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GRADIENT IN OPTIMAL CONTROL PROBLEM WITH NON-LOCAL BOUNDARY CONDITIONS

Abstract

At this paper problems of optimal control with integral boundary conditions are considered. Formula of gradient for the considered problem is derived.

1. Problem statement

We consider the following optimal control problem with non-local conditions and it is required to minimize the functional

$$J(u) = \sum_{i=n}^{N} \varphi(x(t_i)) + \int_{t_0}^{T} F(t, x(t), u(t))\, dt \tag{1}$$

on solutions of the system

$$\mathring{x} = f(t, x, u), \quad t \in [t_0, T], \tag{2}$$

under non-linear conditions

$$x(t_0) + \int_{t_0}^{T} n(t)\, x(t)\, dt = B, \tag{3}$$

where

$$u = u(\cdot) \in U = \left\{ u(t) \in L_2^2 [t_0, T] : u(t) \in V, \quad a.e.\ t \in [t_0, T] \right\}, \tag{4}$$

Assume, that $t_0 \le t_1 < t_2 < ... < t_{N-1} < t_N \le T$ are fixed time, $x \in R^n$ is a phase variable, $u \in R^n$ are the controls, $F(t, x, u)$ and $\varphi(x)$ are real functions of variables $1 + n + r$ and n, respectively. $f(t, x, n)$ is n-dimensional function of variables $1 + n + r$, $V \in R^n$ is a bounded closed set, $n(t)$ is $n \times n$-dimensional function, B is n-dimensional given vector.

Denote a norm in R^n (or R^r) by $|\cdot|$, i.e. $|x| = \left(x_1^2 + x_2^2 + .. + x_n^2\right)^{1/2}$ and a scalar product by $<\cdot,\cdot>$, i.e. $<x, y>= x_1 y_1 + ... + x_n y_n$.

Let a norm and a scalar product in a space be denoted by $\|\cdot\|$ and $(\cdot,\cdot)$, respectively, .i.e.:

$$\|u\| = \left(\int_{t_0}^{T} |u(t)|^2\, dt \right)^{1/2}, \quad (u, v) = \int_{t_0}^{T} < u(t), v(t) > dt.$$

Suppose, that elements of matrix $n(t)$ are piecewise continuous and $\det\left(E+\int\limits_{t_0}^{T} n(t)\,dt\right)\neq 0$. Note, that if the condition $\left\|det\int\limits_{t_0}^{T} n(t)\,dt\right\|<1$ holds, then matrix $E+\int\limits_{t_0}^{T} n(t)\,dt$ is reversible, where E be $n\times n$-dimensional unit matrix.

Let the functions $f(t,x,u)$, $F(t,x,u)$ and $\varphi(x)$ be continuous by $\{t,x,y\}$ for all $x\in R^n$, $u\in V$, $t_0\le t\le T$ and have partial derivatives by (x,u) and this derivatives be bounded. Moreover, we suppose, that all partial derivatives of the function f,F,φ satisfy Lipschitz condition.

Suppose, that sufficient conditions, which provide existence and uniqueness of boundary value problem (2), (3) for each fixed control $u\in V$ [1], are fulfilled.

At this paper we'll get formula of gradient of functional (1) for limitations (2)-(4).

2. Main theorem

Suppose, that some sufficient conditions, which provide existence and uniqueness of non-local boundary value problem (2), (3) for each fixed permissible control $u(\cdot)\in V$, are fulfilled.

Let $(x(t),u(t))$ and $(x(t)+\bar{x}(t),u(t))+\bar{u}$ be two solutions of boundary value problem (2), (3). At this solutions increment of functional (1) is of the form:

$$J(u+\bar{u})-J(u)=\sum_{i=1}^{N}[\varphi(x(t_i)+\bar{x}(t_2)-\varphi(x(t_i)))+$$

$$+\int\limits_{t_0}^{T}[F(t,x(t)+\bar{x}(t),u(t)+\bar{u}(t)-F(t,x(t),u(t)))]\,dt=$$

$$=\sum_{i=1}^{N}\langle\nabla_x\varphi(x(t_i)),y(t_i)\rangle+ \tag{5}$$

$$+\int\limits_{t_0}^{T}[\langle\nabla_xF(t,x(t),u(t)),y(t)\rangle+\langle\nabla_uF(t,x(t),u(t)),\bar{u}(t)\rangle]\,dt+\eta,$$

where

$$\eta=\sum_{i=1}^{N}[\varphi(x(t_i)+\bar{x}(t_i)-\varphi(x(t_i)))-\langle\nabla_x\varphi(x(t_i)),\bar{x}(t_i)\rangle]+$$

$$+\int\limits_{t_0}^{T}[F(t,x(t)+\bar{x}(t),u(t)+\bar{u}(t))-F(t,x(t),u(t))-$$

$$-\langle\nabla_xF(t,x(t),u(t)),\bar{x}(t)\rangle-\nabla_uF(t,x(t),u(t)),\bar{u}(t)]dt+$$

$$+\sum_{i=1}^{N}\langle\nabla_x\varphi(x(t_i)),\bar{x}(t_i)-y(t_i)\rangle+$$

$$+\int_{t_0}^{T}\langle\nabla_x F(t,x(t),u(t)),\bar{x}(t)-y(t)\rangle]dt. \tag{6}$$

Let's introduce a system of equations in variations:

$$\mathring{y}(t)=\nabla_x f(t,x(t),u(t))\,y(t)+\nabla_u f(t,x(t),u(t))\,\bar{u}(t) \tag{7}$$

$$y(t_0)+\int_{t_0}^{T}n(t)\,y(t)\,dt=0. \tag{8}$$

We multiply equation (7) by the still unknown function $\psi(t)$, $t_0\le t\le T$ and integrate from t to T and add to (5), as a result we have

$$J(u+\bar{u})-J(u)=-\int_{t_0}^{T}\langle\nabla_x H(t,x(t),u(t),\psi(t)),y(t)\rangle\,dt-$$

$$-\int_{t_0}^{T}\langle\nabla_u H(t,x(t),u(t),\psi(t)),\bar{u}(t)\rangle\,dt+\sum_{i=1}^{N}\langle\nabla,\varphi(x(t_i)),y(t_i)+\rangle$$

$$-\int_{t_0}^{T}\langle\psi(t),\mathring{y}(t)\rangle\,dt+\eta. \tag{9}$$

From equality (8) after simple transformations, we get:

$$y(t_0)=-\left[E+\int_{t_0}^{T}n(t)\,dt\right]^{-1}\int_{t_0}^{T}\int_{t}^{T}n(t)\,dt\ \mathring{y}(t)\,dt, \tag{10}$$

$$y(t_i)=\int_{t_0}^{T}\left[\chi(t_i-t)E-\left[E+\int_{t_0}^{T}n(t)\,dt\right]^{-1}\int_{t}^{T}n(t)\,dt\right]^{-1}\mathring{y}(t)\,dt, \tag{11}$$

where E is $n\times n$-dimensional unit matrix

$$\chi(t_i-t)=\begin{cases}0 & at \quad t>t_i\\ 1 & at \quad t\le t_i\ .\end{cases}$$

Carry out the following equivalent transformation:

$$\int_{t_0}^{T}\langle\nabla_x H(t,x(t),u(t),\psi(t)),y(t)\rangle\,dt=$$

$$=\left\langle \int_{t_0}^{T} \nabla_x H\left(t, x\left(t\right), u\left(t\right), \psi\left(t\right) dt, y\left(T\right)\right)\right\rangle -$$

$$-\int_{t_0}^{T}\left\langle \int_{t_0}^{t} \nabla_x H\left(\tau, x\left(\tau\right), u\left(\tau\right), \varphi\left(\tau\right) d\tau, \dot{y}\left(t\right)\right)\right\rangle dt. \qquad (12)$$

Considering equalities (10)-(12) in (9), for increment of functional, we get the expression:

$$J\left(u+\bar{u}\right)-J\left(u\right)-\int_{t_0}^{T}\left\langle \int_{t_0}^{t} \nabla_x H\left(\tau, x\left(\tau\right), u\left(\tau\right), \psi\left(\tau\right)\right) d\tau, \dot{y}\left(t\right)\right\rangle dt-$$

$$-\left\langle \int_{t_0}^{T} \nabla_x H\Big(t, x\left(\tau\right), u\left(\tau\right), \psi\left(\tau\right) d\tau,\right.$$

$$\left.\left[E-\left(E+\int_{t_0}^{T} u\left(t\right) dt\right)^{-1}\int_{t_0}^{T}\int_{t}^{T} h\left(\tau\right) d\tau\right]\dot{y}\left(t\right) dt\Big)\right\rangle +$$

$$+\sum_{i=1}^{N}\left\langle \nabla_x\varphi\left(x\left(t_i\right)\right), \int_{t_0}^{T}\left[E\chi\left(t_i-t\right)-\left(E+\int_{t_0}^{T} n\left(t\right) dt\right)^{-1}\times\right.\right.$$

$$\left.\left.\times\int_{t}^{T} n\left(\tau\right) d\tau\right]\dot{y}\left(t\right) dt\right\rangle +\int_{t_0}^{T}\left\langle \psi\left(t\right), \dot{y}\left(t\right)\right\rangle dt-$$

$$-\int_{t_0}^{T}\left\langle \nabla_u H\left(t, x\left(t\right), u\left(t\right), \psi\left(t\right)\right), \bar{u}\left(t\right)\right\rangle dt+\eta.$$

Grouping similar items form the last expressions we get:

$$J\left(u+\bar{u}\right)-J\left(u\right)-\int_{t_0}^{T}\left\langle \int_{t_0}^{t} \nabla_x H\left(\tau, x\left(\tau\right), \psi\left(\tau\right)\right) d\tau-\right.$$

$$-\left(E-\left(E+\int_{t_0}^{T} n\left(t\right) dt\right)^{-1}\int_{t}^{T} n\left(\tau\right) d\tau\right)'\times$$

$$\left.\times\int_{t_0}^{T} \nabla_x H\left(t, x\left(t\right), u\left(t\right), \psi\left(t\right) dt, \dot{y}\right)\right\rangle dt+$$

$$+\sum_{i=1}^{n}\left\langle\left[E\chi\left(t_i-t\right)-\left(E+\int_{t_0}^{T}n\left(t\right)dt\right)^{-1}\int_{t}^{T}n\left(\tau\right)d\tau\right]'\psi\left(t\right),\dot{y}\left(t\right)\right\rangle dt.$$

We require the function $\psi\left(t\right),\ t_0\leq t\leq T$ to be solution of the integral equation

$$\psi\left(t\right)=\left(E-\left(E+\int_{t_0}^{T}n\left(t\right)dt\right)^{-1}\int_{t}^{T}n\left(\tau\right)d\tau\right)'\times$$

$$\times\int_{t_0}^{T}\nabla_x H\left(t,x\left(t\right),u\left(t\right),\psi\left(t\right)\right)dt-\int_{t_0}^{T}\nabla_x H\left(t,x\left(t\right),u\left(t\right),\psi\left(t\right)\right)dt+$$

$$+\sum_{i=1}^{N}\left[\left(E+\int_{t_0}^{t}h\left(\tau\right)d\tau-E\chi\left(t-t_i\right)\right)\right]\nabla_x\varphi\left(x\left(t_i\right)\right). \qquad (13)$$

Then for increment of the functional we get:

$$J\left(u+e\bar{u}\right)-J\left(u\right)=-\int_{t_0}^{T}\left\langle\nabla_u H\left(t,x\left(t\right),u\left(t\right),\psi\left(t\right)\right),\bar{u}\left(t\right)\right\rangle+\eta.$$

It is easy to show, that

$$\left|\eta\right|\leq C\left\|\bar{u}\right\|^2.$$

Thus, the following theorem is proved:

Theorem. *Let the functions $\varphi\left(x\right)$, $F\left(t,x,u\right)$, $f\left(t,x,u\right)$ by totality of their arguments together with their partial derivatives by variables $\left(x,u\right)$ at $\left(t,x,u\right)\in R\left[t_0,T\right]\times R^n\times R^r$ and besides, functions $f,\nabla_x f,\nabla_x F,\nabla_x\varphi,\nabla_u f,\nabla_u F$, satisfy Lipschitz's conditions by variables $\left(x,u\right)$.*

Then functional (1) under limitations (2)-(4) is continuous and differentiable by $u=u\left(t\right)$ in norm $L_2^r\left[t_0,T\right]$, moreover, its gradient $J'\left(u\right)\in L_2^r\left(t_0,T\right)$ at the point $u=u\left(t\right)$ is representable in the form

$$J'\left(u\right)=-\nabla_u H\left(t,x\left(t\right),u\left(t\right),\varphi\left(t\right)\right)=$$
$$=\nabla_u F\left(t,x\left(t\right),u\left(t\right)\right)-\left\langle\psi\left(t\right),\nabla_u f\left(t,x\left(t\right),u\left(t\right)\right)\right\rangle$$

where $x\left(t\right)$, $t_0\leq t\leq T$ is a solution of problem (2), (3), corresponding to the control $u=u\left(t\right)$, and $\psi\left(t\right)$, $t_0\leq t\leq T$ is a solution of adjoint system (13).

References

[1]. Mekhtiev M.F., Molaei H.H., Sharifov Y.A. *On an optimal control problem for nonlinear systems with integral conditions.* Transactions of NASA, series of physical-technical and mathematical sciences, v.XXV, No4, pp.191-198.

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