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ON THE EXISTENCE OF AN ABSORBING SET FOR SEMI-LINEAR PSEUDOHYPERBOLIC EQUATIONS OF HIGHER ORDER

Abstract

In the paper we consider a mixed problem for semi-linear pseudohyperbolic equations of fourth order with Rinke boundary conditions. For this problem we prove the existence of the absorbing set.

Let $\Omega \subset R_n$ be a bounded domain with smooth boundary Γ. In the domain $Q = [0, \infty) \times \Omega$ we consider a mixed problem for a semi-linear pseudohyperbolic equation

$$u_{tt} - \Delta u_{tt} + \Delta^2 u - \Delta u + u_t + |u|^p u + f(u) = g(x), \ t > 0, \ x \in R_n \tag{1}$$

with boundary conditions

$$u(t, x) = 0, \ \Delta u(t, x) = 0, \ t > 0, \ x \in \Gamma \tag{2}$$

and initial conditions

$$u(0, x) = u_0(x), \ u_t(0, x) = u_1(x), \ x \in \Omega, \tag{3}$$

where the number p and the function $f(u)$ and $g(x)$ satisfy the following conditions

1^0. $1 \le p < \infty$ and for $n > 4$, $p \le \dfrac{n+4}{n-4}$

2^0. $f(u)$ is a differentiable function in R and for all $u \in R$ the estimations

$$|f(u)| \le c_1 (1 + |u|^\rho), \ \rho < p. \ uf(u) \ge c_2 \int_0^u f(s)\, ds - c_3 u^2 - c_4,$$

where $c_i > 0$, $i = 1, 2, 3, 4$, are fulfilled.

3^0. $g(\cdot) \in W_2^{-1}(\Omega)$, where $W_2^{-1}(\Omega) = \left(\overset{\circ}{W}{}_2^{-1}(\Omega)\right)'$.

By $\hat{W}_2^r(\Omega)$ we denote a sub.space of Sobolev space $W_2^r(\Omega)$

$$\hat{W}_2^r(\Omega) = \left\{ u : u \in W_2^r(\Omega), \ \Delta^i u\left(\dot{x}\right) = 0, \ i = 0, 1, ..., (r/2), \ x \in \Gamma \right\},$$

where $\left(\dfrac{r}{2}\right) = \begin{cases} k & \text{for} \quad r = 2k+1 \\ k-1 & \text{for} \quad r = 2k \end{cases}$

Introduce the space $H = \hat{W}_2^2(\Omega) \times \hat{W}_2^1(\Omega)$ with scalar product

$$\langle w^1, w^2 \rangle = \int_\Omega \Delta u^1 \cdot \Delta u^2 dx + \int_\Omega \nabla v^1 \cdot \nabla v^2 dx, \tag{4}$$

where $w^i = \begin{pmatrix} u^i \\ v^i \end{pmatrix}$, $i = 1, 2$. We similarly introduce $H_0 = \hat{W}_2^3(\Omega) \times \hat{W}_2^2(\Omega)$.

Following [1] , by substitution $v_1 = u,\ v_2 = u_t$ we can reduce problem $(1)-(3)$ to the Cauchy problem

$$w' = Aw + F(w) \tag{5}$$

$$w(0) = w_0 \tag{6}$$

in the Hilbert space H , where

$$F(w) = \begin{pmatrix} 0 \\ G\left(|u|^{p-1}u - f(u) + g(x)\right) \end{pmatrix}, G = (I-\Delta)^{-1},$$

$$D(A) = H_0, \quad A = \begin{pmatrix} D & I \\ -\Delta^2 G & \Delta G - G \end{pmatrix}.$$

It is known that $G : \hat{W}_2^s(\Omega) \longrightarrow \hat{W}_2^{s+2}(\Omega),\ \ s \geq 0$ realizes isomorphism (see [2],[3]).

It is proved that a linear operator A generates a strongly continuous semi-group in H and $F(w)$ satisfies the local Lipschitz condition, i.e.

$$\left\|F\left(w^1\right) - F\left(w^2\right)\right\| \leq c\left(\left\|w^1\right\|, \left\|w^2\right\|\right)\left\|w^1 - w^2\right\|,$$

where $c(\cdot,\cdot) \in C(R_2)$. (see [1]). So for the problem $(5)-(6)$, all conditions theorem of the local solvability ([4]) are fulfilled therefore for any $w_0 \in H$ problem $(5)-(6)$ has unique solution $w(\cdot) \in C([0,\infty), H)$. If $w_0 \in H_0$ then $w(\cdot) \in C([0,\infty), H) \cap C^1([0,\infty), H)$. Thus, there exists a nonlinear semi-group $W(t)$, where $w(t) = W(t)w_0$. Problem $(5)-(6)$ is equivalent to the integral equation

$$w(t) = U(t)w_0 + \int_0^t U(t-\tau)F(w(\tau))\,d\tau. \tag{7}$$

By $\mathfrak{B}(H)$ we denote a totality of bounded sets in H.

The set B_0 is said to be an absorbing set for a semi-group $W(t)$ if for any $B \subset \mathfrak{B}(H)$ there exists $t_B > 0$ such that $W(t)B \subset B_0,\ \ t \geq t_B$.

In the paper we obtained the following result

Theorem 1. *Let conditions $1^0 - 3^0$ be fulfilled. Then the semi-group $W(t)$ has an absorbing set $B_0 \subset \mathfrak{B}(H)$*

We first prove the following theorem.

Theorem 2. *$U(t)$ is an exponentially decreasing semi-group, i.e. there exists $M \geq 1$ and $w > 0$, such that*

$$\|U(t)\|_{L(H,H)} \leq Me^{-wt}, \quad t > 0.$$

Proof of theorem 2. let's consider a linear pseudohyperbolic operator

$$L(u) = u_{tt} - \Delta u_{tt} + \Delta^2 u - \Delta u_t + u_t$$

with boundary conditions (2).

Multiply $L(u)$ by $u_t + \eta u$ and integrate over the domain Ω. Then, after integration by parts we'll get

$$\langle L(u), u_t + \eta u\rangle = \frac{d}{dt}\left[\frac{1}{2}\|u_t(t,\cdot)\|^2 + \frac{1}{2}\|\nabla u_t(t,\cdot)\|^2 + \frac{1}{2}\|\Delta u_t(t,\cdot)\|^2 + \right.$$

$$\left.\eta\langle u_t(t,\cdot), u(t,\cdot)\rangle + \eta\langle\nabla u_t(t,\cdot), \nabla u(t,\cdot)\rangle + \frac{\eta}{2}\|\nabla u\|^2 + \frac{\eta}{2}\|u\|^2\right] +$$

$$+\left[(1-\eta)\left\|\nabla u_t(t,\cdot)\right\|^2+(1-\eta)\left\|u_t(t,\cdot)\right\|^2+\eta\left\|\Delta u(t,\cdot)\right\|^2\right]. \qquad (8)$$

Equality (8) is valid for any function

$$u\in\tilde{C}^2=C^2\left([0,\infty),\hat{W}_2^1\right)\cap C^1\left([0,\infty),\hat{W}_2^2\right)\cap C\left([0,\infty),\hat{W}_2^3\right).$$

Denoting

$$E(t)=\frac{1}{2}\left\|u_t(t,\cdot)\right\|^2+\frac{1}{2}\left\|\nabla u_t(t,\cdot)\right\|^2+\frac{1}{2}\left\|\Delta u(t,\cdot)\right\|^2+\eta\left\langle u_t(t,\cdot),u(t,\cdot)\right\rangle+$$

$$+\eta\left\langle\nabla u_t(t,\cdot),\nabla u(t,\cdot)\right\rangle+\frac{\eta}{2}\left\|\nabla u\right\|^2+\frac{\eta}{2}\left\|u\right\|^2,$$

we get from (8)

$$\frac{d}{dt}E(t)+\left[(1-\eta)\left\|\nabla u_t(t,\cdot)\right\|^2+(1-\eta)\left\|u_t(t,\cdot)\right\|^2+\right.$$

$$\left.+\eta\left\|\Delta u(t,\cdot)\right\|^2\right]=\left\langle L(u)(t,\cdot),u_t(t,\cdot)+\eta u(t,\cdot)\right\rangle. \qquad (9)$$

Then, using the Hölder inequality we get

$$\left|\left\langle u_t(t,\cdot),u(t,\cdot)\right\rangle\right|\le\tfrac{1}{2}\left\|u_t(t,\cdot)\right\|^2+\tfrac{1}{2}\left\|u(t,\cdot)\right\|^2.$$

$$\left|\left\langle\nabla u_t(t,\cdot),\nabla u(t,\cdot)\right\rangle\right|\le\tfrac{1}{2}\left\|\nabla u_t(t,\cdot)\right\|^2+\tfrac{1}{2}\left\|\nabla u(t,\cdot)\right\|^2. \qquad (10)$$

In view of the known inequalities (see [2], [3]).

$$\|v\|\le\beta_0\|\Delta v\|,\ \|\Delta v\|\le\beta_1\|v\|_{\hat{W}_2^2(\Omega)},\ \|w\|\le\beta_2\|\Delta w\|,$$

$$v\in\hat{W}_2^2(\Omega),\ w\in\hat{W}_2^1(\Omega) \qquad (11)$$

we get from (9), (10) and (11) that

$$\frac{d}{dt}E(t)+\omega E(t)\le\left\langle Lu(t,\cdot),u_t(t,\cdot)+\eta u(t,\cdot)\right\rangle+$$

$$+\left[\frac{\omega}{2}(1+\eta+\eta\beta_1)-1+\eta\right]\left\|u_t(t,\cdot)\right\|^2+$$

$$+\left[\frac{\omega}{2}(1+\eta)-1+\eta\right]\left\|\nabla u_t\right\|^2+\left[\frac{\omega}{2}(1+\eta\beta_1+\eta\beta_0)-\eta\right]\left\|\Delta u(t,\cdot)\right\|^2.$$

Later we'll choose η and ω in the following way :

$$0<\eta<1,\quad\omega=\min\left\{\frac{2(1-\eta)}{1+\eta+\eta\beta_1},\frac{2(1-\eta)}{1+\eta},\frac{2\eta}{1+\eta\beta_1+\eta\beta_0}\right\}.$$

It follows from the last inequality that for any function $u\in\tilde{C}^2$ it is valid the inequality

$$\frac{d}{dt}E(t)+\omega E(t)\le\left\langle Lu(t,\cdot),u_t(t,\cdot)+\eta u(t,\cdot)\right\rangle. \qquad (12)$$

Let $u(t,x)$ be a solution of the equation

$$L(u)=0 \qquad (13)$$

with boundary conditions (2) and initial conditions (3) where $u_0 \in \hat{W}_2^3$,

$u_0 \in \hat{W}_2^2$It follows from [4] that $u \in \tilde{C}^2$. Therefore, on the basis of $(12)-(13)$ we have the inequality

$$\frac{d}{dt}E(t)+\omega E(t)\leq 0.$$

Hence we get

$$E(t)\leq E(0)\,e^{-\omega t}, \tag{14}$$

where

$$E(0)=\frac{1}{2}\|u_1(\cdot)\|^2+\frac{1}{2}\|\nabla u_1(\cdot)\|^2+\frac{1}{2}\|\Delta u_0(\cdot)\|^2+$$
$$+\eta\langle u_1(\cdot),u_0(\cdot)\rangle+\eta\langle\nabla u_1(\cdot),\nabla u_0(\cdot)\rangle.$$

On the other hand, in view of (4)

$$\|w(t)\|^2=\|\nabla u_t(t,\cdot)\|^2+\|\Delta u(t,\cdot)\|^2$$

Therefore, using (11) we can see that

$$c_1^{-1}\|w(t)\|^2\leq E(t)\leq c_1\|w(t)\|^2, \tag{15}$$

where $c_1\geq 1$ is independent of $w(\cdot)$ and t.

It follows from (14) and (15) that

$$\|w(t)\|\leq Me^{-\omega t}\|w_0\|,\quad t>0,\quad w_0\in H_0,$$

where $M=c_1^2$. Since $w(t)=U(t)\,w_0$, then

$$\|U(t)\,w_0\|\leq Me^{-\omega t}\|w_0\|,\ t>0,\quad w_0\in D(A)=H_0. \tag{16}$$

In view of of boundedness of $U(t)$, for any $t\in[0,\infty)$, inequality (16) is valid for all $w_0\in H$, therefore

$$\|U(t)\|_{L(H,H)}\leq Me^{-\omega t},\ t>0.$$

Proof of theorem 1.

Let $u_0\in\hat{W}_2^3$ and $u_1\in\hat{W}_2^2$ and $u(t,x)$ be a solution of problem $(1)-(3)$, then $u(t,x)\in\tilde{C}^2$, and on the basis of (12) we get

$$\frac{d}{dt}E(t)+\omega E(t)\leq\langle g(x)-|u|^p u-f(u),u_t-\eta u\rangle. \tag{17}$$

Using Hölder and Young inequality and inequality (10) we get the inequalities

$$|\langle g(\cdot),u_t(t,\cdot)\rangle|\leq\frac{1}{\varepsilon}\|g\|_{W_2^{-1}(\Omega)}+\varepsilon\|u_t(t)\|_{\hat{W}_2^1(\Omega)} \tag{18}$$

$$|\langle g(\cdot),u(t,\cdot)\rangle\leq|\frac{1}{\varepsilon}\|g\|^2_{W_2^{-1}(\Omega)}+\varepsilon\|u(t)\|^2_{\hat{W}_2^1(\Omega)}\leq\frac{1}{\varepsilon}\|g\|^2_{W_2^{-1}(\Omega)}+$$
$$+\frac{1}{\varepsilon}\|g\|^2_{W_2^{-1}(\Omega)}+\varepsilon\beta_1\|u_t(t,\cdot)\|^2_{\hat{W}_2^1(\Omega)}. \tag{19}$$

On the other hand

$$\int_\Omega|u|^p u\cdot u_t dx=\frac{1}{p+1}\frac{d}{dt}\int_\Omega|u|^{p+1}dx,\quad\int f(u)\,u_t=\frac{d}{dt}\int F(u)\,dx, \tag{20}$$

where $F(u)=\int\limits_0^u f(s)\,ds$. By condition 3^0 we have :

$$-\eta\int\limits_\Omega uf(u)\le-\eta c_2\int\limits_\Omega F(u)\,dx+\eta c_3\int\limits_\Omega u^2+c_4\eta\le$$

$$\le-\eta c_2\int\limits_\Omega F(u)\,dx+\eta c_3\beta_0\|\Delta u\|^2_{\hat W_2^2}+c_4\eta. \tag{21}$$

Allowing for inequalities (18) – (21) in inequality (17) we get

$$\frac{d}{dt}E(t)+\omega E(t)\le\varepsilon\|u_t\|^2_{\hat W_2^1}+\eta\varepsilon\beta_3\|u\|_{\hat W_2^2}-\frac{1}{p+1}\frac{d}{dt}\int\limits_\Omega|u|^{p+1}dx-\eta\int\limits_\Omega|u|^{p+1}dx-$$

$$-\frac{d}{dt}\int F(u)\,dx-\eta c_2\int\limits_\Omega F(u)\,dx+\eta c_3\beta_0\|u\|^2_{\hat W_2^2}+c_4\eta+\frac{1+\eta}{\varepsilon}\|g\|^2_{W_2^{-1}(\Omega)}. \tag{22}$$

Denoting

$$\tilde E(t)=E(t)+\frac{1}{p+1}\int\limits_\Omega|u|^{p+1}dx+\int\limits_\Omega F(u)\,dx$$

and choosing η sufficiently small from (22) we get

$$\frac{d\tilde E}{dt}+\omega_1\tilde E\le\frac{1+\eta}{\varepsilon}\|g\|_{W_2^{-1}}+c_4\eta,$$

where $\omega_1=\min\{\omega-2\varepsilon,\omega-2\eta(\varepsilon\beta_1+c_3\beta_0),\ 1,\ (p+1)\eta,\ \eta c_2\}$

Thus

$$\tilde E(t)\le\tilde E(0)e^{-\omega_1 t}-\left[\frac{1+\eta}{\varepsilon\omega_1}\|g\|^2_{W_2^{-1}}+\frac{c_4\eta}{\omega_1}\right]e^{-\omega_1 t}+\left[\frac{1+\eta}{\varepsilon\omega_1}\|g\|^2_{W_2^{-1}}+\frac{c_4\eta}{\omega_1}\right]. \tag{23}$$

It follows from (15) and (23) that

$$\|w(t)\|^2\le c_1\left[c_1\|w(0)\|^2+\frac{1}{p+1}\int\limits_\Omega|u_0(x)|^{p+1}dx+\int\limits_\Omega F(u_0(x))\,dx\right]e^{-\omega_1 t}+$$

$$+\left[\frac{1+\eta}{\varepsilon\omega_1}\|g\|^2_{W_2^{-1}}+\frac{c_4\eta}{\omega_1}\right]-\left[\frac{1+\eta}{\varepsilon\omega_1}\|g\|^2_{W_2^{-1}}+\frac{c_4\eta}{\omega_1}\right]e^{-\omega_1 t}. \tag{24}$$

Using the embedding theorem (see [5]) we have:

$$\|u_0\|_{L_{p+1(\Omega)}}\le\beta_3\|u_0\|_{\hat W_2^2(\Omega)}. \tag{25}$$

Further, from condition 2^0 it follows the inequality

$$\int\limits_\Omega F(u_0(x))\,dx\le c\int\limits_\Omega|u_0(x)|\left(1+|u_0(x)|^p\right)dx\le c\cdot mes\Omega+$$

$$+c\|u_0\|^2_{L_{2(\Omega)}}+c\|u_0\|^{p+1}_{L_{p+1}}\le c\cdot mes\Omega+c\beta_0^2\|u_0\|^2_{\hat W_2^2}+c\beta_3^{p+1}\|u_0\|^{p+1}_{\hat W_2(\Omega)}. \tag{26}$$

It follows from (24) – (26) that

$$\|w(t)\|^2 \le c_1 \left[c_1 \|w(0)\|^2 + \frac{\beta_3^{p+1}}{p+1} \|u_0\|_{\hat{W}_2^2}^{p+1} + c\beta_0^2 \|u_0\|_{\hat{W}_2^2}^2 + c\beta_3^{p+1} \|u_0\|_{\hat{W}_2^2(\Omega)}^{p+1} \right] e^{-\omega_1 t} +$$

$$+ c_1 \left[\frac{1+\eta}{\varepsilon\omega_1} \|g\|_{W_2^{-1}}^2 + \frac{c_4 \eta}{\omega_1} \right] - \left[\frac{2\eta}{\varepsilon\omega_1} \|g\|_{W_2^{-1}}^2 + \frac{c_4\eta}{\omega_1} + c \cdot mes\Omega \right] e^{-\omega_1 t}. \qquad (27)$$

Inequality (27) is valid for all $w_0 \in H_0$. On the other hand $\bar{H}_0 = H$ and the solution of problem (5) – (6) continuously depends on w_0 in the space H (see [6]), therefore , we can easily see that inequality (25) is valid for any $w_0 \in H$.

Let $B_0 = \{w : w \in H, \|w\| \le r_0\}$ where

$$r_0 = \sqrt{1 + \frac{c_1(1+\eta)}{\varepsilon\omega_1} \|g\|_{W_2^{-1}(\Omega)}^2 + \frac{c_4\eta}{\omega_1}}.$$

If $\|w_0\| \le r$, it follows from (25) that

$$\|w(t)\|^2 \le c_1 \left[c_1 r^2 + \frac{\beta_3^{p+1}}{p+1} r^{p+1} + c\beta_0^2 r^2 + \frac{c}{\varepsilon}\beta_0^2 r^2 + c\beta_3^{p+1} r^{p+1} \right] e^{-\omega_1 t} +$$

$$+ \left[\frac{c_1(1+\eta)}{\varepsilon\omega_1} \|g\|_{W_2^{-1}} + \frac{c_4\eta}{\omega_1} \right].$$

Obviously, if $t \ge t_r = \frac{1}{\omega_1} \ln c_1 \left[c_1 r^2 + \frac{\beta_3^{p+1}}{p+1} r^{p+1} + c\beta_0^2 r^2 + \frac{c}{\varepsilon}\beta_0^2 r^2 + c\beta_3^{p+1} r^{p+1} \right]$ then $\|w(t)\| \le r_0$ for $t \ge t_r$.

Thus B_0 is an absorbing set for the semi-group $W(t)$.

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