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ON THE CONVERGENCE OF A DIFFERENCE SCHEME APPROXIMATING NON-STATIONARY PROBLEM OF VISCOUS INCOMPRESSIBLE LIQUID MOTION IN CLOSED DOMAIN IN THE PRESENCE OF TEMPERATURE DISTRIBUTIONS

Abstract

In the paper we investage the convergence of a difference scheme approximating nonstationary problem of motion of viscous incompressible liquid in closed domain in the presence of temperature distributions. Theorems on boundedness of approximate solutions, on the convergence of iterative process to the solution of the stationary problem are proved and error estimation of the stationary problem is obtained.

Let's consider a system of equations describing the motion of viscous incompressible liquid in closed domain in the presence of temperature distributions on the boundary the domain

$$\begin{gathered} \frac{\partial\theta}{\partial t}+(\nu,\nabla)\,\theta-\vartheta\Delta\theta=H, \\ -\Delta\nu+\nabla p=Gr\theta\overrightarrow{\gamma}-(\nu,\nabla)\,\nu, \\ div\nu=0 \end{gathered} \tag{1}$$

The equations are written in dimensionless quantities: θ is temperature, v is velocity, H is the temperature source function, p is pressure, Gr is Grassgoph reduced number and ϑ is the reduced thermal diffusivity coefficient, $\overrightarrow{\gamma}$ is a unique vector codirected to the free fall acceleration vector:

System of equations (1) is considered in the bounded domain $\Omega\subset R^2$ with fixed impermeable piecewise-smooth boundary $\partial\Omega$ (i.e. consisting of a finite number smooth archs interesting at non-zero angles). On the boundary the following conditions are given:

$$\begin{gathered} v\,|_{\partial\Omega}=0, \\ \left\{\theta,\frac{\partial\theta}{\partial n}\right\}|_{\partial\Omega}=0, \\ \theta\,(x,0)=0, x\in\Omega. \end{gathered} \tag{2}$$

In particular, let the initial domain $\Omega\in R^2$ be a rectangle

$$\Omega=\left\{\,\overrightarrow{x}=(x_1,x_2)|\,0<x<L_i,i=1,2\right\}.$$

Introduce the following grids in

$$R^2:R_0^h=\left\{\,\overrightarrow{x}=(x_1,x_2)|\,x_i=j_ih_i,-\infty<j_i<+\infty,i=1,2\right\}.$$

Moreover, h_i were chosen so that there exists such $N_i=2^{n_i}+1$, that

$$L_i=(N_i-1)\,h_i,\ R_i^h=\left\{\overrightarrow{x}=(x_1,x_2)|\,\overrightarrow{x}=\overrightarrow{x}_0+\frac{h_i}{2}\overrightarrow{e}_i,\overrightarrow{x}_0\in R_0^h\right\},$$

$$i = 1,2,\ R_3^h = \left\{ \overrightarrow{x} = (x_1, x_2) |\ \overrightarrow{x} = \overrightarrow{x}_0 + \frac{h_1}{2}\overrightarrow{e}_1 + \frac{h_2}{2}\overrightarrow{e}_2,\ \overrightarrow{x}_0 \in R_0^h \right\}.$$

Intersection of each of grids $R_i^h, i = \overline{0,3}$ with domain Ω forms the i-th grid domain $\Omega_i^h = \Omega \cap R_i^h$.

Time grid we call a finite set of the numbers $t_0,\ t_1, ..., t_{k+1}$ such that $t_0 = 0$, $t_{k+1} = \sum\limits_1^k \tau_i \le T$. The number τ_k is said to be the k-th time step. In the paper we consider the case $\tau_k \equiv \tau$.

Further we introduce a space of mesh functions.

P is a space of scalar functions determined on R_0^h and vanishing outside of $\overline{\Omega}_0^h$, where $\overline{\Omega}_0^h = \Omega_0^h \cup \partial\Omega_0^h$, $\overset{0}{P}$ is a sub-space of the space P consisting of functions vanishing on $\partial\Omega_0^h$. $P_\perp$ is a sub-space of the space P consisting of functions φ^h that are orthogonal to a unit.

Let the space $\overset{0}{P}_\perp$ be determined as intersection of subspaces $\overset{0}{P}_\perp = P_\perp \cap \overset{0}{P}$, W_i be a space $(i = 1,2)$ of functions determined on R_i^h and vanishing outside of $\overline{\Omega}_i^h$, $i = 1,2$ where $\overline{\Omega}_i^h$ is a join of the set $\overline{\Omega}_i^h$ with that nodes of R_i^h that lie on boundary $\partial\Omega$. $W = W_1 \otimes W_2$ is a direct product of the spaces W_1 and W_2 consisting of vector functions $(u_1, u_2)^T$. $\overset{0}{W}_i$ is a sub-space of the i-th space W_i consisting of functions vanishing on the boundary and in the following nodes: if $u_1 \in \overset{0}{W}_1$,then $u_1\left(\frac{h_1}{2}, h_2(j-1)\right) = u_1\left(L_1 - \frac{h_1}{2}, h_2(j-1)\right) = 0$ for all $j = 1,\ N_2$ and $u_1\left(\frac{h_1}{2} + (i-1)h_1, 0\right) = u_1\left(\frac{h_1}{2} + (i-1)h_1 L_2\right) = 0,\ \ i = \overline{1 N_1 - 1}$.

Similarly, if $u_2 \in \overset{0}{W}_2$then $u_2\left(h_1(i-1), \frac{h_2}{2}\right) = u_2\left(h_1(i-1), L_2 - \frac{h_2}{2}\right) = 0$, $j = \overline{1, N_2}$, $u_2\left(0, \frac{h_2}{2} + (j-1)h_2\right)$, $u_2\left(L_1, \frac{h_2}{2} + (j-1)h_2\right) = 0$, $i = \overline{1, N_2 - 1}$.

$\overset{0}{W} = \overset{0}{W}_1 \otimes \overset{0}{W}_2$ is a direct product of the spaces $\overset{0}{W}_1$and $\overset{0}{W}_2$ that consists of vector-functions vanishing on the boundary. Q is a space of pseudoscalar functions determined on the grid R_3^h and vanishing outside of Ω_3^h.

On the spaces determined above we introduce the network operators ∇^h, $di\nu^h$, Δ^h that provide the second order of $O\left(h_1^2 + h_2^2\right)$ approximation interior to the domain and the first $O\left(h_i + h_{3-i}^2\right)$ on the boundary.

Introduce Rayleigh's generalized number notion that by definition equals $\Re^* = \Re\frac{2^{\frac{3}{2}}}{\mu_1}\widetilde{C}$ that by $\mu_1, \widetilde{C}$ considers initial data of the problem: geometry, initial and boundary conditions where $\Re = \frac{Gr}{\vartheta}$ is a Rayleigh number.

To differential boundary value problem (1)-(2) we assign the operator difference scheme:

$$\left\{ \begin{array}{l} \theta_t + di\nu^h\left(v\bar{\theta}\right) - \vartheta\Delta^h\theta = H, \\ \Delta_i^h v_i + \nabla_i^h p = Gr\bar{\theta}\gamma_i - N_i^h(v, \bar{v}), i = 1,2, \\ di\nu^h v = 0, \end{array} \right. \tag{3}$$

where under $\bar{\theta}$ we understand averaging by the nodes of the grid R_0^h on the, grid $R_i^h, i = 1,2,\ \ N_i^h(v, \bar{v})$ are summands approximating convective term.

Assume $N_i^h(v, \bar{v}) \equiv 0$, that corresponds to linearized motion equations. Consider the following finite-difference initial boundary value problem:

$$\begin{cases} -\theta_t - \vartheta\Delta^h\hat{\theta} + div\nu^h\left(v\ \bar{\bar{\theta}}\right) = H, \\ -\Delta_i^h\hat{v}_i + \nabla_i^h\hat{p} = Gr\hat{\bar{\theta}}\gamma_i, \\ div\nu^h\hat{v} = 0, \\ \theta^0 = \theta(x,0) \in P. \end{cases} \tag{4}$$

It holds

Theorem 1. *For any initial approximation $\theta_0 \in P$ and any right hand side $H \in P$ there will be found such $\bar{\tau}$ and M depending on initial data of the problem that for any $\tau < \bar{\tau}$ the inequality $\|\theta^n\|_1 \le M$ will be fulfilled.*

Proof. Given Ω^h, θ^h and M . Multiply the second and third equations by $\hat{v}_i$ and sum over all $\vec{x} \in \Omega_i^h$. Using Cauchy – Bunyakowskii inequality and grid analogy of Poincare-Steklov inequality with constant C_1 depending on the domain we get the inequalities

$$\|v^n\|_1 \le Gr\sqrt{C_1}\,\|\theta^n\| \tag{5}$$

and

$$\left\|\hat{\theta}\right\|^2 - \|\theta\|^2 + \tau^2\|\theta_t\|^2 + 2\tau\left\|\hat{\theta}\right\|_1^2 \le \frac{\tau}{\vartheta}\|H\|_{-1} + \tau\vartheta\left\|\hat{\theta}\right\|_1^2 + 2\tau^2\left|\left(div\nu^h v\bar{\theta}, \theta_t\right)\right| \tag{6}$$

It holds

Lemma. *Let $\varphi \in \overset{0}{P}$, then* $\max\limits_{x\in\Omega_0^h}|\varphi| \le \frac{C(\Omega)}{h}\|\varphi\|_1$.

Proof. Fix i and j. The followings are true:

a) $\varphi_{ij} = \sum\limits_{p=2}^{i} h_1\varphi_{\bar{x}_1}(p,j)$, b) $\varphi_{ij} = -\sum\limits_{r=N_1-i+1}^{N_1-1} h_1\varphi_{\bar{x}_1}(r,j)$,

c) $\varphi_{ij} = \sum\limits_{t=1}^{j} h_2\varphi_{\bar{x}_2}(i,t)$, d) $\varphi_{ij} = -\sum\limits_{s=N_2-j+1}^{N_2} h_2\varphi_{\bar{x}_2}(i,s)$.

Estimate each equality from above using Cauchy – Bunyakowskii inequality and then square the obtained inequality.

a) $\varphi_{ij}^2 \le \sum\limits_{p_1=1}^{i-1} h_1 \sum\limits_{p=1}^{i} h_1\left(\varphi_{\bar{x}_1}(p,j)\right)^2$, b) $\varphi_{ij}^2 \le \sum\limits_{r_1=N_1}^{N_1+1-i} h_1 \sum\limits_{r=N_1-i+1}^{N_1} h_1\left(\varphi_{\bar{x}_1}(r,j)\right)^2$,

c) $\varphi_{ij}^2 \le \sum\limits_{t_1=1}^{j-1} h_2 \sum\limits_{t=2}^{j} h_2\left(\varphi_{\bar{x}_2}(i,t)\right)^2$, d) $\varphi_{ij} \le \sum\limits_{s_2=N_2+1-j}^{N_2-1} h_2 \sum\limits_{s=N_2-j+1}^{N_2} h_2\left(\varphi_{\bar{x}_2}(i,s)\right)^2$.

Multiply the inequalities:

b) by $x = \sum\limits_{p=1}^{i-1} h_1$, a) by $L_1 - x$, d) by $y = \sum\limits_{t=1}^{j}(j-1)h_2$, c) by $L_2 - y$ and then put together a) and b); c) and d).

We get

A) $\varphi_{ij}^2 \le \dfrac{x(L_1-x)}{L_1}\sum\limits_{p=2}^{N_1} h_1\left(\varphi_{\bar{x}_1}(p,j)\right)^2$, B) $\varphi_{ij}^2 \le \dfrac{y(L_2-y)}{L_2}\sum\limits_{t=2}^{N_2} h_2\left(\varphi_{\bar{x}_2}(i,t)\right)^2$.

Notice that $\dfrac{x(L_1-x)}{L_1} \le \dfrac{L_1}{4}$, $\dfrac{y(L_2-y)}{L_2} \le \dfrac{L_1}{4}$.

Multiply and divide the right hand side of a) into h_2 and sum over all j. This fact only amplifies the inequality

$$\varphi_{ij}^2 \le \frac{L_1}{4h_2}\left\|\varphi_{\bar{x}_1}\right\|_{L_2}^2.$$

We do this with inequality b):

$$\varphi_{ij}^2 \leq \frac{L_2}{4h_1} \left\| \varphi_{\bar{x}_2} \right\|_{L_2}^2 .$$

Putting together the last two inequalities we get

$$\varphi_{ij} \leq \frac{1}{8} \frac{\max \{L_1, L_2\}}{\min \{h_1, h_2\}} \left\| \varphi \right\|_1^2$$

or

$$\max_{\Omega_0^h} \varphi_{ij} \leq \frac{1}{8} \frac{L}{h} \left\| \varphi \right\|_1^2 , \quad L = \max \{L_1, L_2\} , \quad h = \min \{h_1, h_2\} ,$$

or

$$\left\| \varphi \right\|_c \leq \frac{C}{\sqrt{h}} \left\| \varphi \right\|_1 , \tag{7}$$

where $C = \frac{1}{2} \sqrt{\frac{L}{2}}$. Q.E.D.

Introduce the denotation $\widetilde{C}(h) = \frac{C}{\sqrt{h}}$.

We use (7) to estimate the scalar product:

$$2\tau^2 \left| \left(div^h \left(v\bar{\theta} \right), \theta_t \right) \right| \leq 2\tau^2 \widetilde{C}(h) \, Gr \left\| \theta \right\| \sqrt{C_1} \left\| \theta \right\|_1 \left\| \theta_t \right\| ,$$

$$2\tau^2 \left| \left(div^h \left(v\bar{\theta} \right), \theta_t \right) \right| \leq 2\tau^2 \widetilde{C}(h) \, C_1 \, (Gr)^2 \left\| \theta \right\|_1 \left\| \theta_t \right\| + \tau^2 \left\| \theta_t \right\|^2 .$$

By means of the obtained inequality we estimate the right hand side of inequality (6) and get:

$$\left(1 + \frac{\tau \vartheta}{\sqrt{C_1}} \right) \left\| \hat{\theta} \right\| = \left\| \theta \right\|^2 + \frac{\tau}{\vartheta} \left\| H \right\|_{-1} + \tau^2 C_5 \left\| \theta \right\|_1^4 , \tag{8}$$

where $C_5 \widetilde{C}^2(h) \, C_1 \, (Gr)^2$. Denote $\alpha = \frac{\vartheta}{\sqrt{C_1}}, \left\| H \right\|_{-1}^2 = M_1$. Let on the n-th time layer the estimation $\left\| \theta^n \right\|_1^2 \leq \gamma M_1$ be true, then $\left\| \theta^n \right\|^2 \leq C_1 \gamma M_1$.

Show that there will be found $\gamma, \bar{\tau} > 0$ that depend on the problem statement such that $\forall \tau < \tau(h) : \left\| \theta^n \right\|_1 \leq \gamma M_1$.

Amplify inequality (8) by passing to a stronger norm

$$(1 + \alpha \tau) \left\| \hat{\theta} \right\|^2 \leq C_1 \left\| \theta \right\|_1^2 + \frac{\tau}{\vartheta} \left\| H \right\|_{-1} + \tau^2 C_5 \left\| \theta \right\|_1^4 ,$$

$$(1 + \alpha \tau) \left\| \hat{\theta} \right\|^2 \leq \gamma C_1 M_1 + \frac{\tau}{\vartheta} M_1 + \tau^2 C_5 \gamma^2 M_1^2 ,$$

$$\left\| \hat{\theta} \right\|^2 \leq \frac{\gamma C_1 M_1 + \frac{\tau}{\vartheta} + \tau^2 C_5 \gamma^2 M_1^2}{1 + \alpha \tau} . \tag{9}$$

Require the fulfillment of the stronger inequality:

$$C_1 \left\| \hat{\theta} \right\|^2 \leq \frac{\gamma C_1 M_1 + \frac{\tau}{\vartheta} M_1 + \tau^2 C_5 \gamma^2 M_1^2}{1 + \alpha \tau} \leq \gamma C_1 M_1 ,$$

then fulfillment of (9) will be provided.

Allowing for $\alpha = \vartheta C_1^{-1/2}$ we get:

$$\bar{\tau} = \frac{C_1 \vartheta}{4C_5 M_1}, \gamma \le 2,$$

$$M = \max \left\{ \frac{2}{\vartheta} \|H\|_{-1}^2 + \left\|\theta^0\right\|^2 + \frac{\bar{\tau}}{\vartheta} \|H\|_{-1}^2 + \bar{\tau}^2 C_5 \left\|\theta^0\right\|_1^4 \right\}.$$

This completes the proof of the lemma.

Now we give a theorem on the convergence of iterative process.

Theorem 2. *By fulfilling conditions of theorem 1 and inequalities* $\Re^* = (\Omega, H, \Re_\alpha) < 1$ *where* $\Re^*$ *is Rayleigh generalized number, iterative process converges with geometrical progression velocity to the solution of the stationary problem.*

Proof. Let θ^* and corresponding v^* be solutions of the stationary problem. Then error $X = \theta - \theta^*$ and corresponding vector-function $y = v - v^*$, $q = p - p^*$ satisfy the system of operator-difference equations:

$$\begin{cases} \frac{\partial X}{\partial t} + div^h \left(v\overline{X}\right) + div^h \left(y\overline{\theta}^*\right) - \vartheta \Delta^h \hat{X} = 0, \\ -\Delta^h \hat{y} + \nabla^h \hat{q} = Gr\overline{X}\overrightarrow{\gamma}, \\ div^h v = 0, \\ div^h y = 0, \\ X^0 = \theta(x,0) - \theta^*(x,0). \end{cases} \tag{10}$$

The inequality

$$\left\|\hat{X}\right\|^2 - \|X\|^2 + \tau^2 \|X_t\|_1^2 \le 2\tau^2 \left|\left(div^h \left(v\overline{X}\right), X_t\right)\right| + 2\tau \left(div^h y\overline{\theta}^*, \hat{X}\right)\Big|. \tag{11}$$

is true.

Show that $\|y\|_c \le K \|X\|$. Use the inequality:

$$\|y\|_c \le C_\Delta \left\|\Delta^h y\right\|_{L_2}. \tag{12}$$

To estimate $\left\|\Delta^h y\right\|_{L_2}$ we scalarly multiply the second and third equations of (10) by $\Delta^h \hat{y}$,

$$\left\|\Delta^h \hat{y}\right\|^2 Gr \left(\hat{X}\overrightarrow{\gamma}, \Delta^h \hat{y}\right)_W + \left(\nabla^h \hat{q}, \Delta^h \hat{y}\right)_{in}. \tag{13}$$

It is true the identity

$$\left(\nabla^h q, \Delta^h \hat{y}\right) = \sum_{x \in \partial\Omega_0^h} qh \left(\Delta^h \hat{y}, \overrightarrow{n}(\partial\Omega)\right) - \left(q, div^h \Delta^h \hat{y}\right)_{in}, \tag{14}$$

where

$$\left(q, div^h \Delta^h y\right)_{in} = \sum_{\overrightarrow{x} \in \Omega_0^h \backslash \partial\Omega_0^h} h_1 h_2 q(\overrightarrow{x}) \cdot div^h \Delta^h \hat{y}.$$

Show that $div_{in}^h \Delta^h y = 0$. By definition of the operator -Δ^h in the nodes $\Omega_i^h \backslash \partial\Omega_i^h$:

$$\left(div^h \Delta^h y\right)_{in}\Big|_{\overrightarrow{x} \in \Omega_0^h \backslash \Gamma'} = \left(\Delta^h div^h y\right)_{in}\Big|_{\overrightarrow{x} \in \Omega_0^h \backslash \Gamma'} \equiv 0,$$

if $\overrightarrow{x}$ doesn't belong to boundary nodes.

Under boundary nodes we understand the set:

$$\Gamma' = \{\overrightarrow{x} \,|\, x_1 = L_1 - h_1, x_2 = (j_1 - 1)\, h_2\} \cup \{\overrightarrow{x} \,|\, x_1 = (i_2 - 1)\, h_1, x_2 = L_2 - h_2\} \cup$$

$$\cup \{\overrightarrow{x} \,|\, x_1 = h_1, x_2 = (j_3 - 1)\, h_2\} \cup \{\overrightarrow{x} \,|\, x_1 = (i_4 - 1)\, h_1, x_2 = h_2\},$$

$$1 < i_k < N_1, \quad 1 < j_k < N_2.$$

However, by virtue of $y \in \overset{0}{W}$, and definition -Δ^h

$$\left(div^h \Delta^h y\right)_{in}\Big|_{\overrightarrow{x} \in \Gamma'} = \left(\Delta^h div^h y\right)_{in}\Big|_{\overrightarrow{x} \in \Gamma'} = 0.$$

It follows from (14)

$$\left|\left(\nabla^h q, \Delta^h \hat{y}\right)\right| = \sum_{\overrightarrow{x} \in \partial\Omega_0^h} q\left(\overrightarrow{x}\right)\left(\Delta^h \hat{y}, \overrightarrow{n}\right) \cdot h_\tau \le \|q\|_{L_2\left(\partial\Omega_0^h\right)} \cdot \left\|\left(\Delta^h \hat{y}_{in}\right)\right\|_{L_2\left(\partial\Omega_0^h\right)}$$

or

$$\left|\left(\nabla^h q, \Delta^h \hat{y}\right)\right| \le \|q\| \cdot \left\|\Delta^h \hat{y}\right\|. \tag{15}$$

As is known

$$\|q\| \le 2C_0 Gr \cdot C_1 \|X\|.$$

Using (15), from (13) we get

$$\left\|\Delta^h y\right\| \le (1 + 2C_1 C_0)\, Gr \|X\|. \tag{16}$$

Substituting (16) into (12) we get

$$\|\hat{y}\|_c \le C_{10} \left\|\hat{X}\right\|, \; C_{10} = C_\Delta (1 + 2C_0 C_1)\, Gr. \tag{17}$$

The following estimation follows from the previous theorem

$$\max_\Omega \|\theta^n\| \le K_1, \quad \max_\Omega \|v\| \le K_2, K_{1,2} > 0, \tag{18}$$

where $K_{1,2}$ is determined by the problem data.

Taking (17) and (18) into account we have the estimations:

$$2\tau^2 \left|\left(div^h \left(v\overline{X}\right), X_t\right)\right| \le \tau^2 K_1^2 \|X\|_1^2 + \|X_t\|, \tag{19}$$

$$2\tau \left|\left(div^h y \theta^* \hat{X}\right)\right| \le \frac{\tau}{\vartheta} C_{10}^2 \|X\|^2 + \tau\vartheta \left\|\hat{X}\right\|_1^2. \tag{20}$$

Substitute the obtained inequalities into (11):

$$\left\|\hat{X}\right\|^2 + \tau\vartheta \left\|\hat{X}\right\|_1^2 \le \|X\|^2 + \tau C_{11} \|X\|_1^2 + \tau^2 K_1 \|X\|_1^2,$$

where

$$C_{11} = \frac{C_{10}^2 C_1}{\vartheta} = \frac{C_1 C_\Delta^2 (Gr)^2 (1 + 2C_0 C_1)^2}{\vartheta}$$

If $C_{11} < \frac{1}{\vartheta}$, the iterative process coverges with geometrical progression velocity Rayleigh generalized number in this case will be: $\Re^* = C_{12}\Re_\alpha < 1$, $C_{12} = C_\Delta\left(1+2C_0C_1\right)C_1^{\frac{1}{2}}$.

Show that the solution of non-stationary linearized operator-difference problem (3) differs from the corresponding differential boundary value problem by a quality $O\left(\tau+h^2\right)$. It holds.

Theorem 3. *By fulfilling conditions of theorem 2 the distance between the solution of the operator-difference problem and corresponding differential nonstationary linearized problem is of order* $O\left(\tau+h^2\right)$ *i.e.*

$$\left\|\theta_{h\tau}-[\theta]_{h\tau}\right\|_{1,t}+\left\|v_{h\tau}-[v]_{h\tau}\right\|_1+\left\|p_{h\tau}-[p]_{h\tau}\right\|=O\left(\tau+h^2\right).$$

Proof. Make- up an equation for errors:

$$X=\theta_{h\tau}-[\theta]_{h\tau}\,,\quad y=v_{h\tau}-[v]_{h\tau}\,,\quad z=p_{h\tau}-[p]_{h\tau}\,.$$

Let ξ_i be errors of approximation

$$\xi_i=\begin{cases} O\left(h^2\right), & x\in\Omega^h_{i(\mathrm{mod}\,3)},\\ O\left(h\right), & x\in\partial\Omega^h_{i(\mathrm{mod}\,3)}, i=\overline{1,3}\end{cases}$$

$$\xi_0=\begin{cases} O\left(\tau+h^2\right), & x\in\Omega^h_0,\\ O\left(\tau+h^2\right), & x\in\Omega^h_0.\end{cases}$$

$$\begin{cases} X_t-\vartheta\Delta^h\hat{X}+div\nu^h\left(v_h\overline{X}\right)+div\nu^h\left(y\left[\overline{\theta}\right]_{h\tau}\right)=\xi_0,\\ -\Delta^h_i y_i+\nabla^h_i z=GrX\gamma_i+\xi_i,\\ div\nu^h y=\xi_3,\\ div\nu^h\mathbf{v}_h=0,\\ X^0=0.\end{cases}\tag{21}$$

The following estimation is true for y.

$$\|y\|_1\le\left(C_0+1\right)Gr\,\|X\|_{-1}+O\left(h^2\right),\tag{22}$$

for z

$$\|z\|\le\left(2C_0+1\right)Gr\,\|X\|_{-1}+O\left(h^2\right),\tag{23}$$

Multiply the first equation of (21) by $2\tau\hat{X}$

$$\left\|\hat{X}\right\|^2-\left\|\hat{X}\right\|^2+\tau^2\left\|X_t\right\|^2+2\tau\vartheta\left\|\hat{X}\right\|_1^2\le2\tau\left\|\xi_0\right\|\cdot\left\|\ddot{X}\right\|+2\tau\left|\left(div\nu^h y\,[\theta]\,,\hat{X}\right)\right|\tag{24}$$

Allowing for (20) we have:

$$2\tau\left|\left(div\nu^h y\,[\theta]\,\hat{X}\right)\right|\le\frac{2\tau}{\vartheta}\widetilde{C}^2\left(Gr\right)^2MC_1\left\|\hat{X}\right\|^2+\frac{\tau\vartheta}{2}\left\|\hat{X}\right\|_1^2\,.$$

Amplify (22) by adding deliberately non-positive quantity $-\tau^2\left\|X_t\right\|$ to its left hand side. We get

$$\left\|\hat{X}\right\|^2-\|X\|^2+\frac{\tau\vartheta}{2}\left\|\hat{X}\right\|_1^2\le4\tau\left\|\xi_0^{in}\right\|^2+\tau C_7\left\|\hat{X}\right\|_1^2,\ C_7=\frac{C_1\widetilde{C}^2\left(Gr\right)^2M_1}{\vartheta}.$$

Sum this inequality in time grid from 0 to $T-\tau$;

$$\left\|X^N\right\|^2+\frac{\vartheta}{2}\left\|X\right\|_{1,t}^2\leq O\left(\tau+h^2\right)+C_7\left\|X\right\|_{1,t}^2.$$

It is seen from the obtained inequality that if $2C_7=\nu$, then $\left\|X\right\|_{1,t}=O\left(\tau+h^2\right)$.

It follows from (22) and (23) that

$$\left\|y\right\|_1=O\left(\tau+h^2\right),$$

$$\left\|z\right\|=O\left(\tau+h^2\right).$$

The theorem is proved.

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