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ON GENERALIZED n-TH ORDER DERIVATIVE FUNCTIONS SQUARE SUMMABLE ON HILBERT SPACE WITH GAUSS MEASURE

Abstract

The concept of generalization of the n-th order derivative for of square-summable functions on Hilbert space with Gauss measure is considered. The necessary and sufficient condition for existence of generalization of derivative for some class of square summable functions is obtained.

Let X be a Hilbert space with scalar product *(x,y), x, y* $\in X$, $\Im$ be $\Im$ $\sigma-$ algebra of Borel sets from X, μ be Gauss measure on $\Im$ with characteristic functional $\varphi_0(z) = exp\left\{-\frac{1}{2}(Bz,z)\right\}$ where B is a correlation operator. *By* $L_2(X,\mu)$ we denote a space of functions on Xsquare summable. Fourier transformation of the function $f(x) \in L_2(x,\mu)$ is defined as follows: $\varphi(z) = \int e^{i(z,x)} f(x)\mu(dx)$. By $\overline{X}$ we denote complex extension of X whose elements are the formal sums *x+iy, x, y*$\in X$, with a scalar product defined by the relation:

$$(x_1 + iy_1, x_2 + iy_2)_k = [(x_1,x_2) + (y_1,y_2)] + i[(y_1,x_2) - (x_1,y_2)]$$

Thus $\overline{X}$-is linear space. The linear space $\overline{X}$ with such a scalar product will be a complex Hilbert space and X will be its subspace ([1]) and $(x,y)_k = (x,y)$, $(x_1 + \lambda y_1, x)_k = (x_1 x) + \lambda(y_1,x), x_1, y_1, x \in X, \quad \lambda$ is a complex number.

Then the function $\varphi(x_1 + \lambda y_1) = \int e^{i(x_1+\lambda y_1,x)_k} f(x)\mu(dx)$ is an entire analytic function with respect to λ for any fixed $x_1,\ y_1 \in X$, and for real λ this function we have

$$\varphi(x_1 + \xi\, y_1) = \int e^{i(x_1+\xi\, y_1,x)} f(x)\mu(dx) = \int e^{i(x_1,x)+i\xi(y_1,x)} f(x)\mu(dx),$$

where ξ is real and integral at the right side is an analytical function with respect to ξ. Consequently, $\varphi_0(z)$ *and* $\varphi(z)$ are continuable on $\overline{X}$ and for complex λ functions $\varphi_0(x+\lambda y)$, $\varphi(x+\lambda y)$ are entire analytical functions with respect to λ for any fixed *x, y*$\in X$, and at each point $x \in X$, both functions have Freshet derivatives, $\varphi_0^{(k)}(x;y_1,...y_k)$ *and* $\varphi^{(k)}(x;y_1,y_2,...y_k)$ are k-linear forms.

In the paper we find necessary and sufficient condition for the function $f(x) \in L_2(X,\mu)$ for existence of generalized derivative $f^{(m)}(x;a_1,a_2,...a_m)$ in any directions $a_1,a_2,...a_m \in BX$.

1. To each polynomial function $P_n(x) = \sum\limits_{k=1}^{n}\sum\limits_{i_1,...i_k}^{n} c_{i_1,i_2...i_k}(x,e_{i_1})...(x,e_{i_k})$ where $n \geq 1$, $c_{i_1,i_2,...i_k}$ are numbers, $e_{i_1,...,}e_{i_k} \in X$, we associate the differential operator

$$P_n\left(\frac{d}{dx}\right)\varphi(z) = \sum_{k=1}^{n}\sum_{i_1...i_k=0}^{n} c_{i_1...i_k}\varphi^{(k)}(z;e_{i_1},e_{i_2},...e_{i_k}).$$

Let's confider a derivative of $P_n(x)$ in the direction $a_1, a_2, ..., a_m \in X$ and differential operator $P^{(m)}_{n,a_1,...a_m}\left(\frac{d}{dz}\right)$ corresponding to it.

First we study an action of one addend of the polynomial $P_n(x)$. Denote it in the form ${}^kP(x) = (x, e_1)...(x, e_k) \qquad 1 \le k \le n.$

The differential operator ${}^kP\left(\frac{d}{dz}\right)\varphi(z) = \varphi^k(z; e_1, e_2...e_k)$ corresponds to it. It is easy to see that ${}^kP^{(1)}_{a_1}(x) = \sum\limits_{i_1=1}^{k}(a_1, e_{i_1})\prod\limits_{j=1\ \ j\neq i_1}^{k}(x, e_j).$

Consequently taking derivatives with respect to $a_2, a_3 a_m$ we get

$$ {}^kP^{(m)}_{a_1,a_2...a_m}(x) = \sum_{\substack{i_{11}-i_m=1\\ i_1\neq i_2\neq ...=i_m}}^{k} (a_1, e_i)(a_2, e_{i2})...(a_m, e_{i_m}) \prod_{\substack{j=1\\ j\neq i_1...i_m}}^{k} (x, e_j). $$

For $m > k {}^kP^{(m)}_{a_1...a_m}(x) = 0.$

Then we have

$$ {}^kP^{(m)}_{a_1...a_m}\left(\frac{d}{dz}\right)\varphi(z) = \sum_{i_1,...i_m=1}^{k}(a_{11}e_{i_1})...(a_{m1}e_{im})\varphi^{(k-m)}\left(z; e_{i_{m+1}}...e_{ik}\right) $$

which corresponds to the polynomial ${}^kP^{(m)}_{a_1,a_2...a_m}(x)$.

Summation is taken for all collection of indices

$$ (i_1...i_m) \cup (i_{m+1}i_k) = (1, 2, ...k), \quad i_1 \neq i_2 \neq ...i_m, \quad i_{m+1} < i_{m+2} < ... < i_{ik}. $$

Hence

$$ {}^kP^{(m)}_{a_1a_2...a_m}\left(\frac{d}{dz}\right)\varphi(z)|_{z=0} = \sum(a_1e_{i1})...(a_{m1}e_{im})\varphi^{(k-m)}\left(0; e_{i_{m+1}}...e_{iki}\right). \quad (1) $$

Now let's see the action of the operator on $(a_1, z)(a_2, z)...(a_m, z)\varphi(z)$.

First we consider the case $(a_1, z)\varphi(z)$. Consider

$$ {}^kP\left(\frac{d}{dz}\right)[(a_1, z)\varphi(z)] = \sum_{i=1}^{k}(a_1, e_i)\varphi^{(k-1)}(z; e_1, ...e_{i-1}e_{i+1}, ...e_k) + $$

$$ + (a_1, z)\varphi^{(k)}(z; e_1, e_2, ...e_k). \quad (2) $$

Let's prove (2) by induction. Denote $\Psi_1(z) = (a_1, z)\varphi(z)$. Relation (2) is true for $k = 1, 2$

$$ {}^1P\left(\frac{d}{dz}\right)[(a_{11}z)\varphi(z)] = \Psi(1)(z; e_1) = \frac{d}{dt}\Psi_1(z + te_1)|_{t=0} = $$

$$ = (a_1, e_1)\varphi(z) + (a_1, z)\varphi^{(1)}(z; e_1) $$

$$ {}^2P\left(\frac{d}{dz}\right)[(a_1, z)\varphi(z)] = \Psi_1^2(z; e_1, e_2) = (a_1, e_1)\varphi^{(1)}(z; e_2) + (a_1, e_2)\varphi^{(1)}(z; e_1) + $$

$$ + (a, z)\varphi^2(z; e_1, e_2) $$

and so on. Let relation (2) is true for k, show that it is true for $k+1$ as well.

$$^{(k+1)}P\left(\frac{d}{dz}\right)\left[(a_1,z)\varphi(z)\right]=$$

$$=\Psi_1^{(k+1)}(z;e_1,e_2...e_k,e_{k+1})=\frac{d}{dt}\left[\Psi_1^{(k)}(z+te_{k+1};e_1,...e_k)\right]_{t=0}=$$

$$=\frac{d}{dt}\left[\sum_{i=1}^{k}(a_1,e_i)\varphi^{(k-1)}(z+te_{k+1};e_1...e_{i-1},e_{i+1},...e_k)+\right.$$

$$\left.+(a_1,z+te_{k+1})\varphi^{(k)}(z+te_{k+1};e_1,...e_k)\right]_{t=0}=$$

$$=\sum_{i=1}^{k}(a_1,e_i)\varphi^{(k)}(z;e_{11}e_{21}...e_{i-1},e_{i+1},...e_k,e_{k+1})+(a_1,e_{k+1})\varphi^{(k)}(z;e_1,e_2...e_k)+$$

$$+(a_1,z)\varphi^{(k+1)}(z;e_1,e_2...e_k,e_{k+1})=\sum_{i=1}^{k+1}(a_1,e_i)\varphi^{(k)}(z;e_1,e_2...e_{i-1},e_{i+1},...e_{k+1})+$$

$$+(a,z)\varphi^{(k+1)}(z;e_1,e_2,...e_k,e_{k+1}).$$

Thus, relation (2) has proved.

Consider $(a_1,z)(a_2,z)\varphi(z)$. Denote $\Psi_2(z)=(a_2,z)\varphi(z)$. Using (2) we have

$$^{k}P\left(\tfrac{d}{dz}\right)\left[(a_1,z)\Psi_2(z)\right]=\sum_{i_1-1}^{k}(a_1,e_{i_1})\Psi_2^{(k-1)}\left(z;e_{i_2},e_{i_3,...e_{i_k}}\right)+$$
$$+(a_1z)\Psi_2^{(k)}(z;e_1,e_2...e_k). \tag{3}$$

Let $^{(k-1)}P(x)=(x,e_1)...(x_ie_{i-1})(x,e_{i+1})...(x,e_k)$.

Then

$$\Psi_2^{(k-1)}(z;e_{i_2},e_{i_3}...e_{i_k})={}^{(k-1)}P\left(\tfrac{d}{dz}\right)\left[(a_2,z)\varphi(z)\right]=$$
$$\sum_{i_2=1}^{k-1}(a_2,e_{i_2})\varphi^{(k-2)}(z;e_{i_3}...e_{i_k})+(a_2z)\varphi^{(k-1)}(z;e_{i_2}e_{i_3}...e_{i_k}), \tag{4}$$

$$\Psi_2^{(k)}(z;e_1,e_2...e_k)={}^{k}P\left(\tfrac{d}{dz}\right)\left[(a_2,z)\varphi(z)\right]=$$
$$=\sum_{i_1=1}^{k}(a_2,e_{i_1})\varphi^{(k-1)}(z;e_{i_2},...e_{i_k})+(a_2,z)\varphi^{(k)}(z;e_1,e_2,...e_k). \tag{5}$$

Substituting (4) and (5) into (3) we get

$$^{k}P\left(\tfrac{d}{dz}\right)\left[(a_1z)(a_2z)\varphi(z)\right]={}^{k}P\left(\tfrac{d}{dz}\right)\left[(a_1z)\Psi_2(z)\right]=$$
$$=\sum_{i=1}^{k}\left[(a_1e_{i_1})\sum_{i_2=1}^{k-1}(a_2,e_{i_2})\varphi^{(k-1)}(z;e_{i_3}...e_{i_n})+(a_2,z)\varphi^{(k-1)}(z;e_{i_2}...e_{i_k})\right]+$$
$$+(a_1z)\left[\sum(a_2,e_{i_1})\varphi^{(k-1)}(z;e_{i_2}...e_{i_k})+(a_2,z)\varphi^{(k)}(z;e_1e_2...e_k)\right].$$

Opening parenthesis, we get:

$$ {}^kP\left(\tfrac{d}{dz}\right)\left[(a_1,z)(a_2z)\varphi(z)\right]=\sum_{\substack{i_1,i_2=1\\ i_1\neq i_2,i_3<i_4<i_k}}^{k}(a_1,e_{i_1})(a_2,e_{i_2})\varphi^{(k-2)}(z;e_{i_3},...e_{i_k})+ $$

$$ +\sum_{i_1=1}^{k}(a_1,e_{i_1})(a_2,z)\varphi^{(k-1)}(z;e_{i_2},...e_{i_k})+\sum_{i_1=1}^{k}(a_2,e_i)(a_1,z)\varphi^{(k-1)}(z;e_{i_2},...e_{i_k})+ $$
$$ +(a_1z)(a_2,z)\varphi^{(k)}(z;e_1e_2...e_k). $$

Using induction we have:

$$ {}^kP\left(\tfrac{d}{dz}\right)\left[(a_1,z)(a_2,z)...(a_m,z)\varphi(z)\right]= $$
$$ =\sum_{i_1,i_2...i_m=1}^{k}(a_1e_{i_1})(a_2e_{i_2})...(a_m,e_{i_m})\varphi^{(k-m)}\left(z;e_{i_{m+1}}...e_{i_k}\right)+ $$

$$ +\sum_{s=1}^{m}\sum_{i_1...i_m=1}^{k}(a_1e_{i_1})(a_2e_{i_2})...\left(a_{s-1},e_{i_{s-1}}\right)\left(a_{s+1},e_{i_{s+1}}\right)...(a_m,e_{i_m})(a_s,z)\,\mathrm{X} $$
$$ \mathrm{X}\varphi^{(k-(m-1))}\left(z;e_{i_{n+1}}...e_{i_n}\right)+ $$
$$ +\sum_{\substack{s_1,s_2=1\\ s_1<s_2}}^{m}\sum_{i_1...i_{m-1}}\prod_{j\neq s_1,s_2}\left(a_j,e_{i_j}\right)(a_{s_i},z)(a_{s_2},z)\varphi^{(k-(m-2))}\left(z;e_{i_{m-2}}...e_{in}\right)+ $$
$$ +\sum_{\substack{s_1,s_2,s_3=1\\ s_1<s_2<s_3}}^{m}\sum_{i_1i_2...i_{m-3}}\prod\left(a_j,e_{i_j}\right)(a_{s_1},z)(a_{s_2},z)(a_{s_3},z)\varphi^{(k-(m-3))}\left(z,e_{i_{m-3}},...e_{i_k}\right)+ $$
$$ +...+(a_1z)(a_2z)...(a_2z)\varphi^{(k)}(z;e_1,e_2...e_k). $$

Hence

$$ {}^kP\left(\tfrac{d}{dz}\right)\left[(a_1,z)(a_2,z)...(a_m,z)\varphi(z)\right]_{z=0}= $$
$$ =\sum_{\substack{i_1i_2...i_m=1\\ i_1\neq i_2\neq...\neq i_{m-1},i_m<...i_k}}^{k}(a_1,e_{i_1})(a_2,e_{i_2})...(a_m,e_{i_m})\varphi^{(k-m)}\left(0;e_{i_{m+1}}...e_{i_k}\right). \tag{6} $$

We have from (1) and (6)

$$ {}^kP\left(\frac{d}{dz}\right)\left[(a_1,z)...(a_m,z)\varphi(z)\right]_{z=0}={}^kP^{(m)}_{a_1a_2...a_m}\left(\frac{d}{dz}\right)\varphi(z)\left.\right|_{z=0} $$

Since these relations are true for all addends of the polynomial $P_n(x)$, we can say that the following lemma is true.

Lemma 1. *The following equality holds:*

$$ P_n\left(\frac{d}{dz}\right)\left[(a_1z)(a_2,z)...(a_m,z)\varphi(z)\right]_{z=0}=P^{(m)}_{n,a_1...a_m}\left(\frac{d}{dz}\right)\varphi(z)\left.\right|_{z=0} $$

2. Similarly (3) we give the following definition.

Definition *Let for a sequence* $\{f_n(k)\}$ *there exist* $f_n^{(m)}(x;a_1,a_2...a_m)$ *and* $f_n(x)\to f(x)\,,n\to\infty,$ *in* $L_2(X,\mu)\,,$ *and a* $\left\{f_n^{(m)}(x;a_1,a_2...a_m)\right\}$ *converges in* $L_2(X,\mu)$ *to some function* $\rho(x;a_1a_2...a_m)\in L_2(X,\mu)$.

Then we' ll say that f(x) has a generalized derivative $f^{(m)}(x;a_1,a_2...a_m)$ *and by definition we assume* $f^{(m)}(x;a_1,a_2...a_m)=\rho(x;a_1a_2...a_m)$.

Having $f^{(m)}(x;a_1,a_2..a_m)$ *we sequentially apply formula of integration by parts:*

$$\int f^1(x;a)\,g(x)\,\mu(dx)=\int f(x)\left[-g'(x;a)+g(x)\left(B^{-1}a,x\right)\right]\mu(dx)$$

and easily get the following equality:

$$\int f^{(m)}(x;a_1,a_2,...a_m)\,g(x)\,\varpi(dx)=\int f(x)\,G(x,a_1,a_2...a_m)\mu(dx) \tag{7}$$

where

$$G(x,a_1,...a_m)=\sum g^{(n_1)}\left(x;a_i,...a_{i_{n_1}}\right)\prod_{j=1}^{n_2}\left(B^{-1}a_{k_{j-i}}a_{k_{ju}}\right)\prod_{s=1}^{n_3}\left(B^{-1}a_{e_3}x\right)$$

and summation is taken in all indices

$$(i_1i_2...i_{n_1})\cup((k_1k_2),(k_3k_4)...(k_{n_2-1},k_{n_2}))\cup(l_1l_2...l_{n_3})=(1,2,3...n),$$
$$n_1,n_2,n_3\geq 0,\qquad n_1+n_2+n_3=m.$$

Taking this relation as a basis we can formulate the following lemma.

Lemma 2. *Let* $f(x)\in L_2(X,\mu)$ *and there exist such* $\rho(x,a_1...a_m)\in L_2(x,\mu)$ *depending on* $a_1,a_2...a_m\in BX$, *that the equality*

$$\int \rho(x_1a_1,a_2...a_m)g(x)\,\mu(dx)=\int f(x)G(x,a_1,...a_m)\,\mu(dx) \tag{8}$$

is fulfilled for any $g(x)\in L_2(x,\mu)$, *for which* $g^m(x;a_1a_2...a_m)$ *exist the functions* $g(x)$ *are a complete system of functions in* $L_2(x,\mu)$. *Then f(x) has a generalized derivative* $f^{(m)}(x;a_1,a_2...a_m)=\rho(x_1a_1,a_2...a_m)$.

The proof is a similar the proof of lemma from [3].

Generalized derivative, defined in lemma 2 is unigue. Let $f_n(k)$ an arbitrary sequence such that $f_n(x)$ and $f_n^m(x,a_1,a_2,...,a_m)$ converge in $L^2(x,\mu)$ to $f(x)$ and $\widetilde{\rho}(x,a_1,a_2,...,a_m)$. Then the equality (7) for $f_n(x)$ has the following form

$$\int f_n^m(x,a_1,a_2,...,a_m)g(x)\mu(dx)=\int f_n(x)G(x,a_1,a_2,...,a_m)\mu(dx).$$

Passing to limit $(asn\to\infty)$ we get

$$\int \widetilde{\rho}(x,a_1,a_2,...,a_m)g(x)\mu(dx)=\int f(x)G(x,a_1,a_2,...,a_m)\mu(dx)$$

which is true for any $g(x)$ satisfing to condition of lemma 2. Right side of last equality coincides with right side of (8). Hence $\widetilde{\rho}(x,a_1,a_2,...,a_m)=\rho(x,a_1,...,a_m)$ almost everywhere.

Theorem. $f(x)\in L_2(x,\mu)$ *has a generalized derivative* $f^{(m)}(x;a_1,a_2,...a_m)$ *in the directions* $a_1,a_2,...a_m\in BX$ *iff*

$$\left|l_{\varphi,B,a,...a_m}(P_n)\right|^2\leq C_{a_1...a_m}\int P_n^2(x)\,\mu(dx),$$

where

$$l_{\varphi,B,a,...a_m}(P_n)=\left[\sum_{i,i_2...i_{n_1}}^{n_1}(-i)^{m-n}\prod_{j=1}^{n_2}\left(B^{-1}a_{k_{j-i}}a_{k_j}\right)P_{n,a_{i_1}...a_{i_{n_1}}}\left(\frac{d}{dz}\right)\times\right.$$

$$\left.\times\prod_{s=1}^{n_3}\left(\frac{d}{dz};B^{-1}a_{m_s}\right)\right]\varphi(z)\,|_{z=0}$$

(and summation is taken in all collection of indices

$$(i_1i_2...i_{n_1})\cup((m_1m_2-m_{n3})\cup(k_{j-1}k_1)...(k_{n_2-1},k_{n_2}))=(1,2...m),$$
$$n_1,n_2,n_3\geq 0,\qquad n_1+n_2+n_3=m$$

$l_{\varphi,B,a,...a_m}(P_n)$ is a linear functional determined on all polynomials.

Proof. Necessity. We sequentially apply the formula of integration by parts

$$\int e^{i(z,x)}f^{(1)}(x;a)\,\mu(dx)=\int f(x)\left[-i(a,z)+\left(B^{-1}a,x\right)\right]e^{i(z,x)}\mu(dx)$$

and get

$$\int e^{i(zx)}f^{(m)}(x;a_1,a_2...a_m)\,\mu(dx)=\sum_{i_1...i_{n1}}(-i)^{n_1}\prod_{s=1}^{n_1}(a_{i_s},z)\prod_{j=1}^{n_2}\left(B^{-1}a_{k_{j-1}},a_{k_j}\right)\times$$

$$\times\int f(x)\,e^{i(z,x)}\prod_{\rho=1}^{n_3}\left(B^{-1}a_{m_\rho},x\right)\mu(dx)=$$

$$=(-i)^m\sum_{i_1...i_k}\prod_{j=1}^{n_2}\left(B^{-1}a_{k_{j-1}}a_{k_j}\right)\prod_{s=1}^{n_1}(a_{i_s},z)\,\varphi^{(n_3)}\left(z;B^{-1}a_{m_1},...B^{-1}a_{m_{n3}}.\right).\qquad(9)$$

Appling the operator $P_n\left(\frac{1}{i}\frac{d}{dz}\right)$ to the both sides of equality (9), using lemma 1 and get:

$$\int e^{i(z,x)}f^{(m)}(x;a_1,a_2...a_m)\,P_n(x)\,\mu(dx)\,|_{z=0}=$$
$$=\left[\sum_{i_1...i_{n_1}}(-i)^{m-n}\prod_{j=1}^{n_2}\left(B^{-1}a_{k_{j-1}},a_{k_j}\right)P^{(n_i)}_{n_1a_{i_1}...a_{i_{n1}}}\left(\tfrac{1}{i}\tfrac{d}{dz}\right)\prod_{s=1}^{n_3}\left(\tfrac{d}{dz};B^{-1}a_{ms}\right)\right]\varphi(z)_{z=0}.$$

Hence

$$l_{\varphi,B,a_1...a_m}(P_n)=\int f^{(m)}(x;a_1a_2...a_m)\,P_n(x)\,\mu(dx).$$

Hence

$$|l_{\varphi,B,a_1...a_m}(P_n)|^2\leq\int\left|f^{(m)}(x;a_1...a_m)\right|^2\mu(dx)\;\cdot\;\int P_n^2(x)\,\mu(dx).$$

Denote $C_{a_1a_2...a_m}=\int\left|f^{(m)}(x;a_1,a_2...a_m)\right|^2\mu(dx)$, then we have

$$|l_{\varphi,B,a_1...a_m}(P_n)|^2\leq C_{a_1...a_m}\cdot\int P_n^2(x)\,\mu(dx).$$

Sufficiency. Since the polynomials$\{P_n(x)\}_{n\geq 1}$, are complete system in $L_2(X,\mu)$, the linear functional $l_{\varphi,B,a_1...a_m}(P_n)$ is bounded on polynomials and can be continued to all $L_2(X,\mu)$ then accorting to the representation theorem there exists the function $\rho(x,a_1,a_2...a_m)$ such that

$$l_{\varphi,B,a,...a_m}(P_n)=\int\rho(x,a_1...a_m)P_n(x)\mu(dx). \qquad (10)$$

Let's consider the function,

$$\psi(z;a_1a_2...a_m)=\int e^{i(z,x)}\rho(x;a_1,a_2...a_m)\mu(dx).$$

Then

$$P_n\left(\frac{1}{i}\frac{d}{dx}\right)\psi(z;a_1,a_2,...a_m)|_{z=0}=\int\rho(x;a_1,...a_m)P_n(x)\mu(dx). \qquad (11)$$

It follows from (10) and (11) that

$$l_{\varphi,B,a_1...a_m}(P_n)=P_n\left(\frac{1}{i}\frac{d}{dz}\right)\psi(z;a_1...a_m)|_{z=0}$$

Then by the theorem on uniqueness of on entire analytical function and from (9) we have

$$\int\rho(x,a_1,a_2...a_m)e^{i(z,x)}\mu(dx)=$$
$$=\int f(x)\sum(-i)^{n_1}\prod_{s=1}^{n_1}(a_{i_s},z)e^{i(z,x)}\prod_{j=1}^{n_2}\left(B^{-1}a_{k_{j-1}}a_{k_j}\right)\prod_{\rho=1}^{n_3}\left(B^{-1}a_{m_\rho},x\right)\mu(dx)$$

Denoting $g_z(x)=e^{i(z,x)}$ we can rewrite last equality in following form

$$\int\rho(x,a_1,a_2...a_m)g_z(x)\mu(dx)=$$
$$=\int f(x)\left[\sum_{i_1...i_{n_1}}g_z\left(x;a_i,...a_{i_{n_1}}\right)\prod_{j=1}^{n_2}\left(Ba_{k_{j-1}},a_{k_j}\right)\prod_{\rho=1}^{n_3}\left(B^{-1}a_{m_\rho},x\right)\right]\mu(dx).$$

This equality is true for all linear combinations $g(x)=\sum_k C_k g_{z_k}(x)$ and their uniform bounded limits. This family of functions is dense in $L_2(X,\mu)$ [2]. Therefore, by lemma 2 *f(x)* has a generalized derivative and

$$f^{(m)}(x;a_1,a_2...a_m)=\rho(x,a_1a_2...a_m).$$

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Received September 07, 2006; Revised November 29, 2006;