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ON GRADIENT METHODS IN OPTIMAL CONTROL PROBLEMS FOR NONLOCAL PROCESSES

Abstract

A problem of optimal control by systems with the lamped parameters with nonlocal boundary conditions is considered. Formula for gradient is obtained in optimal control problem with nonlocal boundary conditions.

We consider a problem of optimal control by systems with the lamped parameters with nonlocal boundary conditions. An optimal control problem with nonlocal conditions arise in different fields of science, technics, medicine and etc. Numerous mathematical models of nonlocal processes, described by differential equations with nonlocal boundary conditions, are given in [1]. In [2] we get a formula of gradient for various processes of optimal control with nonlocal boundary conditions.

At the present paper we get formula of gradient in optimal control problem with nonlocal boundary conditions.

Problem statement. Let some controlled process be described by the differential equation

$$\dot{x}(t) = f(t,x,u), \quad t \in [0,T], \tag{1}$$

with nonlocal boundary conditions

$$Ax(0) = a, \tag{2}$$

$$\int_0^T B(t)\, x(t)\, dt = b. \tag{3}$$

It is supposed that the process is controlled by the functions, that belong to the functional space L_2:

$$u = u(\cdot) \in U = \{u(t) \in L_2[0,T],\, u(t) \in V_n n.b.\ t \in [0,T],\ V \subset R^r\}. \tag{4}$$

Let quality control criterion be minimization of functional of the following form:

$$J(u) = \Phi(x(T)), \tag{5}$$

where $x = (x_1, x_2, ..., x_n)$ are phase states of the system, $u = (u_1, u_2, .., u_r)$ are controls, functions $f(t,x,u)$, $\Phi(x)$ of variables $(t,x,u) \in [0,T] \times E^n \times E^r$ be considered as known, U is a given set from $L_2[0,T]$, the moment T is fixed. Supposed, that A is a constant rectangular matrix of order $m \times n$, $B(t)$ is a functional matrix $(n-m) \times n$.

Main results. Suppose the following conditions are fulfilled:

1. $B(t)$ is a matrix-function with continuous elements on the segment $[0,T]$ and

$$\det \begin{pmatrix} A \\ \int_0^T B(t)\, dt \end{pmatrix} \neq 0.$$

2. The function $f(t,x,u)$ is continuous by totality of its own arguments with the its own partial derivatives by variables x,u at $(t,x,u)\in[0,T]\times E^n\times E^r$ and, moreover, the following conditions:

$$|f(t,x+\bar{x},u+h)-f(t,x,u)|\le K(|\bar{x}|+|h|),$$
$$||f_x(t,x+\bar{x},u+h)-f_x(t,x,u)||\le K(|\bar{x}|+|h|),$$
$$||f_u(t,x+\bar{x},u+h)-f_u(t,x,u)||\le K(|\bar{x}|+|h|),$$
$$|\Phi_x(x+\bar{x})-\Phi_x(x)|\le L|\bar{x}|$$

are fulfilled for all $(t,x+\bar{x},u+h),(t,x,u)\in[0,T]\times E^n\times E^r$, where $K=const\ge 0$.

3. $KT\left[\|L\|\,\|B(t)\|_C\dfrac{T}{2}+1\right]$, where L is the inverse matrix to the matrix $\begin{pmatrix} A \\ \int_0^T B(t)\,dt\end{pmatrix}$, $\|B(t)\|_C$ is a norm of the matrix $B(t)$ in the space $C(0,T)$.

Theorem. *Let conditions 1-3 be fulfilled. Then, nonlocal boundary problem (1)-(3) at each fixed $u\in U$ has a unique solution. Besides, functional (5) is differentiable by $u=u(t)$ in norm $L_2^r[0,T]$ everywhere on $L_2^r[0,T]$, moreover, its gradient $J'(u)=J'(u,t)\in L_2^r[0,T]$ at the point $u=u(t)$ is representable in the form*

$$J'(u)=-H_u(t,x,u,\psi)\,|_{x=x(t,u),u=u(t),\psi=\psi(t,u)}=$$
$$=(f_u(t,x(t,u),u(t)))'\,\psi(t,u),\quad 0\le t\le T,$$

where $x(t)=x(t,u)$, $0\le t\le T$ is a solution of problem (1)-(3), corresponding to the control $u=u(t)$, and $\psi(t)=\psi(t,a)$ $0\le t\le T$ is a solution of the conjugated system

$$A'\lambda+\int_0^T B'(t)\,dt\mu+\int_0^T H_x(t,x(t),u(t),\psi(t))\,dt+\Phi_x(x(T))=0, \tag{6}$$

$$\psi(t)=\int_0^t B'(\tau)\,d\tau\mu+\int_0^t H_x(\tau,x(t),u(\tau),\psi(\tau))\,d\tau+A'\lambda,$$
$$H(t,x,u,\psi)=\langle f(t,x,u),\psi(t)\rangle, \tag{7}$$

"$'$" means a transposition.

Proof. It is easy to reduce the boundary problem (1)-(3) to the equivalent integral equation:

$$x(t)=\begin{pmatrix} A \\ \int_0^T B(t)\,dt\end{pmatrix}^{-1}\begin{pmatrix} a \\ b\end{pmatrix}-$$
$$-\begin{pmatrix} A \\ \int_0^T B(t)\,dt\end{pmatrix}^{-1}\begin{pmatrix} 0 \\ \int_0^T B(t)\int_0^t f(\tau,x(\tau),u(\tau))\,d\tau\end{pmatrix}. \tag{8}$$

We carry out the proof of the first part of the theorem by the succesive approximation method by formulas:

$$x^0(t)=\begin{pmatrix} A \\ \int_0^T B(t)\,dt\end{pmatrix}^{-1}\begin{pmatrix} a \\ b\end{pmatrix}$$

$$x^{n+1}(t)=\begin{pmatrix} A \\ \int_0^T B(t)\,dt \end{pmatrix}^{-1}\begin{pmatrix} a \\ b \end{pmatrix}-\begin{pmatrix} A \\ \int_0^T B(t)\,dt \end{pmatrix}^{-1}\times$$

$$\times\begin{pmatrix} 0 \\ \int_0^T B(t)\int_0^t f(\tau,x^n(\tau),u(\tau))\,d\tau \end{pmatrix}+\int_0^t f(\tau,x^n(\tau),u(\tau))\,d\tau,\quad n=0,1,2,...$$

To prove the second part of the theorem we use formula of increment for functional (5). Let $u, u+\bar{u}$ be two controls from U, and $x(t)$ and $x(t)+\bar{x}(t)$ be corresponding solutions of problem (1)-(3). Then

$$\begin{aligned} J(u+\bar{u})-J(u)&=\Phi(x(T,u+\bar{u}))-\Phi(x(T,u))=\\ &=\langle\Phi_x(x(T,u)),z(T)\rangle+\Phi(x(T)+\bar{x}(T))-\Phi(x(T))-\\ &-\langle\Phi_x(x(T)),\bar{x}(T)\rangle+\langle\Phi(x(T)),\bar{x}(T)-z(T)\rangle, \end{aligned}\tag{9}$$

where $z(t)$ is a solution of the following boundary value problem:

$$\dot{z}(t)=f_x(t,x(t),u(t))\,z(t)+f_u(t,x(t),u(t))\,\bar{u}(t),\tag{10}$$

$$Az(0)=0,\tag{11}$$

$$\int_0^T B(t)\,z(t)\,dt=0.\tag{12}$$

Let $\psi(t)$ be an arbitrary n-dimensional function from $L_2[0,T]$, λ and μ be arbitrary constant vectors of dimension m and $n-m$, respectively.

With the help of these vectors we can rewrite formula (9) in the form:

$$J(u+\bar{u})-J(u)=\int_0^T\langle H_u(t,x(t),u(t),\psi(t)),\bar{u}(t)\rangle\,dt+$$

$$+\int_0^T\left\langle\psi(t)-\int_0^t B'(\tau)\,d\tau\mu-\int_0^t H_x(\tau,x(\tau),u(\tau),\psi(\tau))\,d\tau-A'\lambda,\dot{z}(t)\right\rangle dt+$$

$$+\left\langle A'\lambda+\int_0^T B'(\tau)\,dt\mu-\int_0^T H_x(t,x(t),u(t),\psi(t))\,dt+\Phi_x(x(T)),z(T)\right\rangle+\eta,\tag{13}$$

where

$$\eta=\Phi(x(T)+\bar{x}(T))-\Phi(x(T))-\langle\Phi_x(x(T)),\bar{x}(T)\rangle+\langle\Phi_x(x(T)),\bar{x}(T)-z(T)\rangle.$$

We can show, that

$$|\eta|\le C\,\|\bar{u}\|^2$$

where $C\ge 0$ is a constant.

The proof of the second part of the theorem follows from representation (13).

Remark. Indeed, a system of the conjugated equations (6), (7) depends on three unknowns: $(\lambda,\mu,\psi(t))$, $0\le t\le T$. But, provided $\det\begin{pmatrix} A \\ \int_0^T B(t)\,dt \end{pmatrix}\neq 0$ we can exclude variables (λ,μ). Indeed, system (6) can be represented in the form

$$\begin{pmatrix} A' \\ \int_0^T B'(t)\,dt \end{pmatrix}\begin{pmatrix} \lambda \\ \mu \end{pmatrix}=-\int_0^T H_x(t,x(t),u(t),\psi(t))\,dt-\Phi_x(x(\tau)).$$

Hence

$$\begin{pmatrix} \lambda \\ \mu \end{pmatrix} = -\begin{pmatrix} A' \\ \int_0^T B'(t)\,dt \end{pmatrix}^{-1} \left[\int_0^T H_x(t, x(t), u(t), \psi(t))\,dt + \Phi_x(x(T)) \right]. \quad (14)$$

Now, considering (14) in (7) we have

$$\psi(t) = \int_0^t H_x(\tau, x(\tau), u(\tau), \psi(\tau))\,d\tau -$$

$$-\begin{pmatrix} A' \\ \int_0^t B'(\tau)\,d\tau \end{pmatrix} \begin{pmatrix} A' \\ \int_0^T B'(t)\,dt \end{pmatrix}^{-1} \int_0^T H_x(t, x(t), u(t), \psi(t))\,dt -$$

$$-\begin{pmatrix} A' \\ \int_0^t B'(\tau)\,d\tau \end{pmatrix} \begin{pmatrix} A' \\ \int_0^T B'(t)\,dt \end{pmatrix}^{-1} \Phi_x(x(T)). \quad (15)$$

Integral equation (15) is a system of conjugated equations. It contains the items of Volter and Fredholm integral equations. Such equations are conditionally solvable. One can show, that at condition 3. a system of integral equations is also solvable, i.e. nonlocal boundary value problem (1)-(3) and a system of equations (15) are simultaneously solvable.

On numerical solution of problem (1)-(5). At the numerical solution of problems of infinite-dimensional optimization, in particular optimal control problems, the known methods, for example gradient methods, let to find approximate solution of initial problem. For that, it's necessary to compute formula of a gradient of the studied problems. In this case, for computing a gradient of functional (5) at limitations (1)-(4) it is necessary to solve sequentially two nonlocal boundary value problems: first we have to determine $x(t,u)$ from problem (1)-(4), and then $\psi(t,u)$ from (15) and finally, to find desired gradient by formula $J'(u)$.

References

[1]. Nakhisheva Z.A. *Differential equations of mathematical models of nonlocal processes.* M.: Nauka, 2006, 176 p. (Russian).

[2]. Vasilyev F.P. *Optimization methods.* Moscow, Factorial Press, 2002, 824 p. (Russian).

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Received September 05, 2006; Revised November 30, 2006.