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TO THE PROBLEM ON MULTIPLE SUMMABILITY OF EXPANSIONS IN EIGEN FUNCTIONS OF IRREGULAR BOUNDARY VALUE PROBLEMS OF FOURTH ORDER

Abstract

The paper is devoted to studying in $L_2[0;1]$ *the problem of* $k-$ *fold summability of expansions in a system of eigen functions of an irregular boundary value problem generated by the differential expression*

$$l(y,\lambda)=\sum_{i+j=n} p_i\lambda^i y^{(j)} \tag{1}$$

with the splitting boundary conditions

$$\begin{cases} u_i(y)\equiv y^{(\chi_i)}(\delta_i)=0, \\ \delta_i=0, \quad i=\overline{1,l}, \\ \delta_i=1, \quad i=\overline{l+1,n}, \\ 0\le\chi_1<...<\chi_l\le n-1 \\ 0\le\chi_{l+1}<...<\chi_n\le n-1 \end{cases} \tag{2}$$

where $p_i\in C \quad \left(\overline{0,n}\right), \quad p_0,\ p_n\neq 0,\ \sum_{i=1}^{n-1}|p_i|>0,\ \lambda$ *is a complex parameter.*

The problem on $k-$ *fold completeness* $(1\le k\le n)$ *of a system of eigen and adjoint functions (s.e.a.f)* $\{Y_i(x,\lambda)\}_{i=\overline{1,\infty}}$ *of the considered problems was studied in a number of papers, for example, in* [2], *where it was shown that multiple completeness of s.e.a.f depends on the type of boundary conditions* (2) *and arrangement of the roots* k_i $\left(i=\overline{1,n}\right)$ *of a characteristic equation. The presence of multiple complete s.e.a.f makes urgent the problem on convergence of multiple expansions in Fourier series by this system for functions from the domain of definition of the operetor* L *generated by the problem* $(1)-(2)$. *So, in* [3] *involving* $n-$ *fold complete s.e.a.f the problem on* $n-$ *fold summability of expansions for differential operetors of even and odd orders was studied in the case* $|2l-n|=0;1$. *In* [4; 5] *similar problem was solved in the case* $2\le|2l-n|\le n-2$.

In the suggested paper we'll see how the problem on multiple summability of these expansions involving $k-$ *fold* $(1\le k\le 3)$ *complete in* $L_2[0;1]$ *s.e.a.f is solved for differential expressions of fourth order.*

Thus, in $L_2[0;1]$ we consider a boundary value problem for the differential expression

$$l(y,\lambda)=\sum_{i+j=4} p_i\lambda^i y^{(j)} \tag{1'}$$

with splitting boundary conditions

$$\begin{cases} y^{(\chi_j)}(0)=0, \quad i=\overline{1,l} \\ y^{(\chi_j)}(1)=0, \quad j=\overline{l+1,4} \end{cases} \tag{2'}$$

and let the arrangement of the roots k_i $\left(i=\overline{1,4}\right)$ of the characteristic equation

$$\sum_{i+j=4} p_i k^j = 0 \tag{3}$$

correspond to the conditions 1)-4) of the paper [2] :

1) $k_i \neq 0$ are simple $\left(i=\overline{1,4}\right)$;

2) there exists a straight line passing through the origin and not containing the roots k_i such that at each half-planes generated by it the number of roots k_i $\left(i=\overline{1,4}\right)$ is not less than $(n-l)$;

3) $\det\{k_j^{\chi_i}\}_{i,j=1,l} \cdot \det\{k_j^{\chi_i}\}_{i,j=\overline{1,n-l}} \neq 0$;

4) Let there exist a ray going out from the origin on which maximal number of the roots of equation (3) equals $m > n-l$. In the considered case this is possible for $m=3$ for $|2l-n|=0$ and $m=2;3$ for $|2l-n|=2$.

As is known [1], equation $(1')$ has a fundamental system of solutions $\{y_i(x,\lambda)\}_{i=\overline{1,4}}$ of the form

$$\frac{d^s y_i(x,\lambda)}{dx^s} = (\lambda k_i)^s \exp(\lambda k_i x), \quad \left(s=\overline{0,3}\right) \tag{4}$$

where λ are the roots of the characteristic determinant

$$\Delta(\lambda) = \det\{u_i(y_i)\}_{i,j=\overline{1,4}} \tag{5}$$

The roots $\Delta(\lambda)=0$ are the eigen values of the problem $(1')-(2')$ and arranged according for example to [8], at the angles of arbitrary small opening containing mean perpendiculars to the sides of indicator diagram J_Δ of the function $\Delta(\lambda)$. They generate $i-$ series $\{\lambda_{ij}\}_{j=\overline{1,\infty}}$ by the number of sides J_Δ.

Denote

$$\widetilde{C} = C \setminus \bigcup_{j=1}^{\infty} \bigcup_i u_\varepsilon(\lambda_{ij})$$

and we'll conduct all the reasonings in $\widetilde{C}$.

We'll use the following [2] :

Definition. *Let on q rays going out from the origin, the number of roots k_i $\left(i=\overline{1,n}\right)$ equal m_i , $m_i > n-l$ $\left(i=\overline{1,q}\right)$. The number*

$$d = \max_{1\le i\le q} (m_i - (n-l))$$

is said to be a defective number of problem $(1)-(2)$.

It holds [2].

Theorem. *It the roots k_i $\left(i=\overline{1,n}\right)$ of problem $(1)-(2)$ satisfy conditions 1)-4), then s.e.a.f of this problem is $k-$ fold complete in $L_2[0;1]$, where*

$$k = n - d$$

and d is a defective number of the problem.

We denote by M the least convex domain containing the roots k_i $\left(i=\overline{1,4}\right)$ of problem $(1')-(2')$. Let's consider two cases:

Case I. Let M be a segment and $(0,0)\in M$

Case II. Let M be a triangle and $(0,0) \in M$

In each of these cases we study expansion in Fourier series by s.e.a.f $\{Y_i(x,\lambda)\}_{i=\overline{1,\infty}}$

$$f(x) = \sum_{i=1}^{\infty} (f, Z_i)\, Y_i \tag{6}$$

where $\{Z_i\}_{i=\overline{1,\infty}}$ is a s.e.a.f of the operator L^*

As the considered problem is irregular in the sense of Tamarkin, in some sectors in $\widetilde{C}$ exponential growth of Green function $G(x,\xi,\lambda)$ adversely affect the convergence of series (6). In this relation we can speak only on natural in this case Abel summation method.

Definition. *We'll say that s.e.a.f of the operator L possesses a property of $n-$fold summability by Abel method of order $\gamma = (\gamma_1, ..., \gamma_p)$ of Fourier series for*

$$f(x) \in L_2^k[0;1] \cap D(L),$$

if in the sense of the norm of the space $L_2[0;1]$ there exists a limit

$$\lim_{t \to 0} u_\nu(x,t) = f_\nu(x), \qquad \left(\nu = \overline{0, k-1}\right),$$

where

$$u_\nu(x,t) = \sum_{s=1}^{\infty} \sum_{k=1}^{p} \left(\lambda_{s,k}^{\nu}\ f_\nu, Z_s\right) Y_s \exp\left(-\left(\lambda_{s,k}\ w_k\right)^{\gamma_k} t\right), \tag{7}$$

where w_k $\left(k = \overline{1,p}\right)$ are such that

$$\operatorname{Re}\left(\lambda_{s,k}\ w_k\right)^{\gamma_k} > 0.$$

Show that in each of the cases 1 and 2 there hold the theorems:

Theorem 1. *Series* (6) *is not summable by Abel method, no matter what order and multiplicity.*

Theorem 2. *Series* (6) *is k_1- fold summable* $(1 \le k_1 \le 3)$ *by Abel method of order $\gamma = (\gamma_1, \gamma_2, ..., \gamma_7)$, where $\gamma_j > 1$ $\left(j = \overline{1,7}\right)$ are the numbers depending on the concrete problem.*

Proof of theorem 1. We'll use the lemma [6]*:*

Lemma. *For the series* (6) *to be uniformly $k-$fold summed by Abel method of order $\gamma = (\gamma_1, ..., \gamma_p)$ to $f(x) \in L_2^k[0;1] \cap D(L)$ it is necessary and sufficient that*

$$\lim_{t \to +0} J_\nu(x,t) = f_\nu(x), \qquad \left(\nu = \overline{0, k-1}\right),$$

where

$$J_\nu(x,t) = \frac{1}{2\pi i} \int_{\Gamma} \left[\exp(-\lambda w)^{\gamma} t\right] \lambda^{\nu} d\lambda \int_{0}^{1} G(x,\xi,\lambda)\, f_\nu(\xi,\lambda)\, d\xi d\lambda. \tag{8}$$

Here Γ is a system of extended contours constructed in an ordinary way $[8]$.

Not loosing generality, in case 1 we put $\operatorname{Im} k_i = 0$ $\left(i = \overline{1,4}\right)$ and numerate k_i proceeding from the condition

$$\operatorname{Re} k_1 < ..... < \operatorname{Re} k_4. \tag{9}$$

We accept the following denotation as well: let $|\Omega|$ denote the number of summands in the set Ω and let $\lambda = R\exp i\varphi$.It holds

Lemma1. *Characteristic determinant* $\Delta(\lambda)$ *of problem* $(1')-(2')$ *is a first order growth entire function of normal type and satisfies the asymptotic estimate.*

$$|\Delta(\lambda)| \geq |\lambda|^{\chi_1+\ldots+\chi_4} \exp \lambda \sum_{i\in\Omega} k_i \ , \tag{10}$$

where

$$|\Omega| = \begin{cases} 1, & \text{for} \quad |2l-n| = 2, \\ 2, & \text{for} \quad |2l-n| = 0. \end{cases}$$

In fact, allowing for numeration (9) and expression (5) the segment

$$J_\Delta = \begin{cases} [k_1; k_4], & \text{for} \quad |2l-n| = 2 \\ [k_1+k_2; k_3+k_4], & \text{for} \quad |2l-n| = 0 \end{cases}$$

will be indicator diagram of the function $\Delta(\lambda)$.

Let's introduce the growth indicator

$$h_\Delta(\varphi) = \lim_{R\to\infty} \frac{\ln |R\exp i\varphi|}{R}.$$

By Polia theorem, in the left and right half-planes

$$h_\Delta(\varphi) = K(-\varphi),$$

where $K(-\varphi)$ is a support function of conjugated diagram,and

$$K(\varphi) = \sup_{x+iy\in J_\Delta} (x\cos\varphi + y\sin\varphi) =$$

$$= \begin{cases} \sup\limits_{j=\overline{1,4}} k_j (\cos\varphi + i\sin\varphi), & \text{for} \quad |2l-n| = 2, \\ \sup\limits_{i\neq j} (k_i+k_j)(\cos\varphi + i\sin\varphi), & \text{for} \quad |2l-n| = 0 \end{cases}$$

The latter proves lemma 1.

Let's construct and study the function $H(x,\xi,\lambda)$ for $x<\xi$

$$H(x,\xi,\lambda) = \lambda^{\chi_1+\ldots+\chi_4-(n-1)} \times$$

$$\times \begin{cases} \begin{vmatrix} g(x,\xi) & y_1(x) & \ldots & y_4(x) \\ u_1(g) & u_1(y_1) & \ldots & u_4(y_4) \\ \ldots & \ldots & \ldots & \ldots \\ u_3(g) & u_3(y_1) & \ldots & u_3(y_4) \\ 0 & u_4(y_1) & \ldots & u_4(y_4) \end{vmatrix}, & \text{for} \ |2l-n| = 2 \\ \\ \begin{vmatrix} g(x,\xi) & y_1(x) & \ldots & y_4(x) \\ u_1(g) & u_1(y_1) & \ldots & u_1(y_4) \\ u_2(g) & u_2(y_1) & \ldots & u_2(y_4) \\ 0 & u_3(y_1) & \ldots & u_3(y_4) \\ 0 & u_4(y_1) & \ldots & u_4(y_4) \end{vmatrix}, & \text{for} \quad |2l-n| = 0 \end{cases} \tag{11}$$

necessary calculations allow to get the following expression for $H(x,\xi,\lambda)$ for $x<\xi$:

$$H(x,\xi,\lambda)=\lambda^{\chi_1+\ldots+\chi_4-(n-1)}\times$$

$$\times\begin{cases}\sum\limits_{\substack{s=1\\ s\neq m}}^{4}\sum\limits_{m=1}^{4}C_{sm}\exp\left(\lambda k_s x+\lambda k_m(1-\xi)\right)+\\ +\sum\limits_{i=1}^{4}D_i\exp\lambda k_i(1+x-\xi), & \text{for}\quad |2l-n|=2\\ \sum\limits_{s=1}^{4}\sum\limits_{m,j=1}^{4}C_{smj}\exp\left(\lambda k_s x+\lambda k_m(1-\xi)+\lambda k_j\right)+\\ +\sum\limits_{\substack{i,j=1\\ i\neq j}}^{4}D_{ij}\exp\lambda k_i(1+x-\xi)+\lambda k_j\ , & \text{for}\quad |2l-n|=0\end{cases}\tag{12}$$

Similarly, for $x\geq\xi$ we have:

$$H(x,\xi,\lambda)=\lambda^{\chi_1+\ldots+\chi_4-(n-1)}\times$$

$$\times\begin{cases}\begin{vmatrix}g(x,\xi) & y_1(x) & \ldots & y_4(x)\\ 0 & u_1(y_1) & \ldots & u_4(y_4)\\ \ldots & \ldots & \ldots & \ldots\\ 0 & u_3(y_1) & \ldots & u_3(y_4)\\ u_4(g) & u_4(y_1) & \ldots & u_4(y_4)\end{vmatrix}, & \text{for}\ |2l-n|=2\\ \\ \begin{vmatrix}g(x,\xi) & y_1(x) & \ldots & y_4(x)\\ 0 & u_1(y_1) & \ldots & u_1(y_4)\\ 0 & u_2(y_1) & \ldots & u_2(y_4)\\ u_3(g) & u_3(y_1) & \ldots & u_3(y_4)\\ u_4(g) & u_4(y_1) & \ldots & u_4(y_4)\end{vmatrix}, & \text{for}\quad |2l-n|=0\end{cases}$$

In this connection we get for $x\geq\xi$

$$H(x,\xi,\lambda)\underset{x<\xi}{=}\lambda^{\chi_1+\ldots+\chi_4-(n-1)}\times$$

$$\times\begin{cases}\sum\limits_{s=1}^{4}\sum\limits_{i=1}^{4}B_{s_i}\exp\lambda k_s(x-\xi)+\lambda k_i+\\ +\sum\limits_{\substack{s=1\\ s\neq m}}^{4}\sum\limits_{m=1}^{4}C_{sm}\exp\left(\lambda k_s x+\lambda k_m(1-\xi)\right), & \text{for}\quad |2l-n|=2\\ \sum\limits_{s=1}^{4}\sum\limits_{\Omega}B_{\Omega}\exp\lambda k_s(x-\xi)+\lambda\sum\limits_{\substack{i\in\Omega\\ |\Omega|=2}}k_i+\\ +\sum\limits_{\substack{s=1\\ s\neq m\neq j}}^{4}\ \sum\limits_{m,j=1}^{4}C_{sm_j}\exp\left(\lambda k_s x+\lambda k_m(1-\xi)+\lambda k_j\ \right), & \text{for}\quad |2l-n|=0\end{cases}$$

The segment

$$J_H = \begin{cases} [k_1+k_2; k_3+k_4], & \text{for} \quad |2l-n|=2 \\ [k_1+k_2+k_3; k_2+k_3+k_4], & \text{for} \quad |2l-n|=0 \end{cases}$$

well be an indicator diagram J_H of the function $H(x,\xi,\lambda)$.

By the Polia theorem in the left and right half-planes the function $H(x,\xi,\lambda)$ will have its own growth indicator

$$h_H = \begin{cases} (k_i+k_{i+1})(\cos\theta + i\sin\theta), & i=1 \ \text{from the left} \\ & \quad i=3 \ \text{from the right} \\ (k_i+k_{i+1}+k_{i+2})(\cos\theta+i\sin\theta), & i=1 \ \text{from the left} \\ & \quad i=2 \ \text{from the right} \end{cases}$$

Then, it holds

Lemma 2. *The function $H(x,\xi,\lambda)$ of boundary value problem$(1')-(2')$ is a first order growth entire function of normal type and has two growth indicators.*

Then $J_\Delta \subset J_H$, if the roots $k_i \quad \left(i=\overline{1,4}\right)$ are arranged by two in each of half-planes and $J_\Delta \subset J_H$ in that of half-planes wherein $(k_i+k_{i+2}+k_{i+3}) > (k_i+k_{i+1})$ in the case when half-planes contain unequal number $k_i - x$.

Then, it is easy to calculate the growth indicator of the Green function $G(x,\xi,\lambda)$:

$$h_G(\theta) = h_H(\theta) - h_\Delta(\theta).$$

The above-stated one allows to assert that, at least, in one of half-planes $h_\Delta(\theta) > 0$, and consequently, the Green function grows there exponentially. Thus, it holds

Theorem 3. *The Green function $G(x,\xi,\lambda)$ of problem $(1')-(2')$ is a first order growth entire function of normal type and satisfies the asymptotic estimate*

$$|G(x,\xi,\lambda)| \underset{|\lambda|\to\infty}{\leq} \frac{M \ \exp \lambda k_i (x-\xi)}{|\lambda|^3} \qquad (x \geq \xi) \tag{13}$$

at least, in one of half-planes.

Consider estimate (13) in (8). Moreover, if we can choose the number γ so that

$$\begin{aligned} &1)\ \gamma > 1 \\ &2)\ \operatorname{Re}\lambda^\gamma > 0 \\ &3)\operatorname{Re}\lambda k_i(x-\xi) - \lambda^\gamma t \leq 0 \end{aligned} \tag{14}$$

then, proceeding from the lemma $[6]$, the summation ($k-$ fold summation) of series (6) should hold, uniformly over $x,\xi \in [0,1]$. However, fullfilment of conditions 1) and 2) from (14) is impossible within the frames of theorem 3, consequently, the assertion of theorem 1 holds.

In case 2 the similar reasonings lead to the following results: with regard to the fact that $(0,0) \in M$ in case 2 only boundary conditions of type $|2l-n| = 2$ are to be considered. Introduce the denotation $k_j = \alpha_j + i\beta_j \quad \left(j=\overline{1,4}\right)$. Not loosing generality, we put $\beta_j = 0 \quad (j=3,4)$ and preserve the numeration (9) of the roots $k_j \quad \left(j=\overline{1,4}\right)$. Then a triangle with vertices at the point k_1, k_2, k_4 will be an indicator diagram J_Δ of the function $\Delta(\lambda)$. It holds

Lemma 1′. *Characteristic determinant $\Delta(\lambda)$ of problem $(1')-(2')$ is a first order growth entire function of normal type and satisfies the asymptotic estimate*

$$|\Delta(\lambda)| \underset{|\lambda|\to\infty}{\geq} \lambda^{\chi_1+\ldots+\chi_4}\exp\lambda k_i$$

$i=1;2;4$, respectively, in the sectors

$$S_1=\left\{\varphi|\,\pi\leq\varphi\leq 2\pi-arctg\frac{\alpha_4+|\alpha_2|}{|\beta_2|}\right\}$$

$$S_2=\left\{\varphi|\,arctg\frac{\alpha_4+|\alpha_1|}{|\beta_1|}\leq\varphi\leq\pi\right\}$$

$$S_3=\left\{\varphi|-arctg\frac{\alpha_4+|\alpha_2|}{|\beta_2|}\leq\varphi\leq arctg\frac{\alpha_4+|\alpha_1|}{|\beta_1|}\right\}.$$

And, consequently the function $\Delta(\lambda)$ has three growth indicators

$$h_\Delta^i(\theta)=K(-\theta)=\sup_{x+iy\in J_\Delta}(x\cos\theta+y\sin\theta)=$$

$$=\begin{cases}\alpha_1\cos\theta+\beta_1\sin\theta, & \text{for}\quad \theta\in S_1,\\ \alpha_2\cos\theta+\beta_2\sin\theta, & \text{for}\quad \theta\in S_2,\\ \alpha_4\cos\theta, & \text{for}\quad \theta\in S_3.\end{cases}$$

For the function $H(x,\xi,\lambda)$ the assertions (11) and (12) are preserved for the case $|2l-n|=2$.

A convex hexagon with verices at the points: $\begin{array}{l}(k_1+k_2);(k_3+k_1);(k_3+k_2);\\(k_4+k_1);(k_4+k_2);(k_3+k_4).\end{array}$
will be an indicator diagram J_Δ of the function $\Delta(\lambda)$.

We'll have respectively, six growth indicators:

$$h_H^1(\theta)=(k_3+k_4)\cos\theta,\qquad \theta\in C_1=\left\{\varphi|-arctg\frac{\alpha_1+\alpha_3}{|\beta_1|}\leq\theta\leq arctg\frac{\alpha_2+\alpha_3}{|\beta_2|}\right\}$$

$$h_H^2(\theta)=(k_2+k_4)(\cos\theta+i\sin\theta),\qquad \theta\in C_2=\left\{\varphi|\,arctg\frac{\alpha_1+\alpha_3}{|\beta_2|}\leq\theta\leq\frac{\pi}{2}\right\}$$

$$h_H^3(\theta)=(k_3+k_2)(\cos\theta+i\sin\theta),\qquad \theta\in C_3=\left\{\varphi|\,\frac{\pi}{2}\leq\theta\leq\pi-arctg\frac{\alpha_2+|\alpha_1|}{|\beta_1|}\right\}$$

$$h_H^4(\theta)=(k_1+k_2)(\cos\theta+i\sin\theta),\qquad \theta\in C_4=$$

$$=\left\{\pi-arctg\frac{\alpha_3+|\alpha_1|}{|\beta_1|}\leq\theta\leq\pi-arctg\frac{\alpha_3+|\alpha_2|}{|\beta_2|}\right\}$$

$$h_H^5(\theta)=(k_1+k_3)(\cos\theta+i\sin\theta),\qquad \theta\in C_5=\left\{\pi-arctg\frac{\alpha_3+|\alpha_2|}{|\beta_2|}\leq\theta\leq\frac{3\pi}{2}\right\}$$

$$h_H^6(\theta)=(k_1+k_4)(\cos\theta+i\sin\theta),\qquad \theta\in C_6=\left\{\frac{3\pi}{2}\leq\theta\leq -arctg\frac{\alpha_4+|\alpha_1|}{|\beta_1|}\right\}.$$

Thus, it holds

Lemma 2′. *The function $H(x,\xi,\lambda)$ of boundary value problem $(1')-(2')$ is an entire function of first order growth and has six grow indicators at six sectors with openings $<\pi$.*

Count the indicators of the function $G(x,\xi,\lambda)$. In the sector $C_1\cap S_3$, we have:

$$h_G(\theta)=h_H(\theta)-h_\Delta(\theta)=k_3\cos\theta,\qquad -arctg\frac{\alpha_1+\alpha_3}{|\beta_1|}\le\theta\le arctg\frac{\alpha_2+\alpha_3}{|\beta_2|}$$

in the sector $C_1\cap S_3$ we have:

$$h_G(\theta)=h_H(\theta)-h_\Delta(\theta)=k_2(\cos\theta+i\sin\theta),\qquad \frac{\alpha_2+\alpha_3}{|\beta_2|}\le\theta\le\frac{\alpha_4+\alpha_3}{|\beta_2|}$$

in the sector $C_2\cap S_2$ we have:

$$h_G(\theta)=h_H(\theta)-h_\Delta(\theta)=k_4\cos\theta,\qquad \frac{\alpha_2+\alpha_4}{|\beta_2|}\le\theta\le\frac{\pi}{2}$$

in the sector $C_3\cap S_2$ we have:

$$h_G(\theta)=h_H(\theta)-h_\Delta(\theta)=k_3\cos\theta<0,\qquad \frac{\pi}{2}\le\theta\le -arctg\frac{\alpha_3+|\alpha_1|}{|\beta_1|}$$

in the sector $C_4\cap S_2$ we have:

$$h_G(\theta)=h_H(\theta)-h_\Delta(\theta)=k_1(\cos\theta+i\sin\theta),\qquad \pi-arctg\frac{\alpha_3+|\alpha_1|}{|\beta_1|}\le\theta\le\pi$$

in the sector $C_4\cap S_1$ we have:

$$h_G(\theta)=h_H(\theta)-h_\Delta(\theta)=k_2(\cos\theta+i\sin\theta),\qquad \pi\le\theta\le arctg\frac{\alpha_3+|\alpha_2|}{|\beta_2|}$$

in the sector $C_5\cap S_1$ we have:

$$h_G(\theta)=h_H(\theta)-h_\Delta(\theta)=k_3\cos\theta<0,\qquad \pi+arctg\frac{\alpha_3+|\alpha_2|}{|\beta_2|}\le\theta\le\frac{3\pi}{2}$$

in the sector $C_6\cap S_1$ we have:

$$h_G(\theta)=h_H(\theta)-h_\Delta(\theta)=k_4\cos\theta,\qquad \frac{3\pi}{2}\le\theta\le 2\pi-arctg\frac{\alpha_4+\alpha_1}{|\beta_1|}$$

in the sector $C_6\cap S_3$ we have:

$$h_G(\theta)=h_H(\theta)-h_\Delta(\theta)=k_1(\cos\theta+i\sin\theta),\qquad arctg\frac{\alpha_4+\alpha_1}{|\beta_1|}\le\theta\le -arctg\frac{\alpha_4+\alpha_1}{|\beta_1|}$$

Consider the obtained estimates in (8) and choose the vector $\gamma=(\gamma_1....\gamma_7)$ proceeding from conditions (14). To this end, we consider one of the sectors with exponential growth $G(x,\xi,\lambda)$. Let this be a sector of the opening ψ. Then, the condition

$$R\ \cos(\varphi+\psi)-R^\gamma t\cos\gamma\varphi\le R-R^\gamma t\cos\gamma\varphi\le 0$$

may be achieved for
$$1<\gamma\leq\frac{\pi}{\psi}. \tag{15}$$
We get a condition on
$$t>\frac{1}{R^{\gamma-1}\cos\gamma\varphi}>\frac{r}{R^{\gamma-1}}. \tag{16}$$
Similar reasonings at other sectors lead to the same estimates $(15),(16)$.

Choose $t>\max\limits_{i=\overline{1,7}}\left(\frac{r_i}{R^{\gamma_i-1}}\right)$ and tending $R\to\infty$ we get that the function
$$G_t(x,\xi,\lambda)=G(x,\xi,\lambda)\,e^{-\lambda^{\gamma}t}$$
we'll everywhere satisfy the conditions $\widetilde{C}$
$$a)\quad |G_t(x,\xi,\lambda)|\leq\frac{M}{|\lambda|^3}$$
$$b)\quad \lim_{t\to 0}G_t(x,\xi,\lambda)=G(x,\xi,\lambda)$$
and convergence of (8) proceeds from these conditions, and consequently, convergence of (6) uniformly with respect to $x,\xi\in[0;1]$. Notice that this convergence is serial i.e. it is carried out by one of systems of extended contours $\Gamma_i=\bigcup\limits_{j=1}^{\infty}\Gamma_j$, containing one of series $\{\lambda_{ij}\}_{j=\overline{1,\infty}}$ and the final result is taken over the totality of contours $\Gamma=\bigcup\limits_{i=1}^{2}\Gamma_i$ (in the case, if M is a segment) and $\Gamma=\bigcup\limits_{i=1}^{3}\Gamma_i$ (in the case, if M is a triangle).

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Received September 22, 2006; Revised December 28, 2006