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NUMERICAL METHOD OF SOLUTION OF NONLOCAL BOUNDARY VALUE PROBLEMS FOR A LINEAR SYSTEM OF DIFFERENTIAL EQUATIONS

Abstract

In the paper we consider a problem on finding initial values of solutions of linear differential equations for nonlocal boundary conditions. A part of boundary conditions has a nonlocal form, and other part is represented in the integral form. The approximate formulas of computation of fundamental matrix for homogeneous system of differential equation are obtained. The algorithm of finding approximate solution of boundary value problem by the Runge-Kutt's two-stage method is constructed.

Let on segment $[0,T]$ numerical solution of a system of differential equations

$$\dot{x}(t) = D(t)\,x + f(t) \tag{1}$$

be required at the given nonlocal boundary conditions

$$Ax(0) = a, \tag{2}$$

$$\int_0^T B(t)\,x(t)\,dt = b, \tag{3}$$

where $x = (x_1, x_2, ..., x_n)$, $f(t) = (f_1(t), f_2(t), .., f_n(t))$, A is a constant matrix of dimension $m \times n$, $D(t)$ and $B(t)$ are the functional matrices of dimension $n \times n$ and $(n-m) \times n$, respectively, a and b are $m-$ and $(n-m)-$ dimensional constant vectors, respectively. Suppose, that elements of matrix $D(t)$, $B(t)$ and $f(t)$ are continuous functions on the segment $[0,T]$.

Assume, that problem (1)-(3) has a unique solution.

It is known, that any solution of equation (1), passing through the point $x = x(t)$ at $t = 0$, is uniquely represented in the form of [1]

$$x(t) = \Phi(t)\,x(0) + \int_0^t \Phi(t)\,\Phi^{-1}(\xi)\,f(\xi)\,d\xi, \tag{4}$$

where $\Phi(t)$ is a functional matrix corresponding to homogeneous system of equations (1), satisfying condition $\Phi(0) = E$.

If from the exact solution (4) of equation (1) we require to fulfill the boundary conditions (2) and (3), we can rewrite the given boundary conditions in the following form:

$$\begin{pmatrix} A \\ \int_0^T B(t)\,\Phi(t)\,dt \end{pmatrix} x(0) = \begin{pmatrix} a \\ b - \int_0^T B(t)\,\Phi(t) \int_0^t \Phi^{-1}(\xi)\,f(\xi)\,d\xi dt \end{pmatrix}. \tag{5}$$

Since by the supposition problem (1)-(3) has a unique solution, then a system of algebraic equations (5) is uniquely solvable.

Hence, an initial value of solution of boundary value problem (1)-(3) is uniquely defined by the roots of a system of linear algebraic equations (3). It is hard to construct exact system of algebraic equations (5) by solving practical problems. For example, usually, it is succeeded to construct only an approximation to a fundamental matrix and actually at once we get an approximation to a system of algebraic equations (5).

One more deficiency of system (5) is calculation of inverse matrices $\Phi^{-1}(\xi)$ or that is the same finding the values of fundamental matrix of a system, adjoined to homogeneous system of differential equations (1). Therefore, approximations to (5) is convenient to construct applying the appropriate methods of numerical integration directly to inhomogeneous system of differential equations (1). It allows to avoid computation of inverse matrices $\Phi^{-1}(\xi)$. Now we construct a numerical method for problems (1)-(3). Applying Runge-Kutt's method of second order to a system of differential equations (1), we get

$$x_k = x_{k-1} + \frac{1}{2}\left(S_1^{(k-1)} + S_2^{(k-1)}\right), \quad k = 1, 3, .., N, \tag{6}$$

where

$$S_1^{(k-1)} = h_k \left[D(t_k) x_{k-1} + f(t_{k-1})\right], \tag{7}$$

$$S_2^{(k-1)} = h_k \left[D(t_k)\left(x_{k-1} + S_1^{(k-1)}\right) + f(t_k)\right], \tag{8}$$

$$0 = t_0 < t_1 < ... < t_{N-1} < t_N = T, \quad h_k = t_k - t_{k-1}. \tag{9}$$

Expressing all $x_i\,(i = 1, 2, ..., k-1)$ through the initial value x_0, by induction we get the formula for approximate solution of equation (1). We have the correlation [2]:

$$x_k = \left(\prod_{i=0}^{k-1} R_i^{(H)}\right) x_0 + \sum_{j=1}^{k-1}\left[\left(\prod_{i=j}^{k-1} R_i^{(H)}\right) K_{j-1}^{(H)}\right] + K_{k-1}^{(H)}, \tag{10}$$

where the matrix $R_i^{(H)}$ is defined by the formula

$$R_i^{(H)} = E + \frac{1}{2}\left[h_{i+1}\left(D(t_i) + D(t_{i+1})\right) + h_{i+1}^2 D(t_{i+1}) D(t_i)\right], \tag{11}$$

and the vector

$$K_{j-1}^{(H)} = \frac{h_j}{2}\left[f(t_{j-1}) + f(t_j)\right] + \frac{h_j^2}{2} D(t_{j-1}) D(t_j), \tag{12}$$

We apply trapezoid formula to condition (3) by nodes (9). Then, we have

$$\sum_{k=1}^{N} \frac{h_k}{2}\left[B(t_{k-1}) x_{k-1} + B(t_k) x_k\right] - b. \tag{13}$$

Considering (10) in (13), we get

$$\sum_{k=1}^{N} \frac{h_k}{2}\left[B(t_{k-1})\left(\prod_{i=0}^{k-2} R_i^{(H)}\right) + B(t_k)\left(\prod_{i=0}^{k-1} R_i^{(H)}\right)\right] x_0 =$$

$$= b - \sum_{k=1}^{N} \frac{h_k}{2} \left\{ B\left(t_{k-1}\right) \left[\sum_{j=1}^{k-2} \left(\prod_{i=j}^{k-2} R_i^{(H)} \right) K_{j-1}^{(H)} + K_{k-2}^{(H)} \right] + \right.$$

$$\left. + B\left(t_k\right) \left[\sum_{j=1}^{k-1} \left(\prod_{i=j}^{k-1} R_i^{(H)} \right) K_{j-1}^{(H)} + K_{k-1}^{(H)} \right] \right\}. \quad (14)$$

Thus, discrete analogy of a system of algebraic equations (5) has the form:

$$\left(\begin{array}{c} A \\ \sum_{k=1}^{N} \frac{h_k}{2} \left[B\left(t_{k-1}\right) \left(\prod_{i=0}^{k-2} R_i^{(H)} \right) + B\left(t_k\right) \left(\prod_{i=0}^{k-1} R_i^{(H)} \right) \right] \end{array} \right) x_0 = \left(\begin{array}{c} a \\ b - \tilde{b} \end{array} \right) \quad (15)$$

where $\tilde{b} = b_1 + b_2$,

$$b_1 = \sum_{k=1}^{N} \frac{h_k}{2} \left\{ B\left(t_{k-1}\right) \left[\sum_{j=1}^{k-2} \left(\prod_{i=1}^{k-2} R_i^{(H)} \right) K_{j-1}^{(H)} + K_{k-2}^{(H)} \right] \right\},$$

$$b_2 = \sum_{k=1}^{N} \frac{h_k}{2} \left\{ B\left(t_k\right) \left[\sum_{j=1}^{k-1} \left(\prod_{i=1}^{k-1} R_i^{(H)} \right) K_{j-1}^{(H)} + K_{k-2}^{(H)} \right] \right\}.$$

Now show, that a matrix prescribed by the expression

$$\prod_{i=k_1}^{k} R_i^{(H)}$$

converges as $h \to 0$, i.e. to its exact fundamental matrix at the point $t = t_k$ $\left(h = \max_k h_k \right)$. Indeed, for a system of differential equations

$$\dot{x} = D\left(t\right) x, \quad (16)$$

we apply scheme (6)-(9). Then, approximate solution of system (16) at

$$x\left(0\right) = x_0 \quad (17)$$

on grid (9) is found by the formulae

$$x_k = x_{k-1} + \frac{1}{2} \left(S_1^{(k-1)} + S_2^{(k-1)} \right), \quad k = 1, 2, ..., N,$$

where

$$S_1^{(k-1)} = h_k A\left(t_k\right) x_{k-1}, \qquad S_2^{(k-1)} = h_k \left[A\left(t_k\right) \left(x_{k-1} + S_1^{(k-1)} \right) \right].$$

Expressing all $x_i \left(i = 1, 2, ..., k-1\right)$ by x_0, by induction for solution of problems (16), (17) at the point $t = t_k$ we get the following approximate formula:

$$x_k = \left(\prod_{i=0}^{k-1} R_i^{(H)} \right) x_0, \quad k = 0, 1, 2, ..., N, \quad (18)$$

where $R_i^{(H)}$ is defined by expression (11).

Consequently, approximate value of the fundamental matrix at the point $t = t_k$ on the basis of Runge-Kutt's method of second order is defined by the formula

$$\prod_{i=k-1}^{k} \left\{ E + \frac{1}{2} \left[h_{i+1} \left(D\left(t_i\right) + D\left(t_{i-1}\right) \right) + h_{i+1}^2 D\left(t_{i+1}\right) D\left(t_i\right) \right] \right\}. \tag{19}$$

By virtue of uniqueness of solution (1)-(3) and from comparison of correlations (4) and (10) it follows, that as $h \to 0$ the approximate fundamental matrix, prescribed by expression (19) converges to its exact value $\Phi\left(t_k\right)$, and the vector

$$\sum_{j=1}^{k-1} \left[\left(\prod_{i=j}^{k-1} R_i^{(H)} \right) K_{j-1}^{(H)} \right] + K_{k-1}^{(H)}$$

converges to the vector

$$\Phi\left(t\right) = \int_0^t \Phi^{-1}\left(\xi\right) f\left(\xi\right) d\xi.$$

Similarly we can prove, that as $h \to 0$ the matrix

$$\sum_{k=1}^{N} \frac{h_k}{2} \left[B\left(t_{k-1}\right) \left(\prod_{i=0}^{k-2} R_i^{(H)} \right) + B\left(t_k\right) \left(\prod_{i=0}^{k-1} R_i^{(H)} \right) \right]$$

converges to the matrix $\int_0^T B\left(t\right) \Phi\left(t\right) dt$.

So, we prove the theorem

Theorem. *Let nonlocal boundary value problem (1)-(3) have a unique solution. Then, at sufficiently small h, $h = \max\limits_{1 \le i \le N}$ of grid (9) an approximate initial value x_0 of the considered boundary value problem is defined as a unique root of a system of linear algebraic equations (15).*

Note, that after unique determination of the initial value for system (1) with condition $x\left(0\right) = x_0$ we can apply any numerical method to the solution of Cauchy problem [3, 4].

References

[1]. Heartman F. *Ordinary differential equations.* M.: Mir, 1970. (Russian)

[2]. Samarskii A.A., Nikolaev E.S. *Method of solution of grid equations.* M.: Nauka, 1978 . (Russian)

[3]. Samarskii A.A., Gulin A.V. *Numerical methods.* M.: Nauka, 1989. (Russian)

[4]. Bakhvalov N.S., Zhidkov N.P., Kobelkov G.M. *Numerical methods.* M.: BINOM, 2003. (Russian)

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