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# BEHAVIOR OF THE SOLUTION OF THE CAUCHY PROBLEM FOR BARENBLATT-ZHELTOV-KOCHINA TYPE EQUATION AT GREAT VALUES OF TIME

**Abstract**

*In the paper we obtain the estimation of the Cauchy problem solution for Barenblatt-Zheltov-Kochina type equation at great values of time.*

By studying liquid filtration in cracked rocks with porosity G.I. Barenblatt, Yu.P. Zheltov and I.N. Kochina in [1] obtained an equation unsolved with respect to time derivative of the form

$$(\eta\Delta - 1)\, D_t u\,(x,t) + \chi\Delta u\,(x,t) = 0, \quad x \in R_3 \;, \tag{I}$$

were $\Delta$ is a Laplace operator with respect to $x = (x_1, x_2, x_3) \in R_3$, $R_3$ is a three-dimensional Eucleadian space, $\eta$ is a permeability coefficient, $\chi$ is a piezoconductivity coefficient. Different boundary value problems for this equation in a bounded domain, mainly in one-dimensional, three-dimensional spaces were stated in the paper [1] and expression for pressure difference in the both sides of the break surface was obtained.

The mixed problem for equation (I) in a multivariate cylindrical domain was studied in the paper [2]. In this paper we obtain the estimation of the Cauchy problem solution for Barenblatt-Zheltov-Kochina type equation at great values of time. In $R_{m+n} \times (0,\infty)$ we consider the following Cauchy problem

$$\left(\sigma^2\Delta_{m,n} - \beta^2\right) D_t u\,(x,t) + \omega^2\Delta_m u\,(x,t) = 0 \tag{1}$$

$$u\,(x,t)|_{t=0} = \varphi\,(x)\,, \tag{2}$$

here $x = (x_1, x_2, ...x_{m+n})$,

$$\Delta_{m,n} = \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_m^2} + \frac{\partial^2}{\partial x_{m+1}^2} + ... + \frac{\partial^2}{\partial x_{m+n}^2},$$

$$\Delta_m = \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_m^2},$$

$\sigma, \beta, \omega$ are positive constants having physical sense. We'll assume that the function $u\,(x,t)$ for each $t$ with respect to $x$ is a distribution over $D\,(R_{m+n})$ ([3], p.40) continuous with respect to $t$, and we'll understand the solution of problem $(1)-(2)$ in the sense of distributions ([3], p. $124-178$), where the space of finite infinitely differentiable function in $R_{m+n}$ is denoted by $D\,(R_{m+n})$. Notice that equation (1) belongs to Sobolev-Galperin class of equations. Solvability problems of the Cauchy problem for this class of equations in the class of distributions was studied by A.G. Kostyuchenko and G.I. Eskin in [4].

**Representation of solution of the Cauchy problem (1)–(2).**
Assuming $u(x,t)$ as a distribution and perfoming Fourier transformation on problem $(1)-(2)$ we get the duality problem

$$\sigma^2\left(|s|^2+\beta^2\right)V_t(s,t)+\omega^2|\bar{s}|^2V(s,t)=0 \tag{3}$$

$$V(s,t)|_{t=0}=\tilde{\varphi}(s), \tag{4}$$

where the sign $\backsim$ denotes Fourier transformation with respect to $x$, $s=(s_1,s_2,...s_{m+n})$ is a duality variable to $x$ with respect to Fourier transformation

$$\bar{s}=(s_1,s_2,...s_m),\ \bar{\bar{s}}=(s_{m+1},s_{m+2},...s_{m+n}),\ |s|^2=|\bar{s}|^2=\left|\bar{\bar{s}}\right|^2\equiv r^2.$$

Having solved problem $(3)-(4)$ we get.

$$V(s,t)=e^{-\dfrac{\omega^2|\bar{s}|^2t}{\sigma^2|s|^2+\beta^2}}\tilde{\varphi}(s).$$

Hence, performing the Fourier inverse transformation on $V(s,t)$ for the solution of Cauchy problem $(1)-(2)$ we get

$$u(x,t)=\frac{1}{(2\pi)^{m+n}}\int_{R_{m+n}}...\int e^{-\dfrac{\omega^2|\bar{s}|^2t}{\sigma^2|s|^2+\beta^2}}\tilde{\varphi}(s)e^{-i(x,s)}ds=G(x,t)*\varphi(x),$$

where

$$G(x,t)=\frac{1}{(2\pi)^{m+n}}\int_{R_{m+n}}...\int e^{-\dfrac{\omega^2|\bar{s}|^2t}{\sigma^2|s|^2+\beta^2}}e^{-i(x,s)}ds.$$

The integral in the expression $G(x,t)$ doesn't converge in the ordinary sense. Therefore, taking into account

$$\tilde{\varphi}(s)=(-1)^{\mu}\left(1+|s|^2\right)^{-m}\widetilde{(1-\Delta_{m,n})^{\mu}\varphi}(s)$$

we represent the solution of Cauchy problem $(1)-(2)$ $u(x,t)$ in the form

$$u(x,t)=G_1(x,t)*(1-\Delta_{m+n})^{\mu}\varphi(s),$$

where

$$G_1(x,t)=\frac{1}{(2\pi)^{m+n}}\int_{R_{m+n}}...\int\left(1+|\bar{s}|^2\right)^{-\mu}e^{-\dfrac{\omega^2|\bar{s}|^2t}{\sigma^2|s|^2+\beta^2}}e^{-i(x,s)}ds. \tag{5}$$

We choose the number $\mu$ so that the integral in (5) converges absolutely. To this end we assume

$$2\mu=\begin{cases} m+n+1, & \text{if } m+n \text{ is odd}\\ m+n+2, & \text{if } m+n \text{ is even}\end{cases}$$

Now we get the estimation of $G_1(x,t)$ at great values of time.

**Asymptotic estimation of the function $\mathbf{G}_1(x,t)$ at great values of time.**

In (5) pass to spherical coordinates

$$\begin{aligned} s_1 &= r\cos\varphi_1 \\ s_2 &= r\sin\varphi_1\cos\varphi_2 \\ s_3 &= r\sin\varphi_1\sin\varphi_2\cos\varphi_3 \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ s_{m-1} &= r\sin\varphi_1\sin\varphi_2\ldots.\sin\varphi_{m-2}\cos\varphi_{m-1} \\ s_m &= r\sin\varphi_1\sin\varphi_2\ldots.\sin\varphi_{m-1}\cos\varphi_m \\ s_{m+1} &= r\sin\varphi_1\sin\varphi_2\ldots.\sin\varphi_m\cos\varphi_{m+1} \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ s_{m+n} &= r\sin\varphi_1\sin\varphi_2\ldots.\sin\varphi_{m+n-2}\sin\varphi_{m+n-1} \end{aligned} \tag{6}$$

then

$$s_1^2+s_2^2+\ldots+s_m^2 = r^2\cos^2\varphi_1+\ldots+r^2\sin^2\varphi_1\sin^2\varphi_2\ldots.\cos^2\varphi_m =$$
$$= r^2\left(1-\sin^2\varphi_1\ldots\sin^2\varphi_m\right)\equiv r^2T\left(\overline{\varphi}\right), \tag{7}$$

where

$$0\le\varphi_j\le\pi,\quad j=1,2,\ldots,m+n-2;\quad 0\le\varphi_{m+n-1}\le 2\pi,$$
$$\overline{\varphi}=\left(\varphi_1,\varphi_2,\ldots,\varphi_m\right),\ \overline{\overline{\varphi}}=\left(\varphi_{m+1},\varphi_{m+2},\ldots,\varphi_{m+n}\right).$$

Using (6) and (7) and passing in (5) to polar coordinates we get

$$G_1(x,t)=\frac{1}{(2\pi)^{m+n}}\int\limits_0^\infty\frac{r^{m+n-1}}{(1+r^2)^\mu}\times$$
$$\times\int\limits_0^\pi\ldots\int\limits_0^\pi\int\limits_0^{2\pi}\sin^{n-2}\varphi_{m+1}\sin^{n-3}\varphi_{m+2}\ldots.\sin\varphi_{m+n-2}d\varphi_{m+1}\ldots.d\varphi_{m+n-2}\times$$
$$\times\int\limits_0^\pi\ldots\int\limits_0^\pi\sin\varphi_1^{m+n-2}\sin\varphi_2^{m+n-3}\ldots.\sin^{n-1}\varphi_m e^{-\frac{\omega^2r^2tT(\overline{\varphi})}{\sigma^2(r^2+\beta^2)}}e^{i(x,r\delta(\varphi))}d\varphi_1\ldots d\varphi_m, \tag{8}$$

where $s=r\delta\left(\varphi_1,\ldots,\varphi_{m+n-1}\right)$.

In (8) denote the internal integral by $J\left(x,r,\overline{\overline{\varphi}},t\right)$

$$J\left(x,r,\overline{\overline{\varphi}},t\right)=\int_{K_m}\ldots\int\sin^{m+n-2}\varphi_1\sin^{m+n-3}\varphi_2\ldots.\sin^{n-1}\varphi_m\times$$
$$\times e^{-\dfrac{\omega^2r^2tT\left(\overline{\varphi}\right)}{\sigma^2\left(r^2+\beta^2\right)}}e^{i(x,r\delta(\varphi))}d\varphi_1\ldots d\varphi_m, \tag{9}$$

where

$$K_m = [0,\pi] \times [0,\pi] \times ... \times [0,\pi]$$

is $m$-dimensional cube. By $K_0$ denote $m$-dimensional cube

$$K_0 = \underbrace{\left[\frac{\pi}{2} - \varepsilon,\ \frac{\pi}{2} + \varepsilon\right] \times .... \times \left[\frac{\pi}{2} - \varepsilon,\ \frac{\pi}{2} + \varepsilon\right]}_{m \text{ times}},$$

and by $O_3 \equiv K_0^* \cup K_1^*$ a finite covering of the cube $K_m$ and write appropriate expansion of the unit

$$1 \equiv \sum_{v=0}^{1} \psi_v(\overline{\varphi}),$$

where $\psi_v(\overline{\varphi})$ are finite infinitely differentiable functions with support in $K_v^*$ .

Denote

$$J_v\left(x, r, \overline{\overline{\varphi}}, t\right) =$$

$$= \int_{K_v^*} ... \int \sin^{m+n-2}\varphi_1 ... \sin^{n-1}\varphi_m e^{-\dfrac{\omega^2 r^2 t T(\overline{\varphi})}{\sigma^2\left(r^2+\beta^2\right)}} \psi_v(\overline{\varphi})\, e^{i(x, r\delta(\varphi))} d\overline{\varphi}. \qquad (10)$$

The point $\varphi_1 = \frac{\pi}{2}, \varphi_2 = \frac{\pi}{2}, ...., \varphi_m = \frac{\pi}{2}$ is a simple saddle point of the function $T(\overline{\varphi})$. Really,

$$\frac{\partial}{\partial \varphi_j} T(\overline{\varphi}) = -2 \sin\varphi_j \cos\varphi_j \sin^2\varphi_1 ... \sin^2\varphi_{j-1} \sin^2\varphi_{j+1} \sin^2\varphi_m. \qquad (10')$$

Hence we get

$$\frac{\partial^2}{\partial \varphi_j^2} T(\overline{\varphi})\,|_{\overline{\varphi}=\overline{\varphi}_0} = -2, \qquad j = 1, 2, ...., m$$

and

$$\frac{\partial^2}{\partial \varphi_\mu \partial \varphi_j} T(\overline{\varphi})\,|_{\overline{\varphi}=\overline{\varphi}_0} = 0, \qquad \mu \neq j,$$

where

$$\overline{\varphi}_0 = \left[\frac{\pi}{2},\ \frac{\pi}{2}, ..., \frac{\pi}{2}\right].$$

Consequently

$$\det \left\| \frac{\partial^2 T(\overline{\varphi}_0)}{\partial \varphi_\mu \partial \varphi_j} \right\| = (-2)^m \neq 0,$$

i.e. the point $\overline{\varphi}_0$ is non-degenerate saddle point of the function $T(\overline{\varphi})$. Applying the saddle point method ([5], p. 418) to the integral $J_0\left(x, r, \overline{\overline{\varphi}}, t\right)$ as $t \to +\infty$ we get

$$J_0\left(x, r, \overline{\overline{\varphi}}, t\right) = 2^{-n} \pi^{-\left(\frac{m}{2}+n\right)} t^{-\frac{m}{2}} e^{i\left(x, r\delta\left(\frac{\pi}{2}\overline{\overline{\varphi}}\right)\right)} + O\left(t^{-\frac{m}{2}-1}\right). \qquad (11)$$

The domain $K_1^*$ doesn't contain the point $\varphi_j = \frac{\pi}{2}$, $j = 1, 2, ..., m$ with some neighborhood. Therefore, there exist the constants $C_1,\ c_1$ such that for $\overline{\varphi} \in K_1^*$

$$0 < c_1 \leq T(\overline{\varphi}) \leq C_1 < 1. \qquad (12)$$

For $G_1(x,t)$ it holds the representation

$$G_1(x,t)=\frac{1}{(2\pi)^{m+n}}\sum_{v=0}^{1}\int_0^{\pi}...\int_0^{\pi}\int_0^{2\pi}B_v\left(x,\overline{\overline{\varphi}},t\right)\times$$

$$\times\sin^{n-2}\varphi_{m+1}\sin^{n-3}\varphi_{m+2}....\sin\varphi_{m+n-2}d\overline{\overline{\varphi}}\equiv G_1^{(0)}(x,t)+G_1^{(1)}(x,t),$$

where

$$B_v\left(x,\overline{\overline{\varphi}},t\right)=\int_0^{\infty}\frac{r^{m+n-1}}{(1+r^2)^{\mu}}J_v\left(x,r,\overline{\overline{\varphi}},t\right)dr,\quad v=0,1. \tag{13}$$

Let's consider $B_1\left(x,\overline{\overline{\varphi}},t\right)$. Changing integration order in (13) we get

$$B_1\left(x,\overline{\overline{\varphi}},t\right)=\int_{K_1^*}...\int\sin^{m+n-2}\varphi_1....\sin^{n-1}\varphi_m\times\psi_1(\overline{\varphi})\times$$

$$\times\Big\{\int\limits_0^{\infty}\frac{r^{m+n-1}}{(1+r^2)^{\mu}}e^{-\dfrac{\omega^2r^2tT(\overline{\varphi})}{\sigma^2\left(r^2+\beta^2\right)}}e^{i(x,r\delta(\varphi))}dr\Big\}d\overline{\varphi}. \tag{14}$$

We represent the internal integral in (14) in the form

$$W(x,\varphi,t)=\left\{\int\limits_0^{a}+\int\limits_a^{\infty}\right\}\frac{r^{m+n-1}}{(1+r^2)^{\mu}}e^{-\dfrac{\omega^2r^2tT(\overline{\varphi})}{\sigma^2\left(r^2+\beta^2\right)}}e^{i(x,r\delta(\varphi))}dr\equiv$$

$$\equiv W^{(I)}(x,\varphi,t)+W^{(II)}(x,\varphi,t), \tag{15}$$

where $a>0$ is a sufficiently small number. In order to reduce $W^{(I)}(x,\varphi,t)$ to the form wherein the Watson lemma is applicable, we make substitution

$$\frac{\omega^2r^2T(\overline{\varphi})}{\sigma^2r^2+\beta^2}=\tau^2.$$

Hence

$$r=\frac{\beta\tau}{\left(\dfrac{\omega^2}{\sigma^2}T(\overline{\varphi})-\tau^2\right)^{\frac{1}{2}}}. \tag{16}$$

Substituting (16) in expression $W^{(I)}(x,\varphi,t)$ we get

$$W^{(I)}(x,\varphi,t)=$$

$$=\frac{\omega^2}{\sigma^2}\beta^{m+n-1}T(\overline{\varphi})\int_0^{\frac{\omega}{\sigma}\frac{aT^{1/2}(\overline{\varphi})}{(a^2+\beta^2)^{1/2}}}\frac{\tau^{m+n-1}e^{i\left(x,\frac{\beta\tau\delta(\varphi)}{\left(\frac{\omega^2}{\sigma^2}T(\overline{\varphi})-\tau^2\right)^{\frac{1}{2}}}\right)}e^{-t\tau^2}}{\left(\frac{\omega^2}{\sigma^2}T(\overline{\varphi})-\tau^2\right)^{\frac{m+n}{2}+1-\mu}}d\tau. \tag{17}$$

Taking into account estimation (12) and sufficient smallness of a from (17) we deduce that intergrand has no singularities in integration interval. Applying Watson lemma ([5], p. 58) to the integral in (17) as $t\to+\infty$ we get

$$W^{(I)}(x,\varphi,t)=\frac{1}{2}\beta^{m+n-1}\Gamma\Big(\frac{m+n}{2}\Big)\times$$

$$\times\left(\frac{\omega^2 T(\overline{\varphi})}{\sigma^2}\right)^{\mu-\frac{m+n}{2}} t^{-\frac{m+n}{2}}\left(1+|x|\,O\left(t^{-\frac{1}{2}}\right)\right). \tag{18}$$

Estimating by modulus $W^{(II)}(x,\varphi,t)$ and considering that the function $\dfrac{r^2}{r^2+\beta^2}$ monotonically increases, and $T(\overline{\varphi})\geq c_1>0$ , we get

$$\left|W^{(II)}(x,\varphi,t)\right|\leq Ce^{-\frac{\omega^2 a^2 T(\overline{\varphi})t}{\sigma^2(a^2+\beta^2)}}. \tag{19}$$

It follows from $(15),(18)$ and $(19)$ that as $t\to+\infty$

$$W(x,\varphi,t)=\frac{\beta^{m+n-1}}{2}\Gamma\left(\frac{m+n}{2}\right)\left(\frac{\omega^2 T(\overline{\varphi})}{\sigma^2}\right)^{\mu-\frac{m+n}{2}} t^{-\frac{m+n}{2}}\left(1+|x|\,O\left(t^{-\frac{1}{2}}\right)\right). \tag{20}$$

Now, let's consider $B_0\left(x,\overline{\overline{\varphi}},t\right)$. Substituting the expression $J_0\left(x,r,\overline{\overline{\varphi}},t\right)$ from (11) in (13) for $v>0$ we get

$$B_0\left(x,\overline{\overline{\varphi}},t\right)=2^{-n}\pi^{-\left(\frac{m}{2}+n\right)}\,t^{-\frac{m}{2}}\int\limits_0^\infty\frac{r^{m+n-1}}{(1+r^2)^\mu}e^{i\left(x,r\delta\left(\frac{\pi}{2}\overline{\overline{\varphi}}\right)\right)}dr\left(1+O\left(t^{-\frac{1}{2}}\right)\right).$$

Substituting the expression $B_0\left(x,\overline{\overline{\varphi}},t\right)$ in $G_1^{(0)}(x,t)$ and estimating by modulus we get

$$\left|G_1^{(0)}(x,t)\right|\leq G(m,n)\,t^{-\frac{1}{2}} \tag{21}$$

uniformly with respect to $x\in R_{m+n}$.

Further, substituting asymptotics $W(x,\varphi,t)$ from (20) in the expression $B_1\left(x,\overline{\overline{\varphi}},t\right)$ from (14) we get

$$B_1\left(x,\overline{\overline{\varphi}},t\right)=C_1(\beta,\omega,\sigma,m,n)\,t^{-\frac{m+n}{2}}\left(1+xO\left(t^{-\frac{1}{2}}\right)\right)$$

uniformly with respect to $\overline{\overline{\varphi}}\in K_{n-1}=\underbrace{[0,\pi]\times...\times[0,\pi]\times[0,2\pi]}_{(n-1)\text{ times}}$ , where

$$C_1(\beta,\omega,\sigma,m,n)=\frac{\beta^{m+n-1}}{2}\left(\frac{\omega^2}{\sigma^2}\right)^{\mu-\frac{m+n}{2}}\Gamma\left(\frac{m+n}{2}\right)\times$$

$$\times\int_{K_1^*}...\int\sin^{m+n-2}\varphi_1....\sin^{n-1}\varphi_m T^{\mu-\frac{m+n}{2}}(\overline{\varphi})\,d\overline{\varphi}. \tag{22}$$

From (13) and (22) it follows that as $t\to+\infty$

$$\left|G_1^{(1)}(x,t)\right|\leq C_1(\beta,\omega,\sigma,m,n)\,t^{-\frac{m+n}{2}} \tag{23}$$

uniformly with respect to $x\in R_{m+n}$.

From $(13),(21)$ and $(23)$ it follows that as $t\to+\infty$

$$|G_1(x,t)|\leq C(m,n,\beta,\sigma)\,t^{-\frac{m}{2}}\left(1+|x|\,t^{-\frac{1}{2}}\right) \tag{24}$$

for all $x \in R_{m+n}$.

From the above said one and (5) it follows the following theorem.

**Theorem 1.** *The Green function $G(x,t)$ of problem $(1)-(2)$ is generalized function on the space $D(R_{m+n})$ of singularity order $\mu$, for it it holds the representation*

$$G(x,t) = \left(1 - \Delta_{m,n}\right)^{\mu} G_1(x,t),$$

*where $G_1(x,t)$ is a continuous function with respect to $(x,t)$ and as $t \to +\infty$ for it estimation* (24) *holds.*

Let's introduce the following space. We denote by $H^{\theta}(\rho(x), R_{m+n})$ $(\theta \geq 1)$ a sub-space of Sobolev-Slobodetskii space $H^{\theta}(R_{m+n})$ (see [6], p. 131) for whose elements

$$\|\varphi(x)\|_{H^{\theta}(\rho(x),R_{m+n})} = \left\{ \int_{R_{m+n}} \cdots \int \rho^2(x) \sum_{|\alpha| \leq \theta} \left|D^2 \varphi(x)\right|^2 dx \right\}^{1/2} < +\infty,$$

where $\rho(x)$ is some measurable function increasing at infinity in a power way.

**Theorem 2.** *Let $D_{x_j}^{\beta_j} \varphi(x) \in H^{2\mu}(\rho(x), R_{m+n})$. Then for the solution of the Cauchy problem* $(1)-(2)$ *as $t \to +\infty$ it holds the estimation*

$$\left|D_t^{\alpha} D_{x_j}^{\beta_j} u(x,t)\right| \leq C(m,n)\, t^{-\frac{m}{2}-2\alpha} \left(1 + |x|^2\right)^{\frac{1}{2}} \left\|D_{x_j}^{\beta_j} \varphi(\xi)\right\|_{H^{2\mu}(\rho(x),R_{m+n})},$$

*where $\rho(x) = (1+|x|)^{m+n+3}$, $0 \leq \alpha \leq 1$, $0 \leq \beta_j \leq 2$.*

**Proof.** Estimate $u(x,t)$ from relation (5). Applying Cauchy-Bunyakovskii inequality to this relation preliminarily multiplying and dividing integrand expression into $\left(1+|\xi|^2\right)^{\frac{m+n+3}{4}}$ we get

$$|u(x,t)| \leq \left\{ \int_{R_{m+n}} \cdots \int \frac{|G_1(x-\xi,t)|^2}{\left(1+|\xi|^2\right)^{\frac{m+n+3}{2}}} d\xi \right\}^{1/2} \times$$

$$\times \left\{ \int_{R_{m+n}} \cdots \int \left(1+|\xi|^2\right)^{\frac{m+n+3}{2}} \left|(1-\Delta)^{\mu} \varphi(\xi)\right|^2 d\xi \right\}^{1/2} =$$

$$= \left\{ \int_{R_{m+n}} \cdots \int \frac{|G_1(x-\xi,t)|^2}{\left(1+|\xi|^2\right)^{\frac{m+n+3}{2}}} d\xi \right\}^{1/2} \|(1-\Delta)^{\mu} \varphi(\xi)\|_{L_2(\rho(x),R_{m+n})}. \tag{25}$$

Estimate the first multiplier in (25) denoting it by $I(x,t)$. Using asymptotic estimation (24) we get

$$I(x,t) \leq Ct^{-\frac{m}{2}} \left\{ \int_{R_{m+n}} \cdots \int \frac{1+|x-\xi|^2}{\left(1+|\xi|^2\right)^{\frac{m+n+1}{2}}} d\xi \right\}^{1/2} \leq Ct^{-\frac{m}{2}} \left(1+|x|^2\right)^{\frac{1}{2}} \times$$

$$\times \left\{ \int_{R_{m+n}} \cdots \int \frac{|\xi|^2 \, d\xi}{\left(1+|\xi|^2\right)^{\frac{m+n+3}{2}}} \right\}^{1/2} = C_1 t^{-\frac{m}{2}} \left(1+|x|^2\right)^{\frac{1}{2}}. \tag{26}$$

From (25) and (26) we get

$$|u(x,t)| \leq C_1(m,n)\, t^{-\frac{m}{2}} \left\|(1-\Delta_{m,n})^{\mu}\varphi(\xi)\right\|_{L_2(\rho(x),R_{m+n})} \left(1+|x|^2\right)^{1/2}.$$

Asymptotics $D_t u(x,t)$ as $t \to +\infty$ is studied in the same way as the asymptotics $u(x,t)$ with a difference that the integrand in the espression $G_1(x,t)$ from (8) is multiplied by $T(\overline{\varphi})$, by differentiating with respect to $t$, that increases order of zero of this function at the point $\overline{\varphi} = \left(\frac{\pi}{2}, \frac{\pi}{2}, ..., \frac{\pi}{2}\right)$ for two units. Taking this into account as $t \to +\infty$ we get

$$D_t J_0\left(x,r,\overline{\overline{\varphi}},t\right) = C(m,n)\, t^{-\frac{m}{2}-2}\left(1+O\left(t^{-1}\right)\right) \tag{27}$$

uniformly with respect to $x, r, \overline{\overline{\varphi}}$.

By differentiating $u(x,t)$ with respect to $x_j$ by the convolution differentiation property we throw the derivative over the initial function $\varphi(x)$. Further, using estimation (25) and acting as in the estimation of $u(x,t)$ we get

$$\left|D_t^{\alpha} D_{x_j}^{\beta_j} u(x,t)\right| \leq C(m,n)\, t^{-\frac{m}{2}-2\alpha}\left(1+|x|^2\right)^{\frac{1}{2}} \left\|D_{x_j}^{\beta_j}\varphi(\xi)\right\|_{H^{2\mu}(\rho(x),R_{m+n})}. \tag{28}$$

The theorem is proved.

**Remark.** For $\beta = 0$ the asymptotics $J_0\left(x,r,\overline{\overline{\varphi}},t\right)$ as $t \to +\infty$ doesn't change, and the estimation $B_1\left(x,\overline{\overline{\varphi}},t\right)$ from (14) gives

$$\left|B_1\left(x,\overline{\overline{\varphi}},t\right)\right| \leq Ce^{-c_1 t}.$$

Therefore the first addend in the expression $G_1(x,t)$ from (13) makes the basic contribution to the asymptotis of the solution of problem (1) – (2) and consequently, asymptotic estimation (27) for great values of time of the solution of Cauchy problem (1) – (2) doesn't change.

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