t))\,\Phi(x,t)\Big)\,dxdt - \int_0^l \varphi_1(x)\,\Phi(x,0)\,dx = 0$$

for all $\Phi(x,t) \in C\left(0,T;W^1,W^0\right)$ and $\Phi(x,T) = 0$.

Definition 2. *$(u(x,t),v(x,t))$ is said to be admissible pair, if $v(x,t)$ is admissible control, $u(x,t) \in C\left(0,T;W^1,W^0\right)$ is a generalized solution of problem (1)-(3) corresponding to this control. Admissible pair is said to be optimal pair if it delivers minimum to functional (4).*

We can prove that under above-mentioned conditions on data for each admissible control the problem (1)-(3) has a unique generalized solution (see [2], [3]).

For the given admissible pair $(u_0(x,t),v_0(x,t))$ we introduce the conjugate system:

$$\frac{\partial^2\psi(x,t)}{\partial t^2} + \frac{\partial^4\psi(x,t)}{\partial x^4} = \frac{\partial H(x,t,u(x,t),v(x,t)\,\psi(x,t))}{\partial u}, \tag{5}$$

$$\psi(x,T) = 0,\quad \frac{\partial\psi(x,T)}{\partial t} = 0, \tag{6}$$

$$\psi(0,t) = \psi(l,t) = 0,\quad \frac{\partial\psi(0,t)}{\partial x} = \frac{\partial\psi(l,t)}{\partial x} = 0 \tag{7}$$

where $H(x,t,u(x,t),v(x,t),\psi(x,t))\,\psi(x,t)\,f(x,t,u(x,t),v(x,t)) -$ $-f_0(x,t,u(x,t),v(x,t))$ is Pontryagin function. Since (5) is a linear equation with respect to $\psi(x,t)$ we get from the conditions imposed on data that problem (5)-(6) has a unique generalized solution from $C\left(0,T;W^1,W^0\right)$ (see [2], [3]).

Estimation of solution increment. Our goal is to derive necessary optimality conditions for the problem under consideration. Let $(\sigma,\tau) \in Q$ be a tame point of all the functions of the problem. Determine the admissible control in the following way:

$$v_\varepsilon(x,t) = \begin{cases} v,\ (x,t) \in \Pi_\varepsilon, \\ v_0(x,t),(x,t) \in Q\backslash\Pi_\varepsilon, \end{cases} \tag{8}$$

where v is an arbitrary number from $[\alpha,\beta]$, $\Pi_\varepsilon = \big\{(x,t) : \sigma < x < \sigma+\varepsilon, \tau < t < < \tau+\varepsilon\big\}$ and $\varepsilon > 0$ is so small that $\Pi_\varepsilon \subset Q$. Denote the solution of problem (1)-(3) for $v_\varepsilon(x,t)$ by $u_\varepsilon(x,t)$. Then $\Delta u_\varepsilon = u_\varepsilon - u_0$ is the solution of the following problem:

$$\frac{\partial^2\Delta u_\varepsilon(x,t)}{\partial t^2} + \frac{\partial^4\Delta u_\varepsilon(x,t)}{\partial x^4} = f(x,t,u_0(x,t)+\Delta u_\varepsilon(x,t),v_\varepsilon(x,t)) -$$

$$-f(x,t,u_0(x,t),v_0(x,t)), \tag{9}$$

$$\Delta u_\varepsilon(x,0) = 0,\quad \frac{\partial\Delta u_\varepsilon(x,0)}{\partial t} = 0, \tag{10}$$

$$\Delta u_\varepsilon(0,t)=\Delta u_\varepsilon(l,t)=0,\quad \frac{\partial\Delta u_\varepsilon(0,t)}{\partial x}=\frac{\partial\Delta u_\varepsilon(l,t)}{\partial x}=0. \tag{11}$$

Lemma. *By fulfilling conditions 1.-2. for the solution of problem (9)-(11) the following estimation holds:*

$$\|\Delta u_\varepsilon\|^2_{L_2(0,l)}+\left\|\frac{\partial\Delta u_\varepsilon}{\partial t}\right\|^2_{L_2(0,l)}+\left\|\frac{\partial\Delta u_\varepsilon}{\partial x}\right\|^2_{L_2(0,l)}+$$

$$+\left\|\frac{\partial\Delta u_\varepsilon}{\partial x^2}\right\|^2_{L_2(0,l)}\le C\varepsilon^3,\ \forall t\in[0,T]. \tag{12}$$

In sequel, by C we'll denote various constants.

Proof. Let $(x,t)\in[0,l]\times[0,\tau]$. In this domain $v_\varepsilon(x,t)=v_0(x,t)$. Therefore,

$$\frac{\partial^2\Delta u_\varepsilon(x,t)}{\partial t^2}+\frac{\partial^4\Delta u_\varepsilon(x,t)}{\partial x^4}=f\left(x,t,u_0(x,t)+\Delta u_\varepsilon(x,t),v_0(x,t)\right)-$$

$$-f\left(x,t,u_0(x,t),v_0(x,t)\right), \tag{13}$$

$$\Delta u_\varepsilon(x,0)=0,\quad \frac{\partial\Delta u_\varepsilon(x,0)}{\partial t}=0, \tag{14}$$

$$\Delta u_\varepsilon(0,t)=\Delta u_\varepsilon(l,t)=0,\quad \frac{\partial\Delta u_\varepsilon(0,t)}{\partial x}=\frac{\partial\Delta u_\varepsilon(l,t)}{\partial x}=0. \tag{15}$$

By uniqueness of the solution of problem (13)-(15)

$$\Delta u_\varepsilon(x,t)=0,\quad (x,t)\in[0,l]\times[0,\tau]. \tag{16}$$

Let $(x,t)\in[0,l]\times[\tau,\tau+\varepsilon]$. Assume that $\{\eta_k(x)\}$ is a fundamental system in $\overset{0}{W}{}_2^2(0,l)$ and $\int_0^l\eta_k(x)\eta_l(x)\,dx=\delta_k^l=\begin{cases}1, k=l,\\ 0, k\ne l.\end{cases}$. We seek for the approximate solution $\Delta u_\varepsilon^N(x,t)$ of problem (13)-(15) in the form $\Delta u_\varepsilon^N(x,t)=\sum_{k=1}^N c_k^N(t)\eta_k(x)$ from the relations

$$\int_0^l\frac{\partial^2\Delta u_\varepsilon^N(x,t)}{\partial t^2}\eta_l(x)\,dx+\int_0^l\frac{\partial^2\Delta u_\varepsilon^N(x,t)}{\partial x^2}\frac{\partial^2\eta_l(x)}{\partial x^2}dx=$$

$$=\int_0^l\left[f\left(x,t,u_0(x,t)+\Delta u_\varepsilon^N(x,t),v_\varepsilon(x,t)\right)-f\left(x,t,u_0(x,t),v_0(x,t)\right)\right]\eta_l(x)\,dx,$$
$$l=1,...,N \tag{17}$$

and

$$\frac{d}{dt}c_k^N(t)\,|_{t=0}=0,\quad c_k^N(0)=0. \tag{18}$$

Equalities (17) are the systems of second order differential equations with respect to t for the unknowns $c_k^N(t)$, $k=1,...,N$, solved with respect to $\frac{d^2c_k^N(t)}{dt^2}$. Since f by the argument u satisfies Lipschitz condition, this system is uniquely solvable under initial data of (18), moreover, $\dfrac{d^2c_k^N(t)}{dt^2} \in L_2(0,T)$. Multiplying each of equalities (17) by its own $\dfrac{d}{dt}c_l^N(t)$ and summing with respect to l from 1 to N, we arrive at the equality

$$\int_0^l \frac{\partial^2 \Delta u_\varepsilon^N(x,t)}{\partial t^2}\frac{\partial \Delta u_\varepsilon^N(x,t)}{\partial t}dx + \int_0^l \frac{\partial^2 \Delta u_\varepsilon^N(x,t)}{\partial x^2}\frac{\partial^3 \Delta u_\varepsilon^N(x,t)}{\partial x^2 \partial t}dx =$$

$$= \int_0^l \left[f\left(x,t,u_0(x,t)+\Delta u_\varepsilon^N(x,t), v_\varepsilon(x,t)\right) - \right.$$

$$\left. -f\left(x,t,u_0(x,t), v_0(x,t)\right)\right]\frac{\partial \Delta u_\varepsilon^N(x,t)}{\partial t}dx. \tag{19}$$

Hence we have

$$\frac{1}{2}\frac{d}{dt}\int_0^l \left[\left(\frac{\partial \Delta u_\varepsilon^N(x,t)}{\partial t}\right)^2 + \left(\frac{\partial^2 \Delta u_\varepsilon^N(x,t)}{\partial x^2}\right)^2\right]dx =$$

$$= \int_0^l \left[f\left(x,t,u_0(x,t)+\Delta u_\varepsilon^N(x,t), v_\varepsilon(x,t)\right) - \right.$$

$$\left. -f\left(x,t,u_0(x,t), v_0(x,t)\right)\right]\frac{\partial \Delta u_\varepsilon^N(x,t)}{\partial t}dx.$$

Considering the conditions $\Delta u_\varepsilon^N(x,\tau)=0$, $\dfrac{\Delta u_\varepsilon^N(x,\tau)}{\partial t}=0$ and integrating the last relation with respect to t from τ to t we get:

$$\int_0^l \left[\left(\frac{\partial \Delta u_\varepsilon^N(x,t)}{\partial t}\right)^2 + \left(\frac{\partial^2 \Delta u_\varepsilon^N(x,t)}{\partial x^2}\right)^2\right]dx =$$

$$= 2\int_\tau^t \int_0^l \left[f\left(x,s,u_0(x,s)+\Delta u_\varepsilon^N(x,s), v_\varepsilon(x,s)\right) - \right.$$

$$\left. -f\left(x,s,u_0(x,s), v_0(x,s)\right)\right]\frac{\partial \Delta u_\varepsilon^N(x,s)}{\partial t}dxds. \tag{20}$$

Keeping in mind the estimation

$$\int_0^l \left(\Delta u_\varepsilon^N\right)^2 dx \le C\int_\tau^t \int_0^l \left(\frac{\partial \Delta u_\varepsilon^N}{\partial t}\right)^2 dxds.$$

and equivalence of norms in $\overset{0}{W}{}_2^2(0,t)$ from (20) by means of some transformations we have:

$$\int_0^l \left[\left(\Delta u_\varepsilon^N(x,t)\right)^2 + \left(\frac{\partial \Delta u_\varepsilon^N(x,t)}{\partial t}\right)^2 + \left(\frac{\partial \Delta u_\varepsilon^N(x,t)}{\partial x}\right)^2 + \left(\frac{\partial^2 \Delta u_\varepsilon^N(x,t)}{\partial x^2}\right)^2 \right] dx \le$$

$$\le C \int_\tau^t \left[\int_0^l \left(\Delta u_\varepsilon^N(x,s)\right)^2 + \left(\frac{\partial \Delta u_\varepsilon^N(x,s)}{\partial t}\right)^2 + \left(\frac{\partial \Delta u_\varepsilon^N(x,s)}{\partial x}\right)^2 + \right.$$

$$\left. + \left(\frac{\partial^2 \Delta u_\varepsilon^N(x,s)}{\partial x^2}\right)^2 \right] dxds + C \int_\tau^t \left[\int_0^l \left(\Delta u_\varepsilon^N(x,s)\right)^2 + \left(\frac{\partial \Delta u_\varepsilon^N(x,s)}{\partial t}\right)^2 + \right.$$

$$\left. + \left(\frac{\partial \Delta u_\varepsilon^N(x,s)}{\partial x}\right)^2 + \left(\frac{\partial^2 \Delta u_\varepsilon^N(x,s)}{\partial x^2}\right)^2 dx \right]^{1/2} \times$$

$$\times \left[\int_0^l \left[f(x,s,u_0(x,s), v_\varepsilon(x,s)) - f(x,s,u_0(x,s), v_0(x,s)) \right]^2 ds \right]^{1/2}. \qquad (21)$$

Introduce the following denotation:

$$A^N(t) = \int_0^l [\left(\Delta u_\varepsilon^N(x,s)\right)^2 + \left(\frac{\partial \Delta u_\varepsilon^N(x,s)}{\partial t}\right)^2 + $$
$$+ \left(\frac{\partial \Delta u_\varepsilon^N(x,s)}{\partial x}\right)^2 + \left(\frac{\partial^2 \Delta u_\varepsilon^N(x,s)}{\partial x^2}\right)^2]ds, \qquad (22)$$
$$g(s) = C[\int_0^l [f(x,s,u_0(x,s), v_\varepsilon(x,s)) - f(x,s,u_0(x,s), v_0(x,s))]^2 ds]^{1/2}.$$

Then we can write inequality (21) in the following form:

$$A^N(t) \le C \int_\tau^t A^N(s)\, ds + \int_\tau^t g(s) \sqrt{A^N(s)} ds.$$

Let

$$\alpha(t) = C \int_\tau^t A^N(s)\, ds + \int_\tau^t g(s) \sqrt{A^N(s)} ds.$$

Then $\alpha(\tau) = 0$.
Hence

$$\alpha(t) = CA^N(t) + g(t) \sqrt{A^N(t)} \le C\alpha(t) + g(t) \sqrt{\alpha(t)}. \qquad (23)$$

Multiplying the both sides of (23) by $\alpha^{-\frac{1}{2}}(t)\exp\left(-\frac{C(t-\tau)}{2}\right)$ and integrating from τ to t and then transforming the result, we get

$$\alpha(t)\le C\left(\int\limits_{\tau}^{\tau+\varepsilon} g(s)\,ds\right)^2=$$

$$=C\left[\int\limits_{\tau}^{\tau+\varepsilon}\left[\int\limits_0^l \left[f(x,s,u_0(x,s),v_\varepsilon(x,s))-f(x,s,u_0(x,s),v_0(x,s))\right]^2dx\right]^{1/2}ds\right]^2.$$

Hence, in view of the fact that (σ,τ) is a tame point of all the functions in the problem and $v_\varepsilon(x,t)$ is of the form (8) we get:

$$A^N\le\alpha(t)\le C\left(\int\limits_{\tau}^{\tau+\varepsilon}\left[\int\limits_{\sigma}^{\sigma+\varepsilon}\left[f(x,s,u_0(x,s),v)-\right.\right.\right.$$

$$\left.\left.\left.-f(x,s,u_0(x,s),v_0(x,s))\right]^2dx\right]^{1/2}ds\right)^2\le C\varepsilon^3,\quad t\in[\tau,\tau+\varepsilon]. \qquad (24)$$

Now, let $(x,t)\in[0,l]\times[\tau+\varepsilon,T]$. For $\Delta u_\varepsilon(x,t)$ as initial condition we take the estimation following from (24) for $t=\tau+\varepsilon$, i.e.

$$\|\Delta u_\varepsilon(x,\tau+\varepsilon)\|^2\le C\varepsilon^3,\quad \left\|\frac{\partial\Delta u_\varepsilon(x,\tau+\varepsilon)}{\partial t}\right\|^2\le C\varepsilon^3. \qquad (25)$$

If we take into account the form (8) of the function $v_\varepsilon(x,t)$ and estimate (25), it follows from equation (19) that

$$\int\limits_0^l\left[\left(\frac{\partial\Delta u_\varepsilon^N(x,t)}{\partial t}\right)^2+\left(\frac{\partial^2\Delta u_\varepsilon^N(x,t)}{\partial x^2}\right)^2\right]dx\le C\varepsilon^3+$$

$$+2\int\limits_{\tau+\varepsilon}^{t}\int\limits_0^l\left|\left[f\left(x,s,u_0(x,s)+\Delta u_\varepsilon^N(x,s),v_0(x,s)\right)-\right.\right.$$

$$\left.\left.-f(x,s,u_0(x,s),v_0(x,s))\right]\frac{\partial\Delta u_\varepsilon^N(x,s)}{\partial t}\right|dxds.$$

Keeping in mind

$$\int\limits_0^l\left(\Delta u_\varepsilon^N\right)^2dx\le C\varepsilon^3+C\int\limits_{\tau+\varepsilon}^{t}\int\limits_0^l\left(\frac{\partial\Delta u_\varepsilon^N}{\partial t}\right)^2dxds$$

and equivalence of the norm in $\overset{0}{W}{}_2^2(0,l)$, we get

$$\int\limits_0^l[\left(\Delta u_\varepsilon^N(x,t)\right)^2+\left(\frac{\partial\Delta u_\varepsilon^N(x,t)}{\partial t}\right)^2+\left(\frac{\partial\Delta u_\varepsilon^N(x,t)}{\partial x}\right)^2+\left(\frac{\partial^2\Delta u_\varepsilon^N(x,t)}{\partial x^2}\right)^2]dx\le$$

$$\leq C\varepsilon^3 + C\int\limits_{\tau+\varepsilon}^{t}\int\limits_0^l [\left(\Delta u_\varepsilon^N(x,s)\right)^2 + \left(\frac{\partial \Delta u_\varepsilon^N(x,s)}{\partial t}\right)^2 +$$

$$+\left(\frac{\partial \Delta u_\varepsilon^N(x,s)}{\partial x}\right)^2 + \left(\frac{\partial^2 \Delta u_\varepsilon^N(x,s)}{\partial x^2}\right)^2]dxds.$$

If we consider the denotation adopted above, from the previous inequality we have:

$$A^N(t) \leq C\varepsilon^3 + C\int\limits_{\tau+\varepsilon}^{t} A^N(s)\,ds.$$

Applying Gronwall lemma we have:

$$A^N(t) \leq C\varepsilon^3, \quad t\in[\tau+\varepsilon, T]. \tag{26}$$

Thus, it follows from (16), (24) and (26) that estimation

$$\left\|\Delta u_\varepsilon^N\right\|^2_{L_2(0,l)} + \left\|\frac{\partial \Delta u_\varepsilon^N}{\partial t}\right\|^2_{L_2(0,l)} +$$

$$+\left\|\frac{\partial \Delta u_\varepsilon^N}{\partial x}\right\|^2_{L_2(0,l)} + \left\|\frac{\partial^2 \Delta u_\varepsilon^N}{\partial x^2}\right\|^2_{L_2(0,l)} \leq C\varepsilon^3, \ \forall t\in[0,T] \tag{27}$$

is true.

Then, as $N\to\infty$ we can assume that $\Delta u_\varepsilon(x,t)$ is a weak limit of sequence $\Delta u_\varepsilon^N(x,t)$ in $C\left(0,T;W^1,W^0\right)$ and it is a generalized solution of problem (13)-(15). Since the norm in Banach space is lower weakly semi-continuous, estimation (12) follows from (27).

Increment of functional and necessary optimality condition. Now, we find increment of the functional $J(v)$. To calculate the increment of the functional we'll use the expansion:

$$f(x,t,u_0(x,t)+\Delta u_\varepsilon(x,t), v_\varepsilon(x,t)) - f(x,t,u_0(x,t), v_\varepsilon(x,t)) =$$

$$=\frac{\partial f(x,t,u_0(x,t), v_\varepsilon(x,t))}{\partial u}\Delta u_\varepsilon(x,t) + \omega(u_0(x,t);\Delta u_\varepsilon(x,t)),$$

$$f_0(x,t,u_0(x,t)+\Delta u_\varepsilon(x,t), v_\varepsilon(x,t)) - f_0(x,t,u_0(x,t), v_\varepsilon(x,t)) =$$

$$=\frac{\partial f_0(x,t,u_0(x,t), v_\varepsilon(x,t))}{\partial u}\Delta u_\varepsilon(x,t) + \omega_0(u_0(x,t);\Delta u_\varepsilon(x,t)).$$

It is clear that

$$\Delta J(v_0) = J(v_\varepsilon) - J(v_0) =$$

$$=\int\limits_Q [f_0(x,t,u_0(x,t)+\Delta u_\varepsilon(x,t), v_\varepsilon(x,t)) - f_0(x,t,u_0(x,t), v_0(x,t))]\,dxdt.$$

Considering that $\Delta u_\varepsilon(x,t)$ and $\psi(x,t)$ are the generalized solutions of problems (9)-(11) and (5)-(7), after some transformations for the increment of the functional we get the expression:

$$\Delta J(v_0) = -\iint_Q \Delta_{v_\varepsilon} H(x,t,u_0(x,t),v_0(x,t),\psi(x,t))\,dxdt + \eta(\varepsilon), \tag{28}$$

where

$$\Delta_{v_\varepsilon} H(x,t,u_0(x,t),v_0(x,t),\psi(x,t)) = H(x,t,u_0(x,t),v_\varepsilon(x,t),\psi(x,t)) -$$

$$-H(x,t,u_0(x,t),v_0(x,t),\psi(x,t)),$$

$$\eta(\varepsilon) = \iint_Q [\psi(x,t)\,\omega(u_0(x,t);\Delta u_\varepsilon(x,t)) + \omega_0(u_0(x,t);\Delta u_\varepsilon(x,t)) +$$

$$+\frac{\partial \Delta_{v_\varepsilon} H(x,t,u_0(x,t),v_0(x,t),\psi(x,t))}{\partial u}\Delta u_\varepsilon(x,t)]dxdt.$$

Show that

$$\iint_Q [\psi(x,t)\,\omega(u_0(x,t);\Delta u_\varepsilon(x,t)) + \omega_0(u_0(x,t);\Delta u_\varepsilon(x,t))]dxdt = 0\left(\varepsilon^2\right). \tag{29}$$

Indeed, we take into account the form $\omega_0(u_0(x,t);\Delta u_\varepsilon(x,t))$, the conditions imposed on $f(x,t,u(x,t),v(x,t))$ and the mean-value theorem (see [4]), we can write

$$\left|\iint_Q [\psi(x,t)\,\omega(u_0(x,t);\Delta u_\varepsilon(x,t))]dxdt\right| \le$$

$$\le \iint_Q |\psi(x,t)|\,|f(x,t,u_0(x,t)+\Delta u_\varepsilon(x,t),v_0) -$$

$$\left.-f(x,t,u_0(x,t),v_0(x,t)) - \frac{\partial f(x,t,u_0(x,t),v_0(x,t))}{\partial u}\Delta u_\varepsilon(x,t)\right| dxdt \le$$

$$\le \iint_Q |\psi(x,t)| \sup_{0\le\theta\le 1}\left|\frac{\partial f(x,t,u_0(x,t)+\theta\Delta u_\varepsilon(x,t),v_0(x,t))}{\partial u} -\right.$$

$$\left.-\frac{\partial f(x,t,u_0(x,t),v_0(x,t))}{\partial u}\right| \times$$

$$\times|\Delta u_\varepsilon(x,t)|\,dxdt \le C\iint_Q |\psi(x,t)|\,|\Delta u_\varepsilon(x,t)|^2\,dxdt.$$

Since $\psi(x,t) \in C\left(0,T;W^1,W^0\right)$, then by the imbedding theorem (see[5] $\psi(x,t)$ is a continuous function on $\bar{Q}$. Then by estimation (12) the right hand side of the last inequality

$$C\iint_Q |\psi(x,t)|\,|\Delta u_\varepsilon(x,t)|^2\,dxdt \le C\iint_Q |\Delta u_\varepsilon(x,t)|^2\,dxdt \le C\varepsilon^3.$$

Hence we get:

$$\iint_Q [\psi(x,t)\,\omega(u_0(x,t);\Delta u_\varepsilon(x,t))]dxdt = o\left(\varepsilon^2\right). \tag{30}$$

Now, we show that

$$\iint_Q \omega_0 \left(u_0 \left(x,t\right); \Delta u_\varepsilon \left(x,t\right)\right) dxdt = o\left(\varepsilon^2\right). \tag{31}$$

In fact, if we take into account the form $\omega_0 \left(u_0 \left(x,t\right); \Delta u_\varepsilon \left(x,t\right)\right)$, conditions imposed on $f_0 \left(x,t,u \left(x,t\right), v \left(x,t\right)\right)$, mean value theorem and estimation (12) we can write

$$\left| \iint_Q [\omega_0 \left(u_0 \left(x,t\right); \Delta u_\varepsilon \left(x,t\right)\right)] dxdt \right| \leq \iint_Q \left| f_0 \left(x,t,u_0 \left(x,t\right) + \Delta u_\varepsilon \left(x,t\right), v_0\right) - \right.$$

$$\left. - f_0 \left(x,t,u_0 \left(x,t\right), v_0 \left(x,t\right)\right) - \frac{\partial f_0 \left(x,t,u_0 \left(x,t\right), v_0 \left(x,t\right)\right)}{\partial u} \right| \left|\Delta u_\varepsilon \left(x,t\right)\right| dxdt \leq$$

$$\leq C \iint_Q \left|\Delta u_\varepsilon \left(x,t\right)\right|^2 dxdt \leq C\varepsilon^3 = o\left(\varepsilon^2\right).$$

Thus, from (30) and (31) we get the validity (29). From estimation (12) and definition of $v_\varepsilon \left(x,t\right)$ we have:

$$\iint_Q \frac{\partial \Delta_{v_\varepsilon} H \left(x,t,u_0 \left(x,t\right), v_0 \left(x,t\right), \psi \left(x,t\right)\right)}{\partial u} \Delta u_\varepsilon \left(x,t\right) dxdt =$$

$$= \iint_{\Pi_\varepsilon} \frac{\partial \Delta_{v_\varepsilon} H \left(x,t,u_0 \left(x,t\right), v_0 \left(x,t\right), \psi \left(x,t\right)\right)}{\partial u} \Delta u_\varepsilon \left(x,t\right) dxdt = o\left(\varepsilon^2\right). \tag{32}$$

Then from (29) and (32) we get $\eta \left(\varepsilon\right) = o\left(\varepsilon^2\right)$. Therefore, from (28) by the increment of the functional $J \left(v\right)$ we get the relation

$$\Delta J \left(v_0\right) = - \iint_{\Pi_\varepsilon} \Delta_v H \left(x,t,u_0 \left(x,t\right), v_0 \left(x,t\right), \psi \left(x,t\right)\right) dxdt + o\left(\varepsilon^2\right).$$

If $\left(u_0, v_0\right)$ is an optimal pair, then $\Delta J \left(v_0\right) \geq 0$. Considering that $\left(\sigma, \tau\right)$ is a tame point of all the functions in the problem, we have:

$$H \left(\sigma,\tau,u_0 \left(\sigma,\tau\right), v, \psi \left(\sigma,\tau\right)\right) \leq H \left(\sigma,\tau,u_0 \left(\sigma,\tau\right), v_0 \left(\sigma,\tau\right), \psi \left(\sigma,\tau\right)\right).$$

Thus, we prove the following theorem:

Theorem. *Let conditions 1.-2. be fulfilled. If* $\left(u_0 \left(x,t\right), v_0 \left(x,t\right)\right)$ *is an optimal pair and* $\psi \left(x,t\right)$ *is an approximate solution of problem (5)-(7), then for almost all* $\left(x,t\right) \in Q$ *and for all* $v \in [\alpha, \beta]$ *the following inequality is valid:*

$$H \left(x,t,u_0 \left(x,t\right), v, \psi \left(x,t\right)\right) \leq H \left(x,t,u_0 \left(x,t\right), v_0 \left(x,t\right), \psi \left(x,t\right)\right). \tag{33}$$

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