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ON BASIS PROPERTIES OF SYSTEM OF ROOT FUNCTIONS OF A FOURTH ORDER SPECTRAL PROBLEM WITH SPECTRAL AND PHYSICAL PARAMETERS IN BOUNDARY CONDITION

Abstract

In the paper the fourth order spectral problem

$$y^{IV}(x) = \lambda y(x), \ x \in (0,1)$$

$$y''(0) = y'''(0) = y''(1) = 0,$$

$$y'''(1) + d\lambda y(1) = 0,$$

is considered, where $\lambda \in C$ *is a spectral parameter, and* $d < 0$ *is a physical parameter.*

The general characteristic of eigenvalues disposition on a real axis is given, the root space structure and the oscillation properties of eigenfunctions are studied, the asymptotic formulae for eigenvalues and eigenfunctions are derived, and the basis properties in $L_p(0,1)$, $p \in (1,\infty)$ *of the system of root functions of this problem is proved.*

Let's consider the following problem

$$y^{IV}(x) = \lambda y(x), \ x \in (0,1), \tag{0.1}$$

$$y''(0) = y'''(0) = y''(1) = 0, \tag{0.2}$$

$$y'''(1) + d\lambda y(1) = 0, \tag{0.3}$$

where $\lambda \in \mathbf{C}$ is a spectral parameter, and $d \in \mathbf{R}$ is a physical parameter. This problem arises, for example, by solving the dynamic boundary value problem describing small transverse vibrations of homogeneous rod of free left end and subjected to the action of tracing force at the first end by the method of variable separation. In particular, the case $d > 0$ describes situation when on the right end of a rod, the additional mass of the quantity d is concentrated. The bibliography of papers, in which we can find more exact information on physical meaning of the similar type problems, is given in [1].

Problem (0.1)-(0.3) in the case $d > 0$ is investigated in [1]. In this paper the oscillation properties of eigenfunctions were studied, the asymptotic formulae for eigenvalues and eigenfunctions were obtained, and the basicity in $L_p(0,1)$, $p \in (1,\infty)$ of the systems of eigenfunctions of this problem with one removed eigenfunction was established.

Everywhere hereinafter we assume that the condition $d < 0$ holds.

The aim of the present paper is to research basic properties in spaces $L_p(0, \ l)$, $p \in (1,\infty)$, of systems of the root functions of boundary value problem (0.1) - (0.3).

1. Some properties of solutions of problem (0.1) - (0.2) for $\lambda < 0$.

As in [1-3] for studying the oscillation properties of eigenfunctions of boundary value problem (0.1)-(0.3) we'll use the Prufer type transformation of the following form

$$\begin{cases} y(x) = r(x)\sin\psi(x)\cos\theta(x), \\ y'(x) = r(x)\cos\psi(x)\sin\varphi(x), \\ y''(x) = r(x)\cos\psi(x)\cos\varphi(x), \\ y'''(x) = r(x)\sin\psi(x)\sin\theta(x). \end{cases} \tag{1.1}$$

Equation (0.1) allows the equivalent formulation in the matrix form

$$V' = MV, \tag{1.2}$$

where

$$V = \begin{pmatrix} y \\ y' \\ y'' \\ y''' \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda & 0 & 0 & 0 \end{pmatrix}.$$

Assuming $w(x) = \tan\psi(x)$ and using transformation (1.1) in equation (1.2), we get a system of the first order differential equations with respect to the functions r, w, θ, φ of the form

$$r' = \left[\sin 2\psi \sin(\theta+\varphi) + \cos^2\psi \sin 2\varphi + \lambda \sin^2\psi \sin 2\theta\right]\frac{r}{2}, \tag{1.3a}$$

$$w' = w^2\cos\theta\sin\varphi - \frac{1}{2}w\sin 2\varphi + \cos\theta\sin\varphi - \frac{1}{2}\lambda w\sin 2\theta, \tag{1.3b}$$

$$\theta' = -w\sin\theta\sin\varphi + \lambda\cos^2\theta, \tag{1.3c}$$

$$\varphi' = \cos^2\varphi - w\sin\theta\sin\varphi. \tag{1.3d}$$

Lemma 1.1. *Let* $y(x,\lambda)$ *be a nontrivial solution of problem (0.1)-(0.2) for* $\lambda < 0$. *Then the Jacobian* $J[y] = r^3\sin\psi\cos\psi$ *of transformation (1.1) is non-zero for* $x \in (0,1)$.

Proof. Let's suppose that the statement of lemma 1.1 isn't true. Let $x_1 \in (0,1)$ be closest point to the origin at which $\sin\psi(x_1,\lambda)\cos\psi(x_1,\lambda) = 0$, whence it follows, that at least one of the indicated factors equals zero. In case $\sin\psi(x_1,\lambda) = 0$ we have $y(x_1,\lambda) = y'''(x_1,\lambda) = 0$. In view of (0.2) and (0.1) there exists the point $x_0 \in (0,x_1)$ such that $y(x_0,\lambda) = 0$. Without loss of generality, we can consider, that $y(x,\lambda)\,y'''(x,\lambda) > 0$ for $x \in (x_0,x_1)$. Since $y(x_0,\lambda) = y(x_1,\lambda) = 0$, there exists the point $\xi_0 \in (x_0,x_1)$, such that $y'(\xi_0,\lambda) = 0$. Assume $\delta_0 = \arctan\left(y(\xi_0,\lambda)/y'''(\xi_0,\lambda)\right)$. Then the function $y(x,\lambda)$ is a solution of the boundary value problem.

$$y^{IV}(x) = \lambda y(x), \quad 0 < x < \xi_0,$$

$$y''(0) = y'''(0) = 0, \quad y'(\xi_0) = 0, \quad y(\xi_0)\cos\delta_0 - y'''(\xi_0)\sin\delta_0 = 0.$$

By virtue of relations (A.1), (A.2) [3] the eigenvalues of this problem are positive, that contradicts the condition $\lambda < 0$.

Analogously we consider the cases $\cos\psi(x_1,\lambda)=0$.

Lemma 1.1. is proved.

Let $y(x,\lambda)$ be a nontrivial solution of problem (0.1)-(0.2), and let $\theta(x,\lambda)$ and $\varphi(x,\lambda)$ be the corresponding angular functions from (1.1). Without loss of generality, the initial values of these functions can be defined in the following way (see the proof of theorem 3.1 [3]).

$$\theta(0,\lambda)=0 \ \text{ at } \psi(0,\lambda)\neq 0, \quad \varphi(0,\lambda)=\frac{\pi}{2}\operatorname{sgn}|\psi(0,\lambda)-\pi/2|. \tag{1.4}$$

Remark 1.1. If $\psi(0,\lambda)=0$, then $\theta(0,\lambda)=0$. Indeed, from (1.1) it follows, that $y(0,\lambda)=y'''(0,\lambda)=0$, but as $y''(0,\lambda)=0$, then $y'(0,\lambda)\neq 0$. Hence, by (1.1) we have

$$\tan\theta(0,\lambda)=\lim_{x\to 0}\frac{y'''(x,\lambda)}{y(x,\lambda)}=\lim_{x\to 0}\frac{y^{IV}(x,\lambda)}{y'(x,\lambda)}=\lim_{x\to 0}\frac{\lambda y(x,\lambda)}{y'(x,\lambda)}=\frac{\lambda y(0,\lambda)}{y'(0,\lambda)}=0.$$

Lemma 1.2. *Let* $\psi(0,\lambda)=0$. *Then*

(i) $\psi(x,\lambda)\in(0,\pi/2)$, $x\in(0,1)$;

(ii) $\lim\limits_{x\to 0}\theta'(x,\lambda)=\lambda/2$.

Proof. By (1.3b) we have $w'(0,\lambda)=1$. Since $w'(x,\lambda)=\dfrac{\psi'(x,\lambda)}{\cos^2\psi(x,\lambda)}$, then $\psi'(0,\lambda)>0$, and hence $w(x,\lambda)\in(0,\pi/2)$, $x\in(0,1)$.

Notice that $\dfrac{1}{w(0,\lambda)}=\cot\psi(0,\lambda)$, and therefore in (1.3c) $\theta'(0,\lambda)$ makes no sense. Using (1.1) we obtain

$$\lim_{x\to 0}\frac{\sin\theta(x,\lambda)\sin\varphi(x,\lambda)}{w(x,\lambda)}=\lim_{x\to 0}\frac{y'(x,\lambda)\,y'''(x,\lambda)}{y^2(x,\lambda)}\cos^2\theta(x,\lambda)=$$

$$=y'(0,\lambda)\lim_{x\to 0}\frac{y'''(x,\lambda)}{y^2(x,\lambda)}=y'(0,\lambda)\lim_{x\to 0}\frac{y^{IV}(x,\lambda)}{2y(x,\lambda)\,y'(x,\lambda)}=$$

$$=y'(0,\lambda)\lim_{x\to 0}\frac{\lambda y(x,\lambda)}{2y(x,\lambda)\,y'(x,\lambda)}=y'(0,\lambda)\lim_{x\to 0}\frac{\lambda}{2y'(x,\lambda)}=\frac{\lambda}{2}. \tag{1.5}$$

Taking into account (1.5), from (1.3c) we find

$$\lim_{x\to 0}\theta'(x,\lambda)=\frac{\lambda}{2}. \tag{1.6}$$

Theorem 1.1. *The function* $y(x,\lambda)$, $x\in[0,1]$, $\lambda<0$, *has exactly one simple zero in the interval* $(0,1)$.

Proof. From (1.3c) it follows, that $\theta(x,\lambda)$ takes the values $k\pi$ $(k\in Z)$ only strictly decreasing. Using relations (1.3c), (1.6), taking into account (1.4) and remark 1.1 we have $\theta(x,\lambda)<0$. By lemma 3 from [1] $y'''(1,\lambda)\,y(1,\lambda)<0$. Then from (1.1) it follows, that $\cos\theta(1,\lambda)\sin\theta(1,\lambda)<0$. From formulae (7), (8) of the paper [1] it follows, that for sufficiently small values of λ the function $y(x,\lambda)$ has one simple zero in the interval $(0,1)$; and so $\theta(1,\lambda)\in(-3\pi/2,-\pi)$.

If $\sin\theta(x_0,\lambda) = -\dfrac{\pi}{2}$ at some point $x_0 \in (0,1)$, then $\theta'(x_0,\lambda) \neq 0$ holds. Indeed, when $\theta'(x_0,\lambda) = 0$, by lemma 1.2 from (1.3c) it follows, that $\sin\varphi(x_0,\lambda) = 0$. Then by (1.1) we have $y(x_0,\lambda) = 0$, $y'(x_0,\lambda) = 0$, that contradicts the condition $\lambda < 0$ (see [3]).

Now let's prove, that the function $y(x,\lambda)$ in the interval $(0,1)$ has exactly one zero. Indeed, if the function $y(x,\lambda)$ in the interval $(0,1)$ has more than one zero, then the function $\theta(x,\lambda)$ has to cross at least one of the lines $\theta = -s\pi - \dfrac{\pi}{2}$, $s = 0,1$ at least two times. Let x_1, x_2, $x_1 < x_2$, be the closest points to the origin, such that $\theta(x_1,\lambda) = \theta(x_2,\lambda) = -s\pi - \dfrac{\pi}{2}$, where $s = 0$ or $s = 1$. Therefore $\theta'(x_1,\lambda) < 0$, $\theta'(x_2,\lambda) > 0$ and $\theta(x,\lambda) \in \left(-\pi - s\pi, -\dfrac{\pi}{2} - s\pi\right)$, $x \in (x_1,x_2)$, where $s = 0$ or $s = 1$. Hence, by (1.1) $y(x_1,\lambda) = y(x_2,\lambda) = 0$ and $y(x,\lambda)\, y'''(x,\lambda) > 0$ at $x \in (x_1,x_2)$. Then $y'(\xi,\lambda) = 0$ at some point $\xi \in (x_1,x_2)$, that contradict the condition $\lambda < 0$ (see the proof of lemma 1.1). Theorem 1.1 is proved.

From (9) [1] we obtain the asymptotic form

$$F(\lambda) = \sqrt[4]{\lambda^3}\,\frac{\cos\sqrt[4]{\lambda}}{\cos\sqrt[4]{\lambda} - \sin\sqrt[4]{\lambda}}\left(1 + O\left(1/\sqrt[4]{\lambda}\right)\right), \quad |\lambda| \to \infty. \tag{1.7}$$

Let $\mu_n(0)$, $n = 1,2,...$, be simple nonnegative eigenvalue of problem (0.1), (0.2) and $y(1)\cos\delta - y'''(1)\sin\delta = 0$, at $\delta = 0$ (see [1,3]). Denote $D = (-\infty, \mu_2(0)) \cup \bigcup\limits_{n=3}^{\infty} (\mu_{n-1}(0), \mu_n(0))$.

Lemma 1.3. *The function $F(\lambda)$ is convex in the interval $(-\infty, \mu_2(0))$.*

Proof. Following the corresponding reasonings [4, suggestion 4] we get, that the meromorphic function $F(\lambda)$ allows the representation

$$F(\lambda) = \sum_{n=2}^{\infty} \frac{\lambda c_n}{\mu_n(0)(\lambda - \mu_n(0))}, \tag{1.8}$$

where $c_n = \underset{\lambda=\mu_n(0)}{res} F(\lambda) < 0$.

Differentiating (1.8) we obtain

$$F'(\lambda) = -\sum_{n=2}^{\infty} \frac{c_n}{(\lambda - \mu_n(0))^2}, \quad F''(\lambda) = 2\sum_{n=2}^{\infty} \frac{c_n}{(\lambda - \mu_n(0))^3},$$

whence it follows, that $F'(\lambda) > 0$ for $\lambda \in D$, $F''(\lambda) > 0$ at $\lambda \in (-\infty, \mu_2(0))$. Lemma 1.3 is proved.

2. Oscillation properties of eigenfunctions of boundary value problem (0.1)-(0.3).

Lemma 2.1. *The eigenvalues of boundary value problem (0.1)-(0.3) are real, and form no more than countable set not having the finite limit point. All nonzero eigenvalues of boundary value problem (0.1)-(0.3) are simple.*

Proof. If λ is nonreal eigenvalue of problem (0.1)-(0.3), then $\overline{\lambda}$ also will be eigenvalue of this problem. At that, as in lemma 2 [2] we can show the validity of the equality

$$\int_0^1 |y(x,\lambda)|^2 dx + d|y(1,\lambda)|^2 = 0 \tag{2.1}$$

Multiplying the both parts of equation (0.1) by the function $\overline{y}(x,\lambda)$ and integrating the obtained equality by parts in the range from 0 to 1, and also taking into account (0.2), (0.3) we get

$$\int_0^1 |y''(x,\lambda)|^2 dx = \lambda \left[\int_0^1 |y(x,\lambda)|^2 dx + d|y(1,\lambda)|^2 \right]. \tag{2.2}$$

Taking into account (2.1) in (2.2), we find that $y''(x,\lambda) = 0$, $x \in [0,1]$, whence by (0.1) we have $\lambda = 0$. And this contradicts nonreality of λ.

The eigenvalues of boundary value problem (0.1)-(0.3) are zeros of the entire function $y'''(1,\lambda) + d\lambda y(1,\lambda)$. This function doesn't vanish at $\lambda \in C/R$. Hence, it doesn't equal zero identically. Therefore its zeros form no more than countable set not having the finite limit point.

If $\lambda^* \neq 0$ is multiple root of the equation $y'''(1,\lambda) + d\lambda y(1,\lambda) = 0$, then as in lemma 3 from [2] we obtain

$$\int_0^1 y^2(x,\lambda^*) dx + dy^2(1,\lambda^*) = 0 \tag{2.3}$$

Multiplying the both parts of equation (0.1) by the function $y(x,\lambda^*)$ and integrating the obtained equality by parts in the range from 0 to 1, and also taking into account (0.2), (0.3) we have

$$\int_0^1 \left(y''(x,\lambda)\right)^2 dx = \lambda^* \left[\int_0^1 y^2(x,\lambda) dx + dy^2(1,\lambda^*) \right]. \tag{2.4}$$

Taking into account (2.3) in (2.4), we obtain $y''(x,\lambda) = 0$, $x \in [0,1]$, whence by (0.1) we have $\lambda^* = 0$. Lemma 2.1 is proved.

Definition 3.1. *(see also [5, §2, i.3]). Let $y(x)$ be eigenfunction of problem (0.1)-(0.3) corresponding to the eigenvalue λ. We call $v(x)$ the associated function to the eigenfunction $y(x)$, if it satisfies the equation*

$$v^{IV}(x) = \lambda v(x) + y(x), \quad 0 < x < 1,$$

and the boundary conditions

$$v''(0) = v'''(0) = v''(1) = 0,$$

$$v'''(1) + d\lambda v(1) + dy(1) = 0.$$

Obviously, that $\lambda = 0$ is eigenvalue of problem (0.1)-(0.3) and to this eigenvalue there correspond two linear-independent eigenfunctions.

It is easy to check, that if $d \neq -1/4$, then there is no associated function regarding the eigenfunctions responding to the eigenvalue $\lambda = 0$, and if $d = -1/4$, then the function $v(x) = \frac{x^5}{120} - \frac{x^4}{72} + b_1 x + b_2$, where b_1, b_2 are arbitrary constants, is assosiated to the eigenfunction $y(x) - 1/3$, responding to the eigenvalue $\lambda = 0$.

Thus, the root space corresponding to the eigenvalue $\lambda = 0$ for $d \neq -1/4$ consists of the functions $c_1 x + c_2$, and for $d = -1/4$ consists of the functions $c_1 x = c_2 + c_3\left(\frac{x^5}{120} - \frac{x^4}{72}\right)$, where c_1, c_2 are arbitrary constants.

Notice, that the eigenvalues of problem (0.1)-(0.3) coinside with roots of the equation

$$F(\lambda) = -d\lambda \tag{2.5}$$

Theorem 2.1. *There exists indefinitely strictly increasing sequence of the eigenvalues* $\lambda_1, \lambda_2, ..., \lambda_n$ *of spectral problem (0.1)-(0.3), at that* $\lambda_1 < 0$, $\lambda_2 = 0$, *if* $d \in (-1/4, 0)$ *and* $\lambda_1 = 0$, *if* $d \in (-\infty, -1/4]$. *The eigenfunctions* $y_n(x)$ *corresponding to the eigenvalues* $\lambda_n \neq 0$, *posses the following oscillation properties: (a) if* $d \in (-\infty, -1/4)$, *then the eigenfunction* $y_n(x)$, $n = 2, 3, ...$, *has* $n-1$ *simple zeros in the interval* $(0, 1)$; *(b) if* $d = -1/4$, *then the eigenfunction* $y_n(x)$, $n = 2, 3, ...$, *has* n *simple zeros in the interval* $(0, 1)$; *(c) if* $d = (-1/4, 0)$, *then the eigenfunction* $y_1(x)$ *has one simple zeros in the interval* $(0, 1)$, *and the eigenfunction* $y_n(x)$, $n = 3, 4, ...$, *has* $n - 1$ *simple zeros in the interval* $(0, 1)$.

Proof. By virtue of lemma 3 [1], asymptotic form (1.7) and equalities $y(1, \mu_n(0)) = 0$, $n = 2, 3, ...$, we have

$$\lim_{\lambda \to -\infty} F(\lambda) = -\infty, \qquad \lim_{\lambda \to \mu_n(0)-0} F(\lambda) = +\infty, \qquad \lim_{\lambda \to \mu_n(0)+0} F(\lambda) = -\infty. \tag{2.6}$$

Taking into account convexity of the function $F(\lambda)$ in the interval $(-\infty, \mu_2(0))$ (see lemma 1.3) and the equality $F'(0) = 1/4$, which follows from (10) [1], we get, that equation (2.5) has the solution $\lambda_1 \in (-\infty, 0)$ only in the case $d \in (-1/4, 0)$ and has the solution $\lambda_2 \in (0, \mu_2(0))$ only in the case $d \in (-\infty, -1/4)$.

Let now $\lambda \in (\mu_n(0), \mu_{n+1}(0))$, $n = 2, 3, ...$ By (28) [6], (2.4) and (2.6) equation (2.5) has a unique solution λ_{n+1} for $d \in (-\infty, 0) \setminus \{-1/4\}$ and λ_n for $d = -1/4$.

Statements (a)-(c) of theorem 2.1 follow from lemma 4 [1] and theorem 1.1. Theorem 3.1 is proved.

Corollary. *For* $n = 2, 3, ...$, *the relation*

$$\mu_n(0) < \mu_n\left(\frac{\pi}{2}\right) < \lambda_{n+1-\sigma} < \mu_{n+1}(0), \tag{2.7}$$

is true, where $\sigma = 1 - \text{sgn}\left|d + \frac{1}{4}\right|$, $\mu_n\left(\frac{\pi}{2}\right)$, $n = 2, 3, ...$, *is simple eigenvalue of problem (0.1), (0.2) and* $y(1)\cos\delta - y'''(1)\sin\delta = 0$, *for* $\delta = \frac{\pi}{2}$.

3. Basicity in $L_p(0,1)$, $p\in(1,\infty)$, of systems of root functions of spectral problem problem (0.1) - (0.3).

To study basicity properties of systems of root functions of boundary value problem (0.1)-(0.3) in the spaces $L_p(0,1)$, $p\in(1,\infty)$, it is nesessary to attract asymptotic form of eigenfunctions of this problem.

Theorem 3.1. *The following asymptotic formulae are true*

$$\sqrt[4]{\lambda_n}=\left(n+\sigma-\frac{3}{4}\right)\pi+O\left(\frac{1}{n}\right) \tag{3.1}$$

$$y_n(x)=\sin\left(n+\sigma-\frac{3}{4}\right)\pi x-\cos\left(n+\sigma-\frac{3}{4}\right)\pi x-e^{-\left(n+\delta-\frac{3}{4}\right)\pi x}+O\left(\frac{1}{n}\right), \tag{3.2}$$

where $\sigma=1-sgn\left|d+\frac{1}{4}\right|$, *at that relation (3.2) holds uniformly on* $x\in[0,1]$.

Proof of theorem 3.1 is carried out on the scheme of proof of theorem 7 from [1] using theorem 2.1 and corollary 2.1. We consider three cases:

Case 1. Let $d\in(-1/4,0)$. By virtue of theorem 2.1 $\lambda_1<0$ $\lambda_2=0$ $\lambda_n>0$, $n=3,4,\ldots$. Denote by $y_{21}(x)$ and $y_{22}(x)$, $x\in[0,1]$, two linear-independent eigenfunctions from root space, corresponding the eigenvalue $\lambda_2=0$.

Consider the following systems of eigenfunctions of boundary value problem (0.1)-(0.3):

(I_1) $y_1(x)$, $y_{2i}(x), y_3(x),\ldots,y_n,\ldots, i=1,2$, where $y_{21}(x)=1$ and $y_{22}(x)=x+c$, $x\in[0,1]$, c is an arbitary constant different from $-1/3$;

(I_2) $y_1(x)$, $y_{21}(x), y_{22}(x), y_3(x),\ldots,y_n,\ldots, n\neq r, r\in N\backslash\{2\}, y_{21}(x)=1$, $y_{22}(x)=x+c$ or $y_{21}(x)=x+\alpha$, $y_{22}(x)=x+\beta$, $x\in[0,1]$, c, α,β $(\alpha\neq\beta)$ are arbitary constants.

Case 2. Let $d\in(-\infty,-1/4)$. By virtue of theorem 2.1 $\lambda_1=0$, $\lambda_n>0$, $n=2,3,\ldots$. Denote by $y_{11}(x)$ and $y_{12}(x)$, $x\in[0,1]$, two linear-independent eigenfunctions from root space, corresponding the eigenvalue $\lambda_1=0$.

Consider the following systems of eigenfunctions of boundary value problem (0.1)-(0.3):

(II_1) $y_{1i}(x)$, $y_2(x),\ldots,y_n,\ldots, i=1,2$, where $y_{11}(x)=1$ and $y_{12}(x)=x+c$, $x\in[0,1]$, c is an arbitary constant different from $-1/3$;

(II_2) $y_{11}(x)$, $y_{12}(x), y_2(x), y_3(x),\ldots,y_n(x),\ldots, n\neq r, r\in N\backslash\{1\}, y_{11}(x)=1$, $y_{12}(x)=x+c$ or $y_{11}(x)=x+\alpha$, $y_{12}(x)=x+\beta$, $x\in[0,1]$, c, α,β $(\alpha\neq\beta)$ are arbitary constants.

Case 3. Let $d=-1/4$. By virtue of theorem 2.1 we have $\lambda_1=0$, $\lambda_n>0$, $n=2,3,\ldots$. Remind (see §2), that $\lambda_1=0$ is triple eigenvalue of boundary value problem (0.1)-(0.3), at that to this eigenvalue there correspond two linear-independent eigenfunctions $y_{11}(x)$ and $y_{12}(x)$, and for the eigenfunction $x-1/3$ there exists the associated function $y_{13}(x)=\frac{x^5}{120}-\frac{x^4}{72}+c_1x+c_2$.

Consider the following systems of root functions of problem (0.1)-(0.3):

(III_1) $y_{11}(x)$, $y_{12}(x), y_2(x),\ldots,y_n(x),\ldots$, where $y_{11}(x)=1$, $y_{12}(x)=x+c$ or $y_{11}(x)=x+\alpha$, $y_{12}(x)=x+\beta$, $x\in[0,1]$, c, α,β $(\alpha\neq\beta)$ are arbitary constants.

(III$_2$) $y_{11}(x)$, $y_{13}(x), y_2(x), ..., y_n(x), ...,$where $y_{11}(x) = 1$, $x \in [0,1]$, $c_1 \neq 41/11880$;

(III$_3$) $y_{11}(x)$, $y_{12}(x), y_2(x), y_3(x), ..., y_n(x), ..., n \neq r, r \in N\backslash\{1\}, y_{11}(x) = 1$, $y_{12}(x) = x - 1/3$, $x \in [0,1]$.

Theorem 3.2. *The systems of root functions (I$_1$), (I$_2$), (II$_1$), (II$_2$), (III$_1$), (III$_2$), (III$_3$) of spectral problem (0.1)-(0.3) are minimal in the space $L_p(0,1)$, $p \in (1,\infty)$.*

Proof. To prove the theorem it suffices to show the existence of systems which are conjugate, respectively, to the systems (I$_1$), (I$_2$), (II$_1$), (II$_2$), (III$_1$), (III$_2$), (III$_3$).

By virtue of (0.1), (0.2) for any different natural n and m we have

$$y_m(1)\, y_n'''(1) - y_n(1)\, y_m'''(1) = (\lambda_n - \lambda_m)(y_n, y_m), \tag{3.3}$$

where $(y_n, y_m) = \int_0^1 y_n(x)\, y_m(x)\, dx$. By (0.3) from (3.3) we obtain

$$(y_n, y_m) = -d y_n(1)\, y_m(1). \tag{3.4}$$

At proof of lemma 2.1 it was estblished, that for $\lambda_n \neq 0$, $n \in N$, the relation $\int_0^1 y^2(x,\lambda_n)\, dx + d y^2(1,\lambda_n) \neq 0$ holds.

Consider the system (I$_1$). At $i = 1$ elements of the system $\upsilon_1(x)$, $\upsilon_{21}(x)$, $\upsilon_3(x)$, ..., $\upsilon_n(x)$,... are defined by the equalities:

$$\upsilon_{21}(x) = 2(1-x),$$

$$\upsilon_n(x) = \frac{1}{\|y_n(x)\|_2^2 + d y_n^2(1)} \left(y_n(x) - y_n(1) - 2(d+1)\, y_n(1)(x-1)\right), \tag{3.5}$$

where $\|\cdot\|_p$ is norm in the space $L_p(0,1)$, and at $i = 2$ the elements of the system $\upsilon_1(x)$, $\upsilon_{22}(x)$, $\upsilon_3(x)$, ..., $\upsilon_n(x)$,... are defined by the equalities:

$$\upsilon_{22}(x) = 2(1-x)/(c+1/3),$$

$$\upsilon_n(x) = \frac{1}{\|y_n(x)\|_2^2 + d y_n^2(1)} \Big(y_n(x) - y_n(1) - \frac{2y_n(1)}{c+1/3}\left(d(c+1) + \frac{1}{2} + c\right)(x-1)\Big). \tag{3.6}$$

Consider the system (I$_2$). At $y_{21}(x) = 1$, $y_{22}(x) = x + c$, $x \in [0,1]$, the elements of the system $\upsilon_1(x)$, $\upsilon_{21}(x)$, $\upsilon_{22}(x)$, $\upsilon_3(x)$, ..., $\upsilon_n(x)$,..., $n \neq r$, are defined by the equalities:

$$\upsilon_{21}(x) = \begin{cases} \dfrac{3(c+1/3)}{2(d+1/4)\, y_r(1)} \left(y_r(x) - y_r(1) - \dfrac{2(d(1+c) + 1/2 + c)\, y_r(1)}{c+1/3}\right), & c \neq -1/3 \\ 2(1-x), \quad c = -1/3, \end{cases},$$

$$\upsilon_{22}(x)=\frac{3}{2(d+1/4)\,y_r(1)}\left(y_r(x)-y_r(1)-2(1+d)\,y_r(1)(x-1)\right),$$

$$\upsilon_n(x)=\frac{1}{\|y_n(x)\|_2^2+dy_n^2(1)}\left(y_n(x)-\frac{y_n(1)}{y_r(1)}y_r(x)\right), \tag{3.7}$$

and at $y_{21}(x)=x+\alpha$, $y_{22}(x)=x+\beta$, $x\in[0,1]$, $\alpha\neq\beta$, the elements of the system $\upsilon_1(x)$, $\upsilon_{21}(x)$, $\upsilon_{22}(x)$, $\upsilon_3(x)$, ..., $\upsilon_n(x)$,..., $n\neq r$, are defined by the equalities:

$$\upsilon_{21}(x)=\frac{3}{2\left(d+\frac{1}{4}\right)(\beta-\alpha)\,y_r(1)}\left(\left(\beta+\frac{1}{3}\right)(y_r(x)-y_r(1))-\right.$$

$$\left.-2\left(d(1+\beta)+\frac{1}{2}+\beta\right)y_r(1)(x-1)\right),$$

$$\upsilon_{22}(x)=\frac{3}{2\left(d+\frac{1}{4}\right)(\alpha-\beta)\,y_r(1)}\left(\left(\alpha+\frac{1}{3}\right)(y_r(x)-y_r(1))-\right.$$

$$\left.-2\left(d(1+\alpha)+\frac{1}{2}+\alpha\right)y_r(1)(x-1)\right),$$

$$\upsilon_n(x)=\frac{1}{\|y_n(x)\|_2^2+dy_n^2(1)}\left(y_n(x)-\frac{y_n(1)}{y_r(1)}y_r(x)\right). \tag{3.8}$$

By equality (3.4) it is easy to see, that the written systems $\upsilon_1(x)$, $\upsilon_{2i}(x)$, $\upsilon_3(x)$, ..., $\upsilon_n(x)$,..., $i=1,2$; $\upsilon_1(x)$, $\upsilon_{21}(x)$, $\upsilon_{22}(x)$, $\upsilon_3(x)$, ..., $\upsilon_n(x)$,..., $n\neq r$, are conjugate to the systems (I_1) and (I_2), respectively.

The systems conjugate to the systems (II_1), (II_2) are written out absolutely by analogous way (at that $\upsilon_{2i}(x)$ and $\upsilon_{1i}(x)$, $i=1,2$, change places, respectively).

Consider the system (III_1). At $y_{11}(x)=1$, $y_{12}(x)=x+c$, $x\in[0,1]$, the elements of the system $\upsilon_{11}(x)$, $\upsilon_{12}(x)$, $\upsilon_2(x)$, ..., $\upsilon_n(x)$,..., are defined by the equalities:

$$\upsilon_{11}(x)=\frac{4}{3}-\frac{1}{2}(25+63c)\left(x-\frac{1}{3}\right)-\left(c+\frac{1}{3}\right)\left(\frac{63}{2}x^5-\frac{105}{2}x^4\right),$$

$$\upsilon_{12}(x)=\frac{7}{2}\left(9x^5-15x^4+9x-3\right),$$

$$\upsilon_n(x)=\frac{1}{\|y_n(x)\|_2^2-\frac{1}{4}y_n^2(1)}\left(y_n(x)-y_n(1)-\frac{3}{2}y_n(1)(x-1)\right), \tag{3.9}$$

and at $y_{11}(x)=x+\alpha$, $y_{12}(x)=x+\beta$, $x\in[0,1]$, $\alpha\neq\beta$, the elements of the system $\upsilon_{11}(x)$, $\upsilon_{12}(x)$, $\upsilon_2(x)$, ..., $\upsilon_n(x)$,..., are defined by the equalities:

$$\upsilon_{11}(x)=\frac{1}{\beta-\alpha}\left(-\frac{4}{3}+\frac{1}{2}(25+63\beta)\left(x-\frac{1}{3}\right)+\left(\beta+\frac{1}{3}\right)\left(\frac{63x^5}{2}-\frac{105x^4}{2}\right)\right),$$

$$\upsilon_{12}(x)=\frac{1}{\alpha-\beta}\left(-\frac{4}{3}+\frac{1}{2}(25+63\alpha)\left(x-\frac{1}{3}\right)+\left(\alpha+\frac{1}{3}\right)\left(\frac{63x^5}{2}-\frac{105x^4}{2}\right)\right),$$

$$\upsilon_n(x)=\frac{1}{\|y_n(x)\|_2^2-\frac{1}{4}y_n^2(1)}\left(y_n(x)-y_n(1)-\frac{3}{2}y_n(1)(x-1)\right). \tag{3.10}$$

Consider the system (III_2). The elements of the system $\upsilon_{11}(x)$, $\upsilon_{13}(x)$, $\upsilon_2(x)$, ..., $\upsilon_n(x)$,..., are defined by the equalities:

$$\upsilon_{11}(x)=\frac{1}{\frac{41}{11880}-c_1}\left(\frac{4}{3}\left(\frac{41}{11880}-c_1\right)+\left(-\frac{7}{297}+\frac{25c_1+63c_2}{2}\right)\left(x-\frac{1}{3}\right)+\right.$$

$$\left.+\left(-2+1260\left(c_1+3c_2\right)\right)\left(\frac{x^5}{120}-\frac{x^4}{72}\right)\right),$$

$$\upsilon_{13}(x)=\frac{1}{\frac{41}{11880}-c_1}\left(\frac{63}{2}\left(x-\frac{1}{3}\right)+\frac{63}{2}x^5-\frac{105}{2}x^4\right),$$

$$\upsilon_n(x)=\frac{1}{\|y_n(x)\|_2^2-\frac{1}{4}y_n^2(1)}\left(y_n(x)-\frac{3/2}{\frac{41}{11880}-c_1}y_n(1)\times\right.$$

$$\left.\times\left(\frac{x^5}{120}-\frac{x^4}{72}+\left(\frac{7}{594}-c_1\right)\left(x-\frac{1}{3}\right)\right)\right) \tag{3.11}$$

Consider the system (III_2). The elements of the system $\upsilon_{11}(x)$, $\upsilon_{12}(x)$, $\upsilon_{13}(x)$, $\upsilon_2(x)$, ..., $\upsilon_n(x)$,..., $n\neq r$ are defined by the equalities:

$$\upsilon_{11}(x)=\frac{4}{3}-3780\left(c_1+c_2\right)\left(x-\frac{1}{3}\right)-\left(\frac{4}{3}-2520\left(c_1+c_2\right)\right)\frac{y_r(x)}{y_r(1)},$$

$$\upsilon_{12}(x)=3780\left(\left(\frac{7}{594}-c_1\right)\left(x-\frac{1}{3}\right)+\frac{x^5}{120}-\frac{x^4}{72}-\frac{2}{3}\left(\frac{41}{11880}-c_1\right)\frac{y_r(x)}{y_r(1)}\right),$$

$$\upsilon_{13}(x)=1260\left(3\left(x-\frac{1}{3}\right)-2\frac{y_r(x)}{y_r(1)}\right),$$

$$\upsilon_n(x)=\frac{1}{\|y_n(x)\|_2^2-\frac{1}{4}y_n^2(1)}\left(y_n(x)-\frac{y_n(1)}{y_r(1)}y_r(x)\right). \tag{3.12}$$

By virtue of (3.4) we establish, that the written out systems $\upsilon_{11}(x)$, $\upsilon_{12}(x)$, $\upsilon_2(x)$, ..., $\upsilon_n(x)$,...; $\upsilon_{11}(x)$, $\upsilon_{13}(x)$, $\upsilon_2(x)$, ..., $\upsilon_n(x)$,...; $\upsilon_{11}(x)$, $\upsilon_{12}(x)$, $\upsilon_{13}(x)$, $\upsilon_2(x)$, ..., $\upsilon_n(x)$,..., $n\neq r$ are conjugate to the systems (III_1), (III_2), (III_3), respectively. Theorem 3.2 is proved.

Lemma 3.1. *For sufficiently great $n\in N$, $n\neq r$, the asymptotic formula*

$$\upsilon_n(x)=y_n(x)+O(1/n). \tag{3.13}$$

is true.

Proof. By formula (3.2) the relations

$$\|y_n(x)\|_2^2=1+O(1/n) \qquad y_n(1)=O(1/n). \tag{3.14}$$

are true.

Using (3.14) in formulae (3.5)-(3.12) we obtain representation (3.13).

Lemma 3.1 is proved.

Theorem 3.3. *The systems of root functions (I_1), (I_2), (II_1), (II_2), (III_1), (III_2), (III_3) of spectral problem (0.1)-(0.3) form bases in the space* $L_p(0,1)$, $p \in (1,\infty)$, *and for* $p = 2$ *these bases are Riesz bases.*

Proof of theorem 3.3 holds on the scheme of the proof of theorem 10 from [1] (see also [7; theorem 5]) using theorem 3.1, 3.2 and lemma 3.1.

Remark 3.1. In the systems (I_1) and (II_1) the choice of constant c is essential. The system (I_1), where $y_{22}(x) = x - \frac{1}{3}$,and the system ($\mathrm{II}_1$), where $y_{12}(x) = x - \frac{1}{3}$, are incomplete and nonminimal. Indeed, the function $\upsilon(x) = x - 1$ is orthogonal to all functions of these systems, and there hold the expansions

$$y_{i2}(x) = \sum_{n=1, n\neq i}^{\infty} \frac{2\left(d + \frac{1}{4}\right) y_n(1)}{\left(\|y_n(x)\|_2^2 + d y_n^2(1)\right)} y_n(x), \ i = 1, 2,$$

whose validity follows from basicity of the system (I_1) for $i = 1$ and the system (II_1) for $i = 1$ in $L_2(0,1)$.

In the system (III_2) the choice of constant c_1 is essential. For $c_1 = 41/11880$ the system (III_2) is incomplete, since the function $\tau(x) = \frac{63}{2}\left(x - \frac{1}{3}\right) + \frac{63}{2}x^5 - \frac{105}{2}x^4$ is orthogonal to all functions of this system.

Note, that similar results for the second order equation with spectral parameter in the boundary condition are received in the paper [8].

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