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INVESTIGATION OF RESOLVENT OF OPERATOR-DIFFERENTIAL EQUATIONS ON SEMI-AXIS

Abstract

In the present paper asymptotics of the Green function for the $2n$-order operator differential equations on $(0,\infty)$ is obtained.

Let H be a separable Hilbert space. Let's consider in the space $H_1 = L_2[H; 0 \le x < \infty]$ a differential operator L, generated by the expression

$$l\left(y\right) = \left(-1\right)^n \left(P\left(x\right) y^{(n)}\right)^{(n)} + \sum_{j=2}^{2n} Q_j\left(x\right) y^{(2n-j)} \tag{1}$$

with the boundary conditions

$$y^{(j)}\left(0\right) = 0, \quad j = \overline{0, n-1}. \tag{2}$$

Here $y \in H_1$ and derivatives are understood in the strong sense. Denote $Q_{2n}\left(x\right)$ by $Q\left(x\right)$ everywhere.

Suppose, that coefficients of expression (1) satisfy the conditions:

1. For all $x \in (0,\infty)$ and for all $h \in H$

$$m\left(h,h\right)_H \le \left(P\left(x\right)h,h\right)_H \le M\left(h,h\right)_H, \ m, \ M > 0.$$

2. The operator function $P\left(x\right)$ is n-time differentiable for all $x \in (0,\infty)$.

3. The operators $P\left(x\right)$ are self-adjoint in H almost for all x, moreover, in H there exists common for all x and dense everywhere in H the set $D\left\{Q\left(x\right)\right\} = D\left(Q\right)$, on which $Q\left(x\right)$ are defined and symmetric*. (* Thus, we assume, that operators $Q\left(x\right)$ can be nonbounded in H).

4. Operators $Q\left(x\right)$ are uniformly bounded below, i.e. there exists such a number $d > 0$, that for all x and $f \in D\left(Q\right)$, $\left(Q\left(x\right)f,f\right)_H \ge d\left(f,f\right)_H$.

5. There exists a constant number $c > 0$, $0 < a < \dfrac{2n+1}{2n}$ such that for all x and $|x-\xi| \le 1$ the following inequality is true:

$$\left\|\left[Q\left(\xi\right) - Q\left(x\right)\right]^{-a} Q\left(x\right)\right\|_H \le c\left|x-\xi\right|$$

6. For $|x-\xi| > 1$

$$\left\|K\left(\xi\right)\exp\left\{-\frac{Jm\varepsilon_1}{2}\left|x-\xi\right|\omega\right\}\right\|_H < C,$$

where $K\left(x\right) = P^{-\frac{1}{2}}\left(x\right) Q\left(x\right) P^{-\frac{1}{2}}\left(x\right)$, $\omega = \left\{K\left(x\right) + \mu P^{-1}\left(x\right)\right\}^{\frac{1}{2n}}$, $\mu > 0$.

$$Jm\varepsilon_1 = \min_i \left\{Jm\varepsilon_i > 0, \ \varepsilon_i^{2n} = -1\right\}.$$

7. For all $x, \xi \in (0, \infty)$

$$\left\| Q(x) P^{\pm\frac{1}{2}}(x) Q^{-1}(x) \right\|_H < C, \quad \left\| Q(\xi) P^{-\frac{1}{2}}(x) Q^{-1}(\xi) \right\|_H < C.$$

8. Operator functions $Q_j(x)$, $j = 3, 4, ..., 2n-1$ are self-adjoint in H and for all $x \in (0, \infty)$

$$\left\| Q_j(x) Q^{\frac{1-j}{2n}+\varepsilon}(x) \right\|_H < C, \quad (j = 3, 4, ..., 2n-1), \quad \varepsilon > 0.$$

Let D' be a totality of all functions of the form $\sum\limits_{k=1}^{m} \varphi_k(x) f_k$, where $\varphi_k(x)$ are $2n$-time continuously-differentiable finite scalar functions in zero, satisfying conditions (2) and $f_k \in D(Q)$. Let's define operator L', generated by expression (1) with the domain of definition D'. L' is a symmetrical positive definite operator in H_1. We'll suppose that closure L of operator L' is self-adjoint.

The main result of this paper is the following:

Theorem. *If conditions 1)-8) are fulfilled, then for sufficiently large $\mu > 0$ there exists a inverse operator $R_\mu = (L + \mu E)^{-1}$, which is a integral operator with the operator kernel $G(x, \eta ç \mu)$, which we'll call Green's function of operator L. $G(x, \eta, \mu)$ is an operator function in H, which depends on two variables x and η $(0 \leq x, \ \ \eta < \infty)$ and parameter μ, satisfying the following conditions*

a) $\dfrac{\partial^k G(x, \eta, \mu)}{\partial \eta^k}$, $k = \overline{0, 2n-2}$ *is a strongly continuous operator-valued function by variables x, η*

b) $\dfrac{\partial^{2n-1} G(x, x+0, \mu)}{\partial \eta^{2n-1}} - \dfrac{\partial^{2n-1} G(x, x-0, \mu)}{\partial \eta^{2n-1}} = (-1)^n P^{-1}(x)$

c) $(-1)^n \left(C_\eta^{(n)} P(\eta) \right)_\eta^{(n)} + \sum\limits_{j=2}^{2n} C_\eta^{(2n-j)} Q_j(\eta) + \mu G(x, \eta, \mu) = 0$

$\left. \frac{\partial^k G(x,\eta,\mu)}{\partial \eta^k} \right|_{\eta=0} = 0, \quad k = \overline{0, n-1}$

d) $G^*(x, \eta, \mu) = G(\eta, x, \mu)$

e) $\int\limits_0^\infty \|G(x, \eta, \mu)\|_H^2 \, d\eta < \infty.$

We briefly state the proof course of the theorem. At first construct Green's function of operator L_1, generated by differential expression

$$l_1(y) = (-1)^n \left(P(x) y^{(n)} \right)^{(n)} + Q(x) + \mu y \tag{3}$$

and boundary conditions (2).

To this aim we use "parametrics" method. We construct Green's function of operator L_1 with the "frozen" in "ξ" coefficients:

$$\begin{cases} \tilde{l}_1(y) = (-1)^n \left(P(\xi) y^{(n)} \right)^{(n)} + Q(\xi) y + \mu y \\ y^{(j)}(0) = 0, \quad j = \overline{0, n-1}. \end{cases} \tag{4}$$

Here "ξ" is a fixed point from $[0, \infty)$.

We'll search Green's function $g(x, \eta; \xi, \mu)$ of the problem (4) in the following form:

$$g(x, \eta; \xi, \mu) = \tilde{g}(x, \eta; \xi, \mu) + V(x, \eta; \xi, \mu), \tag{5}$$

where $\tilde{g}(x,\eta;\xi,\mu)$ is a Green's function of equation $\tilde{l}_1(y)=0$ on the axis. As is known, (see [4]), it has the form:

$$\tilde{g}(x,\eta;\xi,\mu)=\frac{1}{2\pi}P^{-\frac{1}{2}}(\xi)\,\omega_\xi^{1-2n}\sum_{k=1}^{n}\varepsilon_k\exp\left(i\varepsilon_k\left|x-\eta\right|\omega_\xi\right)P^{-\frac{1}{2}}(\xi). \tag{6}$$

Here, by ε_k we determine roots from $\sqrt[2n]{-1}$, lying on the upper half plane.

The function $V(x,\eta;\xi,\mu)$ is bounded as $x\to+\infty$ by solution of the following problem:

$$\tilde{l}_1(V)=0 \tag{7}$$

$$V^{(j)}|_{x=0}=\tilde{g}^{(j)}|_{x=0},\quad j=\overline{0,n-1}. \tag{8}$$

Solution of problem (7), (8) is represented in the form:

$$V(x,\eta;\xi,\mu)=\frac{1}{2ni}P^{-\frac{1}{2}}(\xi)\,\omega_\xi^{1-2n}\sum_{k=1}^{n}\varepsilon_k e^{i\varepsilon_k\omega_\xi(x+\eta)}P^{-\frac{1}{2}}(\xi). \tag{9}$$

Then, Green's function of problem (5) will take the form:

$$\tilde{g}(x,\eta;\xi,\mu)=\frac{1}{2ni}P^{-\frac{1}{2}}(\xi)\,\omega_\xi^{1-2n}\sum_{k=1}^{n}\varepsilon_k e^{i\varepsilon_k\omega_\xi|x-\eta|}P^{-\frac{1}{2}}(\xi)-$$

$$-\frac{1}{2ni}P^{-\frac{1}{2}}(\xi)\,\omega_\xi^{1-2n}\sum_{k=1}^{n}\varepsilon_k e^{i\varepsilon_k\omega_\xi(x+\eta)}P^{-\frac{1}{2}}(\xi).$$

The function $g(x,\eta;\xi,\mu)$ can be transformed in the following form:

$$\tilde{g}(x,\eta;\xi,\mu)=$$

$$=\begin{cases}\frac{1}{2ni}P^{-\frac{1}{2}}(\xi)\,\omega_\xi^{1-2n}\sum_{k=1}^{n}\varepsilon_k e^{i\varepsilon_k\omega_\xi(x-\eta)}\left\{E-e^{2i\varepsilon_k\omega_\xi\eta}\right\}P^{-\frac{1}{2}}(\xi), & x>\eta\\ \frac{1}{2ni}P^{-\frac{1}{2}}(\xi)\,\omega_\xi^{1-2n}\sum_{k=1}^{n}\varepsilon_k e^{i\varepsilon_k\omega_\xi(\eta-x)}\left\{E-e^{2i\varepsilon_k\omega_\xi\eta}\right\}P^{-\frac{1}{2}}(\xi), & x<\eta\end{cases}$$

Since $\left\|e^{i\varepsilon_k\omega_\xi\eta}\right\|_H\to0$, as $\mu\to\infty$, we have:

$$\tilde{g}(x,\eta;\xi,\mu)=$$

$$=\frac{1}{2ni}P^{-\frac{1}{2}}(\xi)\,\omega_\xi^{1-2n}\sum_{k=1}^{n}\varepsilon_k e^{i\varepsilon_k\omega_\xi|x-\eta|}\left\{E-r(x,\eta;\xi,\mu)\right\}P^{-\frac{1}{2}}(\xi) \tag{10}$$

moreover, as $\mu\to\infty$ we have $\|r(x,\eta;\xi,\mu)\|=0(1)$ uniformly by (x,η). Now, let's investigate Green's function of equation (3). For this we rewrite equation (3) in the following form:

$$\begin{aligned}(-1)^n\left(P(x)\,y^{(n)}\right)^{(n)}+Q(x)\,y+\mu y=(-1)^n\left(P(\xi)\,y^{(n)}\right)^{(n)}+Q(\xi)\,y+\mu y+\\ +(-1)^n\left\{\left(P(x)\,y^{(n)}\right)-\left(P(\xi)\,y^{(n)}\right)\right\}^{(n)}+\left\{Q(x)-Q(\xi)\right\}y=0.\end{aligned} \tag{11}$$

Formally search Green's function of operator L_1, $G(x,\eta;\xi,\mu)$ in the form:

$$G_1(x,\eta;\xi,\mu)=g(x,\eta;\xi,\mu)+g_0(x,\eta;\xi,\mu).$$

In the last equation, putting $g+g_0$ instead of y and applying to the both hand sides the operator, generated by kernel $g(x,\eta;\xi,\mu)$ (supposing $x=\xi$), we get

$$G_1(x,\xi,\mu)=g(x,\xi,\mu)-\int_0^\infty g(x,\xi,\mu)\left[Q(\xi)-Q(x)\right]G_1(\xi,\eta,\mu)\,d\xi+$$

$$+\frac{1}{2ni}\int_0^\infty P(x)^{-\frac{1}{2}}\,\omega\sum_{k=1}^{n}\varepsilon_k\exp\left(i\varepsilon_k|x-\xi|\,\omega\right)\left(E-r(x,\xi,\mu)\right)\times$$

$$\times P^{-\frac{1}{2}}(x)\left[P(\xi)-P(x)\right]G_1(\eta,\xi,\mu)\,d\xi+$$

$$+(-1)^n\sum_{m=1}^{n}C_n^m\int_0^\infty g_\xi^{(2n-m)}(x,\xi,\mu)\,P_\xi^{(m)}(\xi)\,G_1(\eta,\xi,\mu)\,d\xi \tag{12}$$

(Here $G_1(x,\eta,\mu)\equiv G(x,\eta;\xi,\mu)$, $g(x,\eta,\mu)\equiv g(x,\eta;\xi,\mu)$).

For investigation of integral equation (12), according to the paper [1], introduce Banach spaces $X_1,X_2,X_3^{(p)},X_2^{(s)},X_4^{(s)}$ and X_5 ($p\geq 1$, $s<$) whose elements are operator functions $A(x,\eta)$ in H for $x,\eta\in(0,\infty)$ and norms are determined as:

$$\|A(x,\eta)\|_{X_1}^2=\int_0^\infty dx\left\{\int_0^\infty \|A(x,\eta)\|_H^2\,d\eta\right\}$$

$$\|A(x,\eta)\|_{X_2}^2=\int_0^\infty dx\left\{\int_0^\infty \|A(x,\eta)\|_2^2\,d\eta\right\}.$$

Here $\|A(x,\eta)\|_2$ is a Hilbert-Schmidt norm (absolute norm) of operator $A(x,\eta)$ in H.

$$\|A(x,\eta)\|_{X_3^{(p)}}=\left[\sup_{0\leq x<\infty}\int_0^\infty \|A(x,\eta)\|_H^p\,d\eta\right]^{1/p},\qquad X_3\equiv X_3^{(1)}$$

$$\|A(x,\eta)\|_{X_2^{(s)}}=\int_0^\infty dx\left\{\int_0^\infty \|A(x,\eta)\,Q^s(\eta)\|_2^2\,d\eta\right\}$$

$$\|A(x,\eta)\|_{X_4^{(s)}}=\sup_{0\leq x\leq\infty}\int_0^\infty \|A(x,\eta)\,Q^s(\eta)\|_H\,d\eta$$

$$\|A(x,\eta)\|_{X_5}=\sup_{0<x<\infty}\ \sup_{0<\eta<\infty}\|A(x,\eta)\|_H$$

(definition and proof of their completeness for $x,\eta\in(-\infty,+\infty)$ are given by B.M. Levitan [1]).

Introduce operator N, defined by equality

$$NA(x,\eta)=\int_0^\infty g(x,\xi,\mu)\left[Q(\xi)-Q(x)\right]A(\xi,\eta)\,d\xi+\int_0^\infty\frac{1}{2ni}\int_0^\infty P^{-\frac{1}{2}}\omega.$$

$$\sum_{k=1}^{n}\varepsilon_k\exp\left(i\varepsilon_k|x-\xi|\,\omega\right)\left(E-r(x,\xi,\mu)\right)^{P^{-\frac{1}{2}}}\left[P(\xi)-P(x)\right]A(\xi,\eta)\,d\xi+$$

$$+\int_0^\infty(-1)^n\sum_{m=1}^{n}C_n^m g_\xi^{(2n-m)}(x,\xi,\mu)\,P_\xi^{(m)}(\xi)\,A(\xi,\eta)\,d\xi. \tag{13}$$

We prove the following

Lemma. *If operator functions $P(x)$ and $Q(x)$ satisfy conditions 1)-7), then for sufficiently large $\mu > 0$ the operator N is a contracting operator in the spaces $X_1, X_2, X_3^{(p)}, X_2^{(s)}, X_4^{(s)}, X_5$, therefore equation (12) for sufficiently large $\mu > 0$ can be solved by iteration method.*

We prove, that its solution is Green's operator function for operator L_1. Using conditions (4), (5), (6) and equation (3) one can show, that as $\mu \to \infty$ the relation:

$$G_1(x,\eta,\mu) = g(x,\eta,\mu)\left[E + \theta(x,\eta,\mu)\right], \tag{14}$$

holds.

Here $\|\theta(x,\eta,\mu)\|_H = o(1)$ as $\mu \to \infty$ uniformly by $(x,\eta) \in (0,\infty)$.

We'll search Green's function of problem (1) and (2) in the form of:

$$G(x,\eta,\mu) = G_1(x,\eta,\mu) + \int_0^\infty G_1(x,\eta,\mu)\,\rho(\xi,\eta)\,d\xi. \tag{15}$$

Using properties of function $G_1(x,\eta,\mu)$ for $\rho(x,\eta)$ we get the equation

$$\rho(x,\eta) + \sum_{j=2}^{2n} Q_j(x)\frac{\partial^{2n-j}G_1(x,\eta,\mu)}{\partial x^{2n-j}} -$$

$$-\sum_{j=2}^{2n} Q_j(x)\frac{\partial^{2n-j}G_1(x,\xi,\mu)}{\partial x^{2n-j}}\rho(\xi,\eta)\,d\xi = 0. \tag{16}$$

If we suppose

$$F(x,\eta,\mu) = -\sum_{j=2}^{2n} Q_j(x)\frac{\partial^{2n-j}G_1(x,\eta,\mu)}{\partial x^{2n-j}}$$

then, equation (16) takes the form

$$\rho(x,\eta) = F(x,\eta,\mu) - \int_0^\infty F(x,\xi,\mu)\,\rho(\xi,\eta)\,d\xi. \tag{17}$$

Using asymptotical representation (14) for the function $G_1(x,\eta,\mu)$ one can estimate the norm $\|F(x,\eta,\mu)\|_H$:

$$\|F(x,\eta,\mu)\|_H \le c_\mu^{-\xi}\cdot e^{-Jm\omega_1 \sqrt[2n]{\mu}|x-\eta|}.$$

Hence $\sup\limits_{0<x<\infty}\int_0^\infty \|F(x,\eta,\mu)\|_H^2\,d\eta \le c\mu^{-2\varepsilon}$.

It follows from this estimation, that the function $F(x,\eta,\mu)$ is an element of the space $X_3^{(2)}$ and as $\mu \to \infty$ tends (by norm of space $X_3^{(2)}$) to zero. Therefore, equation (17) in space $X_3^{(2)}$ has a solution, and this solution is unique.

Hence, particularly, it follows the fact that at sufficiently large μ the solution $\rho(x,\eta)$ of equation (17) behaves itself in the same way as $F(x,\eta,\mu)$.

At sufficiently large μ an integral operator, contained in equation (15), is contractive (and as $\mu \to \infty$ tends to zero), therefore, at $\mu \to \infty$ we have:

$$G(x,\eta,\mu) = G_1(x,\eta,\mu)\left[E + \alpha(x,\eta,\mu)\right], \tag{18}$$

where $\|\alpha(x,\eta,\mu)\|_H = 0(1)$ at $\mu \to \infty$.

Using asymptotical equality (14) from (18) we get the following important equality

$$\|G(x,\eta,\mu)\|_H = \|g(x,\eta,\mu)\|_H (1+0(1)). \tag{19}$$

It is easy to show, that for the function $g(x,\eta,\mu)$ the following estimation is true:

$$\int_0^\infty \left\{ \int_0^\infty \|g(x,\eta,\mu)\|_2^2 \, d\eta \right\} dx < \infty.$$

From this estimation and equality (19) it follows, that the function $G(x,\eta,\mu)$ generates a Hilbert-Schmidt type integral operator. Since the function $G(x,\eta,\mu)$ is a kernel of the operator $R_\lambda = (L+\mu E)^{-1}$, we get, that operator L has a discrete spectrum $\lambda_1, \lambda_2, ..., \lambda_n ...$ with a unique limit point in infinity.

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