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ON IOST SOLUTION OF STURM-LIOUVILLE EQUATION WITH DISCONTINUITY CONDITIONS

Abstract

In the paper integral representation of Iost solution is found and the properties of representation kernel are studied.

Let's consider the differential equation

$$-y'' + q(x)\,y = \lambda^2 y, \quad x \in (0,a) \cup (a,+\infty) \tag{1}$$

with discontinuity conditions

$$\begin{gathered} y(a-0) = \alpha y(a+0), \\ y'(a-0) = \alpha^{-1} y'(a+0), \end{gathered} \tag{2}$$

where $\alpha \neq 1,\ \alpha > 0,\ \lambda$ is a complex parameter, $q(x)$ is a complex valued function and satisfies the condition

$$\int\limits_0^{+\infty} x\,|q(x)|\,dx < +\infty. \tag{3}$$

The function $e(x,\lambda)$ satisfying equation (1), conditions (2) and condition at infinity $\lim\limits_{x\to+\infty} e(x,\lambda)\,e^{-i\lambda x} = 1$ is said to be Iost solution of equation (1). It is easy to show that if $q(x) \equiv 0$ then the Iost solution is the function

$$e_0(x,\lambda) = \begin{cases} e^{i\lambda x}, & x > a, \\ \alpha^+ e^{i\lambda x} + \alpha^- e^{i\lambda(2a-x)}, & 0 < x < a, \end{cases} \tag{4}$$

where $\alpha^{\pm} = \dfrac{1}{2}\left(\alpha \pm \dfrac{1}{\alpha}\right)$.

The main result of the paper is the following

Theorem. *Let a complex valued function $q(x)$ satisfy equation (3). Then for all λ from the upper half-plane there exists the Iost solution $e(x,\lambda)$ of equation (1), it is unique and representable in the form*

$$e(x,\lambda) = e_0(x,\lambda) + \int\limits_x^{+\infty} K(x,t)\,e^{i\lambda t}dt, \tag{5}$$

where for each fixed $x \in (0,a) \cup (a,+\infty)$ the kernel $K(x,\cdot)$ belongs to the space $L_1(x,\infty)$ and satisfies the inequality

$$\int\limits_x^{+\infty} |K(x,t)|\,dt \le e^{c\sigma_1(x)} - 1 \qquad \left(c = \alpha^+ + \left|\alpha^-\right|,\ \ \sigma_1(x) = \int\limits_x^{+\infty} t\,|q(t)|\,dt\right). \tag{6}$$

Besides

$$K(x,x)=\frac{\alpha^{+}}{2}\int_{x}^{+\infty}q(t)\,dt,\quad x\in(0,a)$$

$$K(x,x)=\frac{1}{2}\int_{x}^{+\infty}q(t)\,dt,\quad x\in(a,+\infty)$$

$$K(x,2a-x+0)-K(x,2a-x-0)=$$

$$=\frac{\alpha^{-}}{2}\int_{a}^{+\infty}q(t)\,dt-\frac{\alpha^{-}}{2}\int_{x}^{a}q(t)\,dt,\ x\in(0,a). \tag{7}$$

Notice that for Sturm-Liouville equation at the absence of discontinuity condition, i.e. when in conditions (2) $\alpha=1$, Iost solutions representation was first obtained in the paper [1] (see also [2]). In this case the representation is of "triangular" form. Formula (5) and (7) show that for discontinuity condition the "triangular property" of Iost solution representation is lost and the kernel bears break on the line $t=2a-x$ for $x\in(0,a)$.

The similar problem for Sturm-Liouville equation with disconnected coefficients was solved in the paper [3].

Proof of the theorem. Rewriting equation (1) in the form $y''+\lambda^2y=q(x)\,y$ and assuming the right hand side to be known, in order to find the solution $e(x,\lambda)$ of this equation we can apply the method of arbitrary constands variation. As a result we arrive at the equality

$$e(x,\lambda)=e_0(x,\lambda)+\int_{x}^{+\infty}S_0(x,t,\lambda)\,q(t)\,e(t,\lambda)\,dt, \tag{8}$$

where

$$S_0(x,t,\lambda)=\frac{1}{2i\lambda}\{\overline{e_0(x,\lambda)}e_0(t,\lambda)-e_0(x,\lambda)\,\overline{e_0(t,\lambda)}\}. \tag{9}$$

Equality (8) is an integral equation for the function $e(x,\lambda)$. We'll look for the solution of this equation in the form (5). In order such kind function satisfy equation (8) the equation

$$\int_{x}^{+\infty}K(x,t)\,e^{i\lambda t}dt=\int_{x}^{+\infty}S_0(x,t,\lambda)\,q(t)\,e_0(t,\lambda)\,dt+$$

$$+\int_{x}^{+\infty}S_0(x,t,\lambda)\,q(t)\int_{t}^{+\infty}K(t,s)\,e^{i\lambda s}ds dt \tag{10}$$

be fulfilled and vise versa, if the function $K(x,t)$ satisfies this equality, the function $e(x,\lambda)$ is Iost solution of equation (1).

Transform the right hand side of equality (10) so that it have the form of Fourier transfermation of some function.

First we note that allowing for relation (4) we can write the function $S_0(x,t,\lambda)$ in the form

$$S_0(x,t,\lambda)=\begin{cases}\dfrac{\sin\lambda(t-x)}{\lambda}, & a<x<t,\\ \alpha^+\dfrac{\sin\lambda(t-x)}{\lambda}+\alpha^-\dfrac{\sin\lambda(t-2a+x)}{\lambda}, & x<a<t,\\ \dfrac{\sin\lambda(t-x)}{\lambda}, & x<t<a\end{cases}$$

For the first addend from the right hand side of (10) for $x<a$ we have

$$\int_x^{+\infty}S_0(x,t,\lambda)\,q(t)\,e_0(t,\lambda)\,dt=$$

$$=\int_x^a\frac{\sin\lambda(t-x)}{\lambda}q(t)\left[\alpha^+e^{i\lambda t}+\alpha^-e^{i\lambda(2a-t)}\right]dt+$$

$$+\int_a^{+\infty}\left[\alpha^+\frac{\sin\lambda(t-x)}{\lambda}+\alpha^-\frac{\sin\lambda(t+x-2a)}{\lambda}\right]q(t)\,e^{i\lambda t}dt=$$

$$=\frac{1}{2}\alpha^+\int_x^{+\infty}\left(\int_x^{2t-x}e^{i\lambda\xi}d\xi\right)q(t)\,dt+\frac{1}{2}\alpha^-\int_x^a\left(\int_{2a-2t+x}^{2a-x}e^{i\lambda\xi}d\xi\right)q(t)\,dt-$$

$$-\frac{1}{2}\alpha^-\int_a^{2a-x}\left(\int_{2t-2a+x}^{2a-x}e^{i\lambda\xi}d\xi\right)q(t)\,dt+\frac{1}{2}\alpha^-\int_{2a-x}^{\infty}\left(\int_{2a-x}^{2t-2a+x}e^{i\lambda\xi}d\xi\right)q(t)\,dt.$$

Changing integration order and then in the obtained equality changing denotation for integration variables, we get (for $x<a$).

$$\int_x^{+\infty}S_0(x,t,\lambda)\,q(t)\,e_0(t,\lambda)\,dt=$$

$$=\frac{1}{2}\alpha^+\int_x^{+\infty}\left(\int_{\frac{t+x}{2}}^{+\infty}q(\xi)\,d\xi\right)e^{i\lambda t}dt+\frac{1}{2}\alpha^-\int_x^{2a-x}\left(\int_{\frac{2a+x-t}{2}}^{a}q(\xi)\,d\xi\right)e^{i\lambda t}dt-$$

$$-\frac{1}{2}\alpha^-\int_x^{2a-x}\left(\int_a^{\frac{t+2a-x}{2}}q(\xi)\,d\xi\right)e^{i\lambda t}dt+\frac{1}{2}\alpha^-\int_{2a-x}^{+\infty}\left(\int_{\frac{t+2a-x}{2}}^{+\infty}q(\xi)\,d\xi\right)e^{i\lambda t}dt. \qquad (11)$$

For $x<a$ we act similarly and for the second addend from the right hand side of (10) and continuing the function $K(t,s)$ by zero for $s<t$, we have

$$\int_x^{+\infty} S_0(x,t,\lambda)\,q(t)\int_t^{+\infty} K(t,s)\,e^{i\lambda s}dsdt=$$

$$=\frac{1}{2}\int_{-\infty}^{+\infty}\left(\int_x^a q(\xi)\int_{t-\xi+x}^{t+\xi-x} K(\xi,s)\,dsd\xi\right)e^{i\lambda t}dt+$$

$$+\frac{1}{2}\alpha^{+}\int_{-\infty}^{+\infty}\left(\int_a^{+\infty} q(\xi)\int_{t-\xi+x}^{t+\xi-x} K(\xi,s)\,dsd\xi\right)e^{i\lambda t}dt-$$

$$-\frac{1}{2}\alpha^{-}\int_{-\infty}^{+\infty}\left(\int_a^{2a-x} q(\xi)\int_{t+\xi-2a+x}^{t-\xi+2a-x} K(\xi,s)\,dsd\xi\right)e^{i\lambda t}dt+$$

$$+\frac{1}{2}\alpha^{-}\int_{-\infty}^{+\infty}\left(\int_{2a-x}^{+\infty} q(\xi)\int_{t-\xi+2a-x}^{t+\xi-2a+x} K(\xi,s)\,dsd\xi\right)e^{i\lambda t}dt. \tag{12}$$

Notice that for $t<x$ for the integrals from the right hand side of (12) we have

$$\int_{t-\xi+x}^{t+\xi-x} K(\xi,s)\,ds=0$$

$$\int_{t+\xi-2a+x}^{t-\xi+2a-x} K(\xi,s)\,ds=0 \quad \text{for } \xi>a$$

$$\int_{t-\xi+2a-x}^{t+\xi-2a+x} K(\xi,s)\,ds=0 \quad \text{for } \xi>2a-x.$$

Considering this remark, it follows from relations (11) and (12) that equality (10) for $x<a$ is equivalent to the equality

$$K(x,t)=K_0(x,t)+\frac{1}{2}\int_x^a q(\xi)\int_{t-\xi+x}^{t+\xi-x} K(\xi,s)\,dsd\xi+$$

$$+\frac{1}{2}\alpha^{+}\int_a^{+\infty} q(\xi)\int_{t-\xi+x}^{t+\xi-x} K(\xi,s)\,dsd\xi-$$

$$-\frac{1}{2}\alpha^{-}\int\limits_{a}^{2a-x}q(\xi)\int\limits_{t+\xi-2a+x}^{t-\xi+2a-x}K(\xi,s)\,dsd\xi+$$

$$+\frac{1}{2}\alpha^{-}\int\limits_{2a-x}^{+\infty}q(\xi)\int\limits_{t-\xi+2a-x}^{t+\xi-2a+x}K(\xi,s)\,dsd\xi, \tag{13}$$

where

$$K_0(x,\xi)=\begin{cases}\frac{1}{2}\alpha^{+}\int\limits_{\frac{x+t}{2}}^{+\infty}q(\xi)\,d\xi+\frac{1}{2}\alpha^{-}\int\limits_{\frac{2a+x-t}{2}}^{a}q(\xi)\,d\xi-\\ -\frac{1}{2}\alpha^{-}\int\limits_{a}^{\frac{t+2a-x}{2}}q(\xi)\,d\xi, \quad x<t<2a-x\\ \frac{1}{2}\alpha^{+}\int\limits_{\frac{x+t}{2}}^{+\infty}q(\xi)\,d\xi+\frac{1}{2}\alpha^{-}\int\limits_{\frac{t+2a-x}{2}}^{+\infty}q(\xi)\,d\xi, \quad t>2a-x\end{cases}. \tag{14}$$

The above-mentioned arguments in the case $x>a$ reduce to the equation

$$K(x,t)=K_0(x,t)+\frac{1}{2}\int\limits_{x}^{+\infty}q(\xi)\int\limits_{t-\xi+x}^{t+\xi-x}K(\xi,s)\,dsd\xi \tag{15}$$

$$K_0(x,t)=\frac{1}{2}\int\limits_{\frac{x+t}{2}}^{+\infty}q(\xi)\,d\xi, \quad x>a. \tag{16}$$

Thus, in order to complete the proof of the theorem it suffices to show that for each fixed $x\in(0,a)\cup(a,+\infty)$ the system of equations $(13),(15)$ has the solution $K(x,\cdot)\in L_1(x,\infty)$ satisfying inequality (6).

Assume

$$K_n(x,t)=\frac{1}{2}\int\limits_{x}^{a}q(\xi)\int\limits_{t-\xi+x}^{t+\xi-x}K_{n-1}(\xi,s)\,dsd\xi+$$

$$+\frac{1}{2}\alpha^{+}\int\limits_{a}^{+\infty}q(\xi)\int\limits_{t-\xi+x}^{t+\xi-x}K_{n-1}(\xi,s)\,dsd\xi-$$

$$-\frac{1}{2}\alpha^{-}\int\limits_{a}^{2a-x}q(\xi)\int\limits_{t+\xi-2a+x}^{t-\xi+2a-x}K_{n-1}(\xi,s)\,dsd\xi+$$

$$+\frac{1}{2}\alpha^{-}\int\limits_{2a-x}^{+\infty}q(\xi)\int\limits_{t-\xi+2a-x}^{t+\xi-2a+x}K_{n-1}(\xi,s)\,dsd\xi, \quad x\in(0,a)$$

$$K_n(x,t)=\frac{1}{2}\int\limits_x^{+\infty} q(\xi)\int\limits_{t-\xi+x}^{t+\xi-x} K_{n-1}(\xi,s)\,dsd\xi,\ x\in(a,+\infty),\quad n=1,2,... \tag{17}$$

where $K_0(x,t)$ is determined by relations (14) and (16)

Show that

$$\int\limits_x^{+\infty} |K_n(x,t)|\,dt\le\frac{c^{n+1}\{\sigma_1(x)\}^{n+1}}{(n+1)!} \tag{18}$$

whence, obviously , it will follow that the series $K(x,\cdot)=\sum\limits_{n=0}^{+\infty}K_n(x,\cdot)$ converges in the space $L_1(x,\infty)$, and its sum $K(x,t)$ is the solution of the system of equations (13) and (15) and satisfies inequality (6).

From the definition $K_n(x,t)$ (see relations (17)) it follows

$$|K_n(x,t)|\le\frac{c}{2}\int\limits_x^{+\infty}|q(\xi)|\int\limits_{t-\xi+x}^{t+\xi-x}|K_{n-1}(\xi,s)|\,dsd\xi,\quad x\in(0,a)\cup(a,+\infty)$$

consequently

$$\int\limits_x^{+\infty}|K_n(x,t)|\,dt\le c\int\limits_x^{+\infty}\xi\,|q(\xi)|\int\limits_\xi^{+\infty}|K_{n-1}(\xi,s)|\,dsd\xi. \tag{19}$$

Now in order to establish (18) we apply the mathematical induction method. For $n=0$, using (14), (16) we have

$$\int\limits_x^{+\infty}|K_0(x,t)|\,dt\le\int\limits_x^{+\infty}\xi\,|q(\xi)|\,d\xi=\sigma_1(x),\quad x>a$$

$$\int\limits_x^{+\infty}|K_0(x,t)|\,dt\le\frac{1}{2}\alpha^+\int\limits_x^{2a-x}\left(\int\limits_{\frac{x+t}{2}}^{+\infty}|q(\xi)|\,d\xi\right)dt+$$

$$+\frac{1}{2}\left|\alpha^-\right|\int\limits_x^{2a-x}\left(\int\limits_a^{\frac{t+2a-x}{2}}|q(\xi)|\,d\xi\right)dt+$$

$$+\frac{1}{2}\left|\alpha^-\right|\int\limits_x^{2a-x}\left(\int\limits_{\frac{2a+x-t}{2}}^{a}|q(\xi)|\,d\xi\right)dt+\frac{1}{2}\alpha^+\int\limits_{2a-x}^{+\infty}\left(\int\limits_{\frac{x+t}{2}}^{+\infty}|q(\xi)|\,d\xi\right)dt+$$

$$+\frac{1}{2}\left|\alpha^-\right|\int\limits_{2a-x}^{+\infty}\left(\int\limits_{\frac{t+2a-x}{2}}^{+\infty}|q(\xi)|\,d\xi\right)dt\le\left(\alpha^++\left|\alpha^-\right|\right)\sigma_1(x),\quad \text{for } 0<x<a.$$

Thus, estimation (6) for $n = 0$ is valid and if it is valid for $\|K_n(x,\cdot)\|_{L_1(x,\infty)}$ then using inequality (19) we have

$$\int_x^{+\infty} |K_{n+1}(x,t)|\,dt \le c\int_x^{+\infty} \xi\,|q(\xi)|\,\frac{c^{n+1}\{\sigma_1(\xi)\}^{n+1}}{(n+1)!}d\xi = \frac{c^{n+2}\{\sigma_1(x)\}^{n+2}}{(n+2)!}.$$

Validity of relations (7) follows directly from (13) – (16).

The theorem is proved.

Remark. It follows from estimation (6) that the solution of the system of integral equations (13), (15) is coutinuous in the domains $\{0 < x < a,\ \ x < t < 2a - x\}$ $\{0 < x < a,\ t > x\}$ and $\{x > a,\ t > x\}$ and have partial derivatives $\frac{\partial K}{\partial x}, \frac{\partial K}{\partial t}$. If the function $q(x)$ is differentiable, then for the function $K(x,t)$ to be the kernel of repersentation (5) it is necessary and sufficient that the function be twice differentiable and satisfy the equation

$$\frac{\partial^2 K(x,t)}{\partial x^2} - \frac{\partial^2 K(x,t)}{\partial t^2} = q(x)\,K(x,t)$$

and conditions

$$\frac{dK(x,x)}{dx} = -\frac{\alpha^+}{2}q(x), \quad x \in (0,a)$$

$$\frac{dK(x,x)}{dx} = -\frac{1}{2}q(x), \quad x \in (a,+\infty)$$

$$\frac{d}{dx}\{K(x,t)|_{t=2a-x-0}^{2a-x+0}\} = \frac{\alpha^-}{2}q(x), \quad 0 < x < a$$

$$K(a-0,t) = \alpha K(a+0,t)$$

$$K'_x(a-0,t) = \alpha^{-1}K'_x(a+0,t)$$

$$\lim_{x+t\to\infty}\frac{\partial K(x,t)}{\partial x} = \lim_{x+t\to\infty}\frac{\partial K(x,t)}{\partial t} = 0.$$

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Received July 17, 2006; Revised October 11, 2006.