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ON ASYMPTOTICS OF SOLUTION OF THE BOUNDARY VALUE PROBLEM FOR SINGULARLY PERTURBED NONLINEAR PARABOLIC EQUATION WITH CORNER PARABOLIC BOUNDARY LAYER

Abstract

In corner point domain we consider the boundary value problem for a second order nonlinear parabolic equation containing a small parameter at higher derivatives. Asymptotic expansion of generalized solution of the considered problem is constucted to within any power of small parameter with corner parabolic boundary layer, and residual term is estimated.

In the rectangle $D=\{0\le t\le T,\ 0\le x\le 1\}$ we consider the following boundary value problem:

$$L_\varepsilon U\equiv \frac{\partial U}{\partial t}-\varepsilon^p\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}\right)^p-\varepsilon\frac{\partial^2 U}{\partial x^2}+a\frac{\partial U}{\partial x}+F\left(t,x,U\right)=0, \tag{1}$$

$$U|_{t=0}=0, \qquad (0\le x\le 1) \tag{2}$$

$$U|_{x=0}=U|_{x=1}=0, \qquad (0\le x\le 1) \tag{3}$$

where $\varepsilon>0$ is a small parameter, $a>0$ is a constant, $p=2k+1$, k is arbitrary natural number, $F\left(t,x,U\right)$ is the given smooth function satisfying the condition

$$\frac{\partial F\left(t,x,U\right)}{\partial U}\ge\gamma>0, \qquad (t,x,U)\in D\times(-\infty,+\infty). \tag{4}$$

Here the function $F\left(t,x,U\right)$ may depend on U both linearly, i.e. $F\left(t,x,U\right)=b\left(t,x\right)U-f\left(t,x\right)$, $b\left(t,x\right)\ge\gamma'>0$, and nonlinearly.

It is known that at each fixed ε there exists a unique generalization of the solution of problem (1)-(3) to the class $L_{p+1}\left[0,T,\overset{0}{W}_{p+1}\left(0,1\right)\right]$ (see [3]). Obviously, if $F\left(t,x,0\right)\equiv 0$, problem (1)-(3) has only trivial solution.

Therefore, assume that

$$F\left(t,x,0\right)\not\equiv 0. \tag{5}$$

In the paper [1] the first terms of the asymptotics of the solution of problem (1)-(3) are constructed. Construction of consequent terms of the asymptotics of the solution of this problem was hampered with existence of nonlineear terms in equation (1) singularities that were met for $x=at$ for the solution of degenerate problem and in availability of corner points of the considered domain. Generalizing the results of [1] complete asymptotics of the solution of problem (1)-(3) was constructed in the paper [2]. By constructing complete asymptotics of the solution (1)-(3) we had to impose necessary conditons on the function $F\left(t,x,U\right)$ at the line $x=at$ and corner point $t=0$, $x=1$. In the present paper we construct complete asymptotics of solution of problem (1)-(3) withy corner boundary layer and reject fron conditions imposed on $F(t,x,U)$ at corner point $t=0$, $x=1$ in [2].

Before we construct asymptotics of the solution (1)-(3) we formulate several affirmations that we will need in future.

Lemma 1. *Let* $F(t,x,U)\in C^k(D\times(-\infty,+\infty))$ and satisfy the condition

$$\left.\frac{\partial^i f(t,x)}{\partial t^{i_1}\partial x^{i_2}}\right|_{x=at}=0;\quad i=i_1+i_2;\quad i=0,1,...,k,\quad (0\le t\le T), \tag{6}$$

in case of linear dependence of F from U, the condition

$$\left.\frac{\partial^i F(t,x,0)}{\partial t^{i_1}\partial x^{i_2}\partial U^{i_3}}\right|_{x=at}=0;\quad i=i_1+i_2+i_3;\quad i=0,1,...,k,\quad (0\le t\le T), \tag{7}$$

in case of nonlinear dependence of F from U. Then the problem

$$AW\equiv\frac{\partial W}{\partial t}+a\frac{\partial W}{\partial x}+F(t,x,W)=0,\qquad W|_{t=0}=0,\qquad W|_{x=0}=0 \tag{8}$$

has a unique solution, moreover, $W(t,x)\in C^k(D)$ and

$$\left.\frac{\partial^i W(t,x)}{\partial t^{i_1}\partial x^{i_2}}\right|_{x=at}=0;\quad i=i_1+i_2;\quad i=0,1,...,k,\quad (0\le t\le T). \tag{9}$$

Lemma 2. *Let $\varphi(t)\in C^k[0,T]$; $a=const>0$. Then for each fixed value of $t\in[0,T]$ the problem*

$$BV\equiv\frac{\partial}{\partial}\left(\frac{\partial V}{\partial\tau}\right)^p+\frac{\partial^2 V}{\partial\tau^2}+a\frac{\partial V}{\partial\tau}=0, \tag{10}$$

$$V|_{\tau=0}=\varphi(t),\qquad \lim_{\tau\to+\infty}V=0 \tag{11}$$

has a unique solution, moreover $V(t,\tau)$ with respect to τ is infinitely differentiable, a with respect to t has continuous derivatives to the k-th order inclusively. Here the estimates of the form

$$\left|\frac{\partial^i V(t,\tau)}{\partial t^{i_1}\partial\tau^{i_2}}\right|\le ce^{-a\tau};\quad i=i_1+i_2;\quad i_1=0,1,2,...,k \tag{12}$$

are valid uniformly with respect to $t\in[0,T]$.

Lemma 3. *Let $\psi(y)\in C^k[0,T]$; $a=const>0$, $h(t,\tau)$ be the known function having continuous derivatives with respect to t to the k-th order inclusively, and with respect to variable $\tau\in[0,+\infty)$ be infinitely differentiable and for each fixed value of $t\in[0,T]$ satisfy the estimate of the form*

$$|h(t,\tau)|\le c_1\left(a_0+a_1\tau+a_2\tau^2+...+a_{i-1}\tau^{i-1}\right)e^{-a\tau}, \tag{13}$$

where $c_1>0$, $a_1>0$, $a_2>0$, ..., $a_{i-1}>0$ are the constants, i is any fixed natural number. If the function $V(t,\tau)$ is the solution of problem (10), (11) then for each fixed value of t the problem

$$C\mu\equiv P\frac{\partial}{\partial\tau}\left[\left(\frac{\partial V}{\partial\tau}\right)^{p-1}\frac{\partial\mu}{\partial\tau}\right]+\frac{\partial^2\mu}{\partial\tau^2}+a\frac{\partial\mu}{\partial\tau}=h(t,\tau), \tag{14}$$

$$\mu|_{\tau=0} = \psi(t), \qquad \lim_{\tau\to+\infty} \mu = 0 \tag{15}$$

has a unique solution, moreover $\mu(t,\tau)$ *with respect to* t *has continuous derivatives to the* k*-th order inclusively, but with respect to* τ *is infinitely differentiable. The estimates*

$$\left|\frac{\partial^s \mu(t,\tau)}{\partial t^{s_1}\partial \tau^{s_2}}\right| \le c_2\left(b_0 + b_1\tau + b_2\tau^2 + ... + b_i\tau^i\right)e^{-a\tau}, \tag{16}$$

where $c_2, b_0, b_1, ..., b_i$ *are non-negative constants,* $s = s_1 + s_2$; $s_1 = 0,1,...,k$; $s_2 = 0,1,...$ *are true.*

The proofs of lemmas 1-3 are cited in the paper [2].

Lemma 4. *Let* $(\xi,\tau) \in [0,+\infty)\times[0,+\infty)$, $a = const > 0$, $\varphi(\tau)$ *be infinitely differentiable and*

$$\left|\varphi^{(k)}(\tau)\right| \le c_1 e^{-a\tau}; \quad k = 0,1,...; \quad c_1 > 0. \tag{17}$$

Then the problem

$$D\eta \equiv \frac{\partial\eta}{\partial\xi} - \frac{\partial^2\eta}{\partial\tau^2} - a\frac{\partial\eta}{\partial\tau} = 0, \tag{18}$$

$$\eta|_{\xi=0} = \varphi(t), \tag{19}$$

$$\eta|_{\tau=0} = 0, \quad \lim_{\tau\to+\infty}\eta = 0 \tag{20}$$

has a unique solution, and the estimates

$$\left|\frac{\partial^i \eta(t,\tau)}{\partial\xi^{i_1}\partial\tau^{i_2}}\right| \le c_2 e^{-\frac{a}{2}\tau - \frac{1}{4}a^2\xi}, \tag{21}$$

where $c_2 > 0$*;* $i = i_1 + i_2$*,* i_1, i_2 *are arbitrary non-negative numbers, are true.*

Proof. By substitution

$$\eta = e^{-\frac{a}{2}\tau - \frac{1}{4}a^2\xi}\bar{\eta}(\xi,\tau)$$

problem (18)-(20) for η is reduced to the problem for heat conductivity equation whose resticted solution is obviously written out.

Lemma 5. *Let* $(\xi,\tau) \in [0,+\infty)\times[0,+\infty)$, $\varphi(\tau)$ *be infinitely differentiable, satisfy condition (17), the function* $H(\xi,\tau)$ *with respect to* τ *be infinitely differentiable, and with respect to* ε *have continuous derivatives to the* k*-th order inclusively, and*

$$|H(\xi,\tau)| \le c_1\left(a_0 + a_1\tau + a_2\tau^2 + ... + a_{i-1}\tau^{i-1}\right)e^{-\frac{a}{2}\tau - \frac{1}{4}a^2\xi}, \tag{22}$$

where $c_1, a_1, a_2, ..., a_{i-1}$ *are non-negatiive constants. Then the problem*

$$Dg = H(\xi,\tau), \tag{23}$$

$$g|_{\xi=0} = \varphi(\tau), \tag{24}$$

$$g|_{\tau=0} = 0, \quad \lim_{\tau\to+\infty} g = 0 \tag{25}$$

has a unique solution and this solution with respect to τ is infinitely differentiable, and with respect to ξ has continuous derivatives to the k-th order inclusively, and the estimates of the form

$$\left|\frac{\partial^j g(\xi,\tau)}{\partial\xi^{j_1}\partial\tau^{j_2}}\right| \leq c_2\left(b_0+b_1\xi+b_2\xi^2+...+b_i\xi^i\right)e^{-\frac{a}{2}\tau-\frac{1}{4}a^2\xi}, \tag{26}$$

where $c_2, b_0, b_1, ..., b_i$ are non-negative numbers, $j = j_1 + j_2$; $j_1 = 0, 1, ..., k+1$; $j_2 = 0, 1, ...$ are true.

Proof. We can find the function g in the form of the sum $g = \eta + h$, where g is the solution of problem (18)-(20), and h is the solution of the problem

$$Dh = -H(\xi,\eta), \tag{27}$$

$$h|_{\xi=0} = 0, \tag{28}$$

$$h|_{\tau=0} = 0, \quad \lim_{\tau\to+\infty} h = 0 \ . \tag{29}$$

After substitution $h = e^{-\frac{a}{2}\tau-\frac{1}{4}a^2\xi}\bar{h}(\xi,\tau)$ problem (27)-(29) is reduced to the problem for heat conductivity equation with a right hand side that can be solved by applying Fourier sine transformation.

Lemma 6. *If $U(t,x)$ is a generalized solution of problem (1)-(3), and $\tilde{U} \in L_{p+1}\left[0,T;\ \overset{\circ}{W}{}^1_{p+1}(0,1)\right] \cap C^2(D)$ is the solution of the equation*

$$L_k\tilde{U} = 0\left(\varepsilon^{n+1}\right) \tag{30}$$

satisfying conditions (2), (3), for the difference $z = U - \tilde{U}$ it is true the estimation

$$\varepsilon^p \iint\limits_D \left(\frac{\partial z}{\partial x}\right)^{p+1} dxd\tau + \varepsilon \iint\limits_D \left(\frac{\partial z}{\partial x}\right)^2 dxd\tau + c_1 \iint\limits_D z^2 dxd\tau \leq c_2\varepsilon^{2n+2}, \tag{31}$$

where $c_1 > 0, c_2 > 0$ are the constants independendent of ε.

Proof. Obviously z satisfies conditions (2), (3), too. Subtracting (30) from (1), multiplying the both hand sides of the obtained equality by z and integrating by parts we have

$$\varepsilon^p \iint\limits_D \left[\left(\frac{\partial U}{\partial x}\right)^p - \left(\frac{\partial \tilde{U}}{\partial x}\right)^p\right]\left(\frac{\partial U}{\partial x} - \frac{\partial \tilde{U}}{\partial x}\right) dxdt + \varepsilon \iint\limits_D \left(\frac{\partial z}{\partial x}\right)^2 dxdt +$$

$$+ \iint\limits_D \frac{\partial F}{\partial U} z^2 dxdt = \varepsilon^{n+1} \iint\limits_D H \cdot zdxdt,$$

where $H(t,x,\varepsilon)$ is a restricted function in D uniformly with respect to $\varepsilon \in [0,\varepsilon_0)$. Using in the last equation condition (4) and elementary inequality

$$\left(a^p - b^p\right)(a-b) \geq 2^{-p}|a-b|^{-p},$$

after some transformations we get estimate (31).

Now let's construct the asymptotics with respect to small parameter of the solution of generalized problem (1)-(3).

First we shall search for the approximate solution of equation (1) in the form

$$W = W_0 + \varepsilon W_1 + ... + \varepsilon^n W_n, \tag{32}$$

and the functions $W_i(t,x)$; $i = 0, 1, ..., n$ will be chosen so that

$$L_\varepsilon W = 0\left(\varepsilon^{n+1}\right). \tag{33}$$

Substituting expression (32) for W into (33), expanding $\dfrac{\partial}{\partial x}\left(\dfrac{\partial W}{\partial x}\right)^p$, $F(t,x,W)$ in powers of ε and comparing the terms with the same powers of ε, to determine W_i we'll get the equations:

$$\frac{\partial W_0}{\partial t} + a\frac{\partial W_0}{\partial x} + F(t,x,W_0) = 0, \tag{34}$$

$$\frac{\partial W_i}{\partial t} + a\frac{\partial W_i}{\partial x} + b(t,x) W_i = f_i(t,x), \tag{35}$$

where $b(t,x) = \dfrac{\partial F(t,x,W_0)}{\partial W_0}$, the functions $f_i(t,x) = H_i(W_0, W_1, ..., W_{i-1})$ depend on W_0, $W_1, ..., W_{i-1}$; $i = 1, 2, ..., n$ and their derivatives.

We'll solve equations (34) and (35) undre the following boundary conditions

$$W_0|_{t=0} = 0, \qquad W_0|_{x=0} = 0, \tag{36}$$

$$W_i|_{t=0} = 0, \qquad W_i|_{x=0} = 0, \qquad i = 1, 2, ..., n. \tag{37}$$

Assume that function $F(t,x,W_0)$ satisfies the conditions of lemma 1 for $k = 2(n+1)$. Then by the same lemma problem (34), (36) has a unique solution, moreover $W_0 \in C^{2(n-i+1)}(D)$.

Problems (35), (37) wherein the functions W_i; $i = 1, 2, ..., n$ are determined sequentially, are linear. Solutions of these problems are written as follows:

$$W_i(t,x) = \begin{cases} \dfrac{1}{a}\displaystyle\int_0^x f_i\left(t - \frac{1}{a}x + \frac{1}{a}\tau, \tau\right)\exp\left[-\frac{1}{a}\int_\tau^x b\left(t - \frac{1}{a}x + \frac{1}{a}\xi, \xi\right)d\xi\right]d\tau \\ \qquad \text{at } x < at, \\ \displaystyle\int_0^t f_i(\tau, x - at + a\tau)\exp\left[-\int_\tau^t b(\xi, x - at + a\xi)\,d\xi\right]d\tau \\ \qquad \text{at } x > at. \end{cases} \tag{38}$$

Considering obvious form of the functions $f_i = H_i(W_0, W_1, ..., W_{i-1})$ in (38) we can prove that $W_1 \in C^{2(n-i+1)}(D)$; $i = 1, 2, ..., n$.

The function W determined by formula (32), generally speaking, doesn't satisfy the boundary condition for $x = 1$. To eliminate this hitchy in fulfilment of the boundary condition we are to constuct the function

$$V = V_0 + \varepsilon V_1 + ... + \varepsilon^{n+1} V_{n+1} \tag{39}$$

of boundary layer type near $x = 1$ so that $W + V$ satisfy the boundary condition

$$(W + V)|_{x=1} = 0. \tag{40}$$

By constructing the function V we must take care of fulfilment of the equality

$$L_{\varepsilon,1}(W + V) - L_{\varepsilon,1}W = 0\left(\varepsilon^{n+1}\right), \tag{41}$$

where $L_{\varepsilon.1}$ is a new splitting of the operator L_ε near the boundary $x = 1$.

In order to write a new splitting of the operator L_ε we make change of variables $1 - x = \varepsilon\tau$, $t = t$. Expanding each function $W(t, 1 - \varepsilon\tau)$ by Taylor formula at the point $(t, 1)$ we get a new expansion in powers of ε of the function W in the form of

$$W = \sum_{j=0}^{n+1} \varepsilon^j \omega_j(t, \tau) + 0\left(\varepsilon^{n+2}\right), \tag{42}$$

where $\omega_0 = W_0(t, 1)$ is independent of τ, and the remaining functions ω_k are determined by the formulas

$$\omega_k = \sum_{i+j=k} (-1)^i \frac{\partial' W_j(t, 1)}{\partial x'} \tau'; \quad k = 1, 2, ..., n + 1.$$

Then, allowing for (39), (42) we expand the expressions

$$\frac{\partial}{\partial\tau}\left[\frac{\partial(W + V)}{\partial\tau}\right]^p, \qquad \frac{\partial}{\partial\tau}\left(\frac{\partial W}{\partial\tau}\right)^p,$$
$$F(t, 1 - \varepsilon\tau, W + V), \qquad F(t, 1 - \varepsilon\tau, W)$$

in powers of ε and put their expansions in (41), after some transformations we get the following recurrently connected equations to determine the functions $V_0, V_\cdot, ..., V_{n+1}$:

$$BV_0 = 0, \tag{43}$$

$$CV_j = h_j(t, \tau), \tag{44}$$

where $h_j = \Phi_j(t, \tau, \omega_0, \omega_1, ..., \omega_{j-1}, V_0, V_1, ..., V_{j-1})$ are the known functions that are defined by the functions ω_k, V_k; $k = 0, 1, ..., j - 1$.

Boundary conditions for equations (43) and (44) are found from (40) by putting into it expression (32) for W and (39) for V. As we also search for boundary layer solutions of equations (43), (44), boundary conditions for these equations will be of the form

$$V_0|_{\tau=0} = -W_0|_{x=1}, \quad \lim_{\tau\to+\infty} V_0 = 0, \tag{45}$$

$$\begin{aligned} &V_i|_{\tau=0} = -W_i|_{x=1}, \quad \lim_{\tau\to+\infty} V_i = 0; \quad i = 1, 2, ..., n; \\ &V_{n+1}|_{\tau=0} = 0, \quad \lim_{\tau\to+\infty} V_{n+1} = 0. \end{aligned} \tag{46}$$

A unique existence of the solution of problem (43), (45) is solved by lemma 2. The functions $V_1, V_2, ..., V_{n+1}$ are constructed consequently by lemma 3.

Let's multiply all V_j by the smoothing functions and denote the new obtained functions again by V_j; $j = 0, 1, ..., n + 1$.

The function V_0 without any additional conditions on $F(t,x,u)$ satisfies the condition $V_0|_{t=0}=0$. But the functions V_j may be different from zero for $t=0$. Therefore, in spite of the fact that for the sum $W+V$ the both bundary conditions are satisfied, initial condition (2) may be not satisfied. In this connection we construct the function

$$\eta=\eta_0+\varepsilon\eta_1+...+\varepsilon^{n+1}\eta_{n+1} \tag{47}$$

of boundary layer type near the corner point $t=0$, $x=1$, so that the sum $W+V+\eta$ satisfy the conditions

$$(W+V+\eta)|_{t=0}=0, \qquad (W+V+\eta)|_{x=1}=0. \tag{48}$$

By constructing the function η the equality

$$L_{\varepsilon,2}(W+V+\eta)-L_{\varepsilon,2}(W+V)=0\left(\varepsilon^{n+1}\right), \tag{49}$$

should be fulfilled, where $L_{\varepsilon,2}$ is a new splitting of the operator L_ε near the corner point $t=0$, $x=1$.

To write a new splitting of the operator L_ε near the corner point of $t=0$, $x=1$ we make change of variables: $t=\varepsilon\xi$, $1-x=\varepsilon\tau$. Consider the auxiliary function

$$r=\sum_{j=0}^{n+1}\varepsilon^j r_j(\xi,\tau),$$

where $r_j(\xi,\tau)$ are the functions determined near the corner point $t=0$, $x=1$. Writing the expansion of $L_\varepsilon r$ in coordinates (ξ,τ) we have

$$L_{\varepsilon,2}r\equiv\varepsilon^{-1}\left\{\frac{\partial r_0}{\partial\xi}-\frac{\partial}{\partial\tau}\left(\frac{\partial r_0}{\partial\tau}\right)^p-\frac{\partial^2 r_0}{\partial\tau^2}-a\frac{\partial r_0}{\partial\tau}+\right.$$

$$+\sum_{j=0}^{n+1}\varepsilon^j\left[\frac{\partial r_j}{\partial\xi}-p\frac{\partial}{\partial\tau}\left(\left(\frac{\partial r_0}{\partial\tau}\right)^{p-1}\frac{\partial r_j}{\partial\tau}\right)-\frac{\partial^2 r_j}{\partial\tau^2}-a\frac{\partial r_j}{\partial\tau}+\right.$$

$$\left.\left.+\Phi_j(\xi,\tau,r_0,r_1,...,r_{j-1})\right]+0\left(\varepsilon^{n+2}\right)\right\}, \tag{50}$$

where Φ_j are the known functions that depend on ξ,τ and also on the functions $r_0,r_1,...,r_{j-1}$ and their first and second derivatives. We can obviously write formulae for Φ_j, but they are very bulky. Here we give only formulae for Φ_1 and Φ_2:

$$\Phi_1=F(0,1,r_0), \tag{51}$$

$$\Phi_2=-\frac{p(p-1)}{2!}\frac{\partial}{\partial\tau}\left(\left(\frac{\partial r_0}{\partial\tau}\right)^{p-1}\left(\frac{\partial r_1}{\partial\tau}\right)^2\right)+\frac{\partial F(0,1,r_0)}{\partial r_0}r_1+$$

$$+\frac{\partial F(0,1,r_0)}{\partial t}\xi-\frac{\partial F(0,1,r_0)}{\partial x}\tau. \tag{52}$$

Expanding the functions $W_i(\varepsilon\xi,1-\varepsilon\tau)$, $V_j(\varepsilon\xi,\tau)$ by Taylor formula at the points $(0,1)$ and $(0,\tau)$ we respectively get a new expansion of $W+V$ in powers of ε in coordinates (ξ,τ) in the form of

$$W+V=\sum_{j=0}^{n+1}\varepsilon^j\theta_j(\xi,\tau)+0\left(\varepsilon^{n+2}\right). \tag{53}$$

It should be noted that the first term in expansion (53) identically equals zero. Indeed, it follows from (36) and relation $V_0|_{t=0} = 0$, that

$$\theta_0 = W_0(0,1) + V_0(0,\tau) = 0. \tag{54}$$

Hence, from (47) and (53) it follows that

$$W + V + \eta = \sum_{j=0}^{n+1} \varepsilon^j \nu_j(\xi,\tau) + 0\left(\varepsilon^{n+2}\right), \tag{55}$$

where $\nu_0 = \eta_0$, $\nu_i = \theta_\iota + \eta_i$, $i = 1,2,...,n+1$.

Putting expressions (53), (55) into (49) we get that the functions η_j contained in the right hand side of (47) are the solutions of the following equations:

$$\frac{\partial\eta_0}{\partial\xi} - \frac{\partial}{\partial\tau}\left(\frac{\partial\eta_0}{\partial\tau}\right)^p - \frac{\partial^2\eta_0}{\partial\tau^2} - a\frac{\partial\eta_0}{\partial\tau} = 0, \tag{56}$$

$$\frac{\partial\eta_j}{\partial\xi} - p\frac{\partial}{\partial\xi}\left(\left(\frac{\partial\eta_0}{\partial\tau}\right)^{p-1}\frac{\partial\eta_j}{\partial\tau}\right) - \frac{\partial^2\eta_j}{\partial\tau^2} - a\frac{\partial\eta_j}{\partial\tau} = H_j(\xi,\tau), \tag{57}$$

respectively, where the right hand sides of (57) are determined by the formulae

$$H_j = \Phi_j\left(\xi,\tau,\theta_0+\eta_0,\theta_1+\eta_1,...,\theta_{j-1}+\eta_{j-1}\right) - \Phi_j\left(\xi,\tau,\theta_0,\theta_1,...,\theta_{j-1}\right). \tag{58}$$

From (48) and the fact that η_j; $j = 0,1,...,n+1$ should be boundary layer type functions, we get the following boundary conditions for equations (56), (57):

$$\eta_0|_{\xi=0} = -V_0|_{t=0}, \quad \eta_0|_{\tau=0} = 0, \quad \lim_{\tau\to+\infty}\eta_0 = 0, \tag{59}$$

$$\eta_j\big|_{\xi=0} = -V_j|_{t=0}, \quad \eta_j\big|_{\tau=0} = 0, \quad \lim_{\tau\to+\infty}\eta_j = 0. \tag{60}$$

Using conditions $V_0|_{t=0} = 0$ we can easily prove that problem (56), (59) has only a trivial solution that essentially simplifies finding of the following functions $\eta_1, \eta_2, ..., \eta_{n+1}$.

It follows from $\eta_0 \equiv 0$ and (51), (58) that $H_1 = F(0,1,\theta_0+\eta_0) - F(0,1,\theta_0) = 0$. Therefore the equation obtained from (57) for $j = 1$ takes the form

$$D\eta_1 = 0. \tag{61}$$

The remaining equations of (57) for $j = 2,3,...,n+1$ will be same but with right hand sides:

$$D\eta_j = H_j(\xi,\tau). \tag{62}$$

Since $|V_1|_{t=0}| \le ce^{-a\tau}$, by lemma 4 problem (61), (60) (for $j = 1$) has a unique solution and for η_1 estimate of the form (21) is valid. It follows from this estimate that the function η_1 satisfies the condition

$$\lim_{\tau\to+\infty}\eta_1 = 0$$

as well.

Knowing the functions η_0, η_1 we can determine the function η_2 as a solution of problem (62), (60) for $j = 2$. Using (52),(58) (for $j = 2$) we can easily see that the right hand side of equation (62) (for $j = 2$) will be of the form

$$H_2(\xi, \tau) = \frac{\partial F(0,1,0)}{\partial u} \eta_1. \tag{63}$$

Since estimate (21) is valid for η_1, it follows from (63) that the right hand side of equaion (62) satisfies estimate (22) for $i = 1$. Then applying lemma 5 to problem (62), (60) (for $j = 2$) we get that for η_2 estimate (26) is valid at $j = 1$, i.e.

$$\left| \frac{\partial' \eta}{\partial \xi^{i_1} \partial \tau^{i_2}} \right| \leq c_2 (b_0 + b_1 \xi) e^{-\frac{a}{2}\tau - \frac{1}{4}a^2 \xi},$$

whence it follows that η_2 satisfies the conditions

$$\lim_{\xi \to +\infty} \eta_2 = 0.$$

Lemma 5 allows to construct the remaining functions $\eta_3, \eta_4, ..., \eta_{n+1}$. Multiply η_j by the smoothing functions that differ from zero only at rectangular vicinity of corner point $t = 0$, $x = 1$ and keep the previous denotation.

Thus, we construct the sum $\tilde{U} = W + V + \eta$, that satisfies conditions (2) and (3). It follows from (33), (41) and (49) that $\tilde{U}$ is the solution of equation (30). Therefore, by lemma 6 for difference $z = U - \tilde{U}$ estimate (31) is valid.

Combining all the obtained results we arrive at the following statement.

Theorem. *Let $F(t, x, U) \in C^{2(n+1)}(D \times (-\infty, +\infty))$ satisfy conditions (4), (5) and (6) in the case when F linearly depends on U, condition (7), when F nonlinearly depends on U ($k = 2n+2$). Then for generalized solution of problem (1)-(3) it holds the asymptotic representation*

$$U = \sum_{i=0}^{n} \varepsilon^i W_i + \sum_{j=0}^{n+1} \varepsilon^j V_j + \sum_{S=0}^{n+1} \varepsilon^S \eta_S + z,$$

where the functions W_i are determined by the first iteration process, V_j, η_S are the boundary layer type functions near the boundary $x = 1$ and corner point $t = 0$, $x = 1$, respectively, z is a remainder term and estimate (31) is valid for it.

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