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ON BEHAVIOUR OF SOLUTION OF CAUCHY'S PROBLEM FOR ONE CORRECT BY PETROVSKY EQUATION AT LARGE TIME VALUES

Abstract

At this paper estimation of solution of Cauchy problem for one correct by Petrovsky equation is obtained at large tame values.

Introduction

By studying disturbance distribution in viscous gas when no viscosity exists there arises the equation

$$\frac{\partial^2}{\partial t^2}u(x,t)-\omega\frac{\partial}{\partial t}\Delta_3 u(x,t)-a^2\Delta_3 u(x,t), \quad x\in R_3, \quad t\geq 0, \tag{1}$$

where $\omega=\frac{4}{3}v$, v is kinematic coefficient of viscosity, a is a sound velocity in gas [1]. In the paper [2], as result of investigations of Cauchy's problem solution for operator-differential equation it is given a sufficient condition of stabilization at large time values of solution of Cauchy's problem for equation (1) with periodic initial data.

Consider the following Cauchy problem

$$\frac{\partial^2}{\partial t^2}u(x,t)-\omega\frac{\partial}{\partial t}\Delta_n u(x,t)=a^2\Delta_n u(x,t)+f(x,t), \tag{2}$$

$$u(x,t)\,|_{t=0}=\varphi_0(x), \quad \frac{\partial}{dt}u(x,t)\,|_{i=0}=\varphi_0(x), \tag{3}$$

where $\varphi_0(x)$, $\varphi_1(x)$, $f(x,t)$ are given functions, and

$$\Delta_n=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+...+\frac{\partial^2}{\partial x_2^2}.$$

Conditions on the data of problem will be formulated below.

Note, that equation (2) is correct by Petrovsky equation. Existence and uniqueness of solution in classes of destributions for correct by Petrovsky equations are studied in [3] (pp.124-183). In this paper estimation of solution of Cauchy problem (2), (3) for large time values is obtained.

§1. Definition, representation of Cauchy problem (2), (3) and auxiliary lemmas

Denote by $D(R_n)$ a space of finite infinitely differentiable functions. Considering solution of problem (1), (2) $u(x,t)$ for each $t>0$ as a distribution over $D(R_n)$, solution of this problem we'll understand in distributions.

Applying Fourier transformation to problem (2), (3) we get dual problem with parameter $s=(s_1,s_2,...,s_n)$

$$\frac{\partial^2}{\partial t}V(s,t)+|s|^2\omega\frac{\partial}{\partial t}V(s,t)+a^2|s|^2V^2(s,t)=\widetilde{f}(s,t), \tag{4}$$

$$V(s,t)|_{t=0}=\widetilde{\varphi}_0(s),\quad \frac{\partial^2}{\partial t}V(s,t)|_{t=0}=\widetilde{\varphi}_1(s), \tag{5}$$

where the sign $\sim$ over the function means Fourier transformation of this function with respect to variable $x=(x_1,x_2,...,x_n)$, $|s|=\sqrt{s_2^1+s_2^2+...+s_n^2}$. Solving problem (4)-(5), we get

$$V(s,t)=\frac{\lambda_2(s)e^{t\lambda_1(s)}-\lambda_1(s)e^{t\lambda_2(s)}}{\lambda_2(s)-\lambda_1(s)}\widetilde{\varphi}_0(s)+\frac{e^{t\lambda_2(s)}-e^{t\lambda_1(s)}}{\lambda_2(s)-\lambda_1(s)}\widetilde{\varphi}_0(s)+$$

$$+\int_0^t\left[\frac{e^{(t-\tau)\lambda_2(s)}}{\lambda_2(s)-\lambda_1(s)}-\frac{e^{(t-\tau)\lambda_1(s)}}{\lambda_2(s)-\lambda_1(s)}\right]\widetilde{f}(s,\tau)\,d\tau, \tag{6}$$

where

$$\lambda_{1,2}=-\frac{|s|^2\omega}{2}\pm\sqrt{\frac{\omega^2|s|^4}{4}-a^2|s|^2} \tag{7}$$

are the roots of the characteristic equation $\lambda+|s|^2\omega\lambda+a^2|s|^2=0$. Solution of problem (2), (3) is defined as inverse Fourier transformation of function $V(s,t)$. By Fourier transformation formula of convolution from (6) we get

$$u(x,t)=\int_{R_n}G_0(x-\xi,t)\varphi_0(\xi)\,d\xi+\int_{R_n}G_1(x-\xi,t)\varphi_1(\xi)\,d\xi+$$

$$\int_0^t d\tau\int_{R_n}G_1(x-\xi,t-\tau)f(\xi,\tau)\,d\xi\equiv u_0(x,t)+u_1(x,t)+u_f(x,t), \tag{8}$$

where

$$G_0(x,t)\equiv\frac{1}{(2\pi)^n}\int_{R_n}Q_0(s,t)e^{-i(x,s)}ds$$

$$=\frac{1}{(2\pi)^n}\int_{R_n}\frac{\lambda_2(s)e^{t\lambda_1(s)}-\lambda_1(s)e^{t\lambda_2(s)}}{\lambda_2(s)-\lambda_1(s)}e^{-i(x,s)}ds, \tag{9}$$

$$G_1(x,t)\equiv\frac{1}{(2\pi)^n}\int_{R_n}Q_1(s,t)e^{-i(x,s)}ds=\frac{1}{(2\pi)^n}\int_{R_n}\frac{e^{t\lambda_2(s)}-e^{t\lambda_1(s)}}{\lambda_2(s)-\lambda_1(s)}e^{-i(x,s)}ds. \tag{10}$$

$G_0(x,t)$, $G_1(x,t)$ are Green functions for Cauchy problems for equation (2) with initial data

$$G_0(x,0)=\delta(x),\qquad G'_{0t}(x,0)=0,$$

$$G_1(x,0)=0,\qquad G'_{1t}(x,0)=\delta(x),$$

where $\delta(x)$ is a Dirac function. Integrals in (9) and (10), generally speaking, don't converge in the ordinary sense. They should be understood in distributions [3] (pp.125-130). Since for any sufficiently smooth function $\varphi(x)$

$$\widetilde{\varphi}(s)=(-1)^\mu\left(1+\rho^2\right)^{-\mu}\widetilde{(1-\Delta_n)^\mu\varphi(x)},\quad \rho=|s|, \tag{11}$$

where μ is a natural number, which we will choose below, besides, we assume, that $\varphi(x)$ decreases sufficiently good at infinity and decrease order we will also appoint below. Using theory of Fourier transformation for distributions [4] (pp.152-179) and (11) integrals (9) and (10) can be regularized in the following way

$$u_j(x,t) = (-1)^{\mu} \int_{R_n} G_j^{(*)}(x-\xi,t)(1-\Delta)^{\mu-j+1}\varphi_j(\xi)\,d\xi, \tag{12}$$

where

$$G_j^*(x,t) = \frac{1}{(2\pi)^n}\int_{R_n} Q_j(s,t)\left(1+|s|^2\right)^{\mu-1+j} e^{-i(x,s)}ds, \quad j=0,1. \tag{13}$$

Integrand functions in (13) are spherically symmetric functions. Passing in (13) to spherical coordinates [5] (p.376), we'll get

$$G_j^*(x,t) = (2\pi)^{\frac{n}{2}-1}|x|^{1-\frac{n}{2}}\int_0^{\infty}\rho^{\frac{n}{2}}J_{\frac{n}{2}-1}(\rho|x|)\,Q_j(s,t)\left(1+\rho^2\right)^{-\mu-1+j}d\rho, \tag{14}$$

where

$$2\mu = \left\{\left[\begin{array}{ll} \left[\frac{n}{2}\right], & \text{if } \left[\frac{n}{2}\right] \text{ is an even number} \\ \left[\frac{n}{2}\right]+1, & \text{if } \left[\frac{n}{2}\right] \text{ is an odd number}\end{array}\right]\right. \tag{14*}$$

where $J_\nu(z)$ is ν order Bessel function.

Functions $\lambda_1(\rho)$ and $\lambda_2(\rho)$ have three branching points: $\rho_1 = 0$, $\rho_2 = \frac{2a}{\omega}$, $\rho_3 = -\frac{2a}{\omega}$. In the neighbourhood of a point $\rho = 0$ the functions $\lambda_1(\rho)$ and $\lambda_2(\rho)$ are regular functions and the point $\rho = -\frac{2a}{\omega}$ is not contained in integration domain. Write expansion of unit $1 \equiv \varphi_1(\rho) + \varphi_2(\rho)$, where $\varphi_1(\rho)$ and $\varphi_2(\varphi)$ are infinitely differentiable functions and

$$\psi_1(\rho) = \begin{cases} 1, & \text{at } \rho \in \left[0, \frac{2a}{\omega}-\varepsilon\right], \\ 0, & \text{at } \rho \geq \frac{2a}{\omega}, \end{cases}$$

and

$$\psi_2(\rho) = \begin{cases} 0, & \text{at } \rho \in \left[0, \frac{2a}{\omega}-\varepsilon\right], \\ 1, & \text{at } \rho \geq \frac{2a}{\omega}. \end{cases}$$

Then $G_j^{(*)}(x,t)$ is represented in the form

$$G_j^{(*)}(x,t) = (2\pi)^{\frac{n}{2}-1}|x|^{1-\frac{n}{2}} \times$$

$$\times \left\{ \int_0^{\frac{2\alpha}{\omega}} \rho^{\frac{n}{2}}J_{\frac{n}{2}-1}(\rho|x|)\,Q_j(s,t)\left(1+\rho^2\right)^{-\mu-1+j}\psi_1(\rho)\,d\rho + \right.$$

$$\left. + \int_{\frac{2\alpha}{\omega}-\varepsilon}^{\infty} \rho^{\frac{n}{2}}J_{\frac{n}{2}-1}(\rho|x|)\,Q_j(s,t)\left(1+\rho^2\right)^{-\mu-1+j}\psi_2(\rho)\,d\rho \right] \equiv$$

$$= G_j^{(*)}(x,t) + G_j^{(*)}(x,t). \tag{15}$$

Let's estimate each summand in (16) at large values of t. Mark out the cases $|x| \leq A$, $|x| > A$, where $A > 0$ is a sufficiently large number. Let's previously prove two lemmas.

Lemma 1. *For $\lambda_1(\rho)$ and $\lambda_2(\rho)$ the following estimations hold:*

$$\lambda_1(\rho) \le -\frac{a^2}{\omega}, \qquad \text{at} \quad \rho \ge \frac{2a}{\omega},$$

$$\operatorname{Re}\lambda_1(\rho) \le -\frac{\omega}{2}\rho^2, \qquad \text{at} \quad \rho \in \left[0, \frac{2a}{\omega}\right]$$

$$\operatorname{Re}\lambda_2(\rho) \le -\frac{\omega}{2}\rho^2, \quad \text{at} \quad \rho \in [0, \infty).$$

Proof. Represent $\lambda_1(\rho)$ in the form

$$\lambda_1(\rho) = \frac{a^2\rho^2}{\frac{\omega}{2}\rho^2 + \sqrt{\frac{\omega^2}{4}\rho^4 - a^2\rho^2}}$$

Hence, at $\rho \ge \frac{2a}{\omega}$ we get $\lambda_1.(\rho) \le -\frac{a^2}{\omega}$. Second and third estimations are obvious.

Lemma 2. *Let*

$$I(t) = \int\limits_0^t \frac{(1+\tau)^\alpha\, d\tau}{[1+(t-\tau)]^{\frac{n-1}{2}}}, \tag{16}$$

where $n \ge 1$; α is any real number. Then as $t \to +\infty$

$$I(t) = O\left(t^{1+\gamma_1}\right),$$

where $\gamma_1 = \max\left\{-\frac{n-1}{2}, \alpha\right\}$.

Proof. Let's make replacement $\tau = ty$ in (16). Then

$$I(t) = t\int\limits_0^t \frac{(1+ty)^\alpha\, dy}{[1+(t-y)]^{\frac{n-1}{2}}} \tag{17}$$

Represent integral in (17) in the form:

$$I(t) = t\left\{\int\limits_0^{1/2} + \int\limits_{1/2}^1\right\} \frac{(1+ty)}{[1+(t(1-y))]^{\frac{n-1}{2}}} \equiv J_1(t) + J_2(t).$$

Let $\alpha \le 0$, n be a natural number. Then

$$I_1(t) = O\left(t^{1-\frac{n-1}{2}}\right), \quad I_2(t) = O\left(t^{1+\alpha}\right). \tag{18}$$

If $\alpha > 0$, n is any natural number, then at large t

$$I_1(t) = O\left(t^{1+\alpha-\frac{n-1}{2}}\right), \quad I_2(t) = O\left(t^{1+\alpha}\right). \tag{19}$$

From (18) and (19) it follows, that at large t

$$I(t) = O\left(t^{1+\gamma_1}\right).$$

The lemma is proved.

§2. Estimation of solution for Cauchy problem (2), (3) at large values of time

Using lemma 1 and expansion of Bessel function at $|x| \le A$ we get

$$\left|G_{1,1}^{(*)}(x,t)\right| \le C(n,A)\int_0^{\frac{2a}{\omega}} e^{-\frac{\omega}{2}t\rho^2}\frac{\rho^{n-2}}{\sqrt{\frac{\omega^2}{2}\rho-a^2}}\left(1+\rho^2\right)^{-\mu}\varphi_1(\rho)\,d\rho \tag{20}$$

Applying Watson's lemma [7] (p.57) to the integral in (20) as $t\to+\infty$, we get

$$\left|G_{1,1}^{(*)}(x,t)\right| \le C(n,A)\,t^{-\frac{n-1}{2}} \tag{21}$$

where $C(n,A)$ is a constant, dependent on n and A.

At $|x| > A$ using asymptotic of Bessel function [6] (p.219) from expression $G_{1,1}^{*}(x,t)$ we get

$$\left|G_{1,1}^{(*)}(x,t)\right| \le C(n)\,|x|^{-\frac{1-n}{2}}\int_0^{\frac{2a}{\omega}} e^{-\frac{\omega}{2}t\rho^2}\rho^{\frac{n-1}{2}}\left(1+\rho^2\right)^{-\mu}\varphi_1(\rho)\,d\rho \tag{22}$$

Applying Watson's lemma [7] (p.57) to the integral in (22) as $t\to+\infty$ we get

$$\left|G_{1,1}^{(*)}(x,t)\right| \le C(n,\omega)\,|x|^{-\frac{1-n}{2}}t^{-\frac{n+1}{4}} \tag{23}$$

From estimations (21) and (23) we get, that for any $x\in R_n$ and $t\to+\infty$

$$\left|G_{1,1}^{(*)}(x,t)\right| \le C(n,\omega)\,|x|^{-\frac{1-n}{2}}t^{-\gamma_2} \tag{24}$$

where $\gamma_2=\min\left\{\frac{n-1}{2},\frac{n+1}{4}\right\}$.

Now consider $G_{1,2}^{(*)}(x,t)$ from (15). Let's represent it in the form

$$G_{1,2}^{(*)}(x,t)=(2\pi)^{\frac{n}{2}-1}|x|^{1-\frac{n}{2}}\left\{\int_{\frac{2\alpha}{\omega}-\varepsilon}^{2\alpha/\omega}+\int_{2\alpha/\omega}^{\infty}\right\}\rho^{\frac{n}{2}}J_{\frac{n}{2}-1}(\rho|x|)\frac{e^{t\lambda_2(\rho)}+e^{t\lambda_1(\rho)}}{\rho\sqrt{\frac{\omega^2}{4}\rho^2-a^2}}\times$$

$$\times\left(1+\rho^2\right)^{-\mu}\psi_2(\rho)\,d\rho=C_{1,2}^{(*,1)}(x,t)+G_{1,2}^{(*,2)}(x,t). \tag{25}$$

Investigate $G_{1,2}^{(*,1)}(x,t)$ as $t\to+\infty$. For $|x|\le A$, expanding Bessel function in series [6] (p.168) and restrictied only by the first member of expansion, contribution of the rest members of expansion as $t\to+\infty$ is small in comparison with contribution of the first member of expansion and estimating by module, we get

$$\left|G_{1,2}^{(*,1)}(x,t)\right| \le C(n,a)\int_{\frac{2\alpha}{\omega}-\varepsilon}^{2\alpha/\omega}\rho^{n-1}\left(1+\rho^2\right)^{-\mu}\psi_2(\rho)\frac{e^{-\frac{\omega}{2}t\rho^2}}{\sqrt{\frac{\omega^2}{4}\rho^2-a^2}}d\rho. \tag{26}$$

Since singularity at the point $\rho=\dfrac{2a}{\omega}$ in (26) is summable, supposing $\varepsilon=\dfrac{a}{\omega}$ from (26) as $t\to+\infty$ we get

$$\left|G_{1,2}^{(*,1)}(x,t)\right| \le C(n,a,\omega)\,e^{-\frac{a^2}{2\omega}t}. \tag{27}$$

Now consider $G_{1,2}^{(*,1)}(x,t)$ for $|x|>A$ and $t\to+\infty$. Considering lemma 1 and asymptotic of Bessel function at large values of argument for $G_{1,2}^{(*,1)}(x,t)$, we get

$$\left|G_{1,2}^{(*,1)}(x,t)\right|\leq C|x|^{\frac{1-n}{2}}\int\limits_{\frac{2\alpha}{\omega}-\varepsilon}^{2\alpha/\omega}\rho^{\frac{n-3}{2}}\left(1+\rho^2\right)^{-\mu}\psi_2(\rho)\frac{e^{-\frac{\omega}{2}t\rho^2}d\rho}{\sqrt{\frac{\omega^2}{4}\rho^2-a^2}}. \tag{28}$$

At the point $\rho=\dfrac{2a}{\omega}$ integrand function in (28) has summable singularity. Since function $\psi_2(\rho)$ at this point is equal to zero together with any order derivatives, then as $t\to+\infty$ contribution of point $\rho=\dfrac{2a}{\omega}$ in integral (28) is equal to zero. Supposing $\varepsilon=\dfrac{a}{\omega}$ and estimating the integral in (28) by module, we get

$$\left|G_{1,2}^{(*,1)}(x,t)\right|\leq C(\omega)|x|^{\frac{1-n}{2}}e^{-\frac{a^2}{2\omega}t} \tag{29}$$

From estimations (28) and (29) it follows, that at all $x\in R_n$ for $G_{1,2}^{(*,1)}(x,t)$ estimation (29) holds

Now consider $G_{1,2}^{(*,2)}(x,t)$ at large values of t and $|x|\geq\delta>0$, where δ is sufficiently small number. Estimating by module and taking into account asymptotic of Bessel function at large values of argument, we get

$$\left|G_{1,2}^{(*,2)}(x,t)\right|\leq C(\omega)|x|^{\frac{1-n}{2}}e^{-\frac{a^2t}{\omega}}\int\limits_{\frac{2\alpha}{\omega}}^{\infty}\rho^{\frac{n}{2}-2-2\mu}\frac{d\rho}{\left(\frac{\omega\rho}{2}-a\right)^{\frac{1}{2}}}\leq C(\omega)|x|^{\frac{1-n}{2}}e^{-\frac{a^2t}{\omega}} \tag{30}$$

By virtue of (14*) the integral in (30) converges. From (29) and (30) it follows, that for all $x\in R^n$

$$\left|G_{1,2}^{(*)}(x,t)\right|\leq C(\omega)|x|^{\frac{1-n}{2}}e^{-\frac{a^2t}{2\omega}} \tag{31}$$

From (24) and (31) it follows, that at large t and for any $x\in R^n$ estimation

$$\left|G_1^{(*)}(x,t)\right|\leq C(n,\omega)|x|^{\frac{1-n}{2}}t^{-\gamma_2}. \tag{32}$$

holds.

Now consider $G_0(x,t)$

$$G_0(x,t)=\frac{1}{(2\pi)^n}\int\limits_{R_n}\frac{\lambda_2(s)e^{t\lambda_1(s)}-\lambda_1(s)e^{t\lambda_2(s)}}{\lambda_2(s)-\lambda_1(s)}e^{-i(x,s)}ds\equiv$$

$$\equiv\frac{1}{(2\pi)^n}\int\limits_{R_n}Q_0(s,t)e^{-i(x,s)}ds \tag{33}$$

Regularizing integral in (35) as in (10) and passing to spherical coordinates, we get

$$G_0^{(*)}(x,t)=(2\pi)^{\frac{n}{2}-1}|x|^{1-\frac{n}{2}}\int\limits_0^{\infty}\rho^{\frac{n}{2}}J_{\frac{n}{2}-1}(\rho|x|)Q_0(s,t)\left(1+\rho^2\right)^{-\mu-1}d\rho. \tag{34}$$

Function $G_0^{(*)}(x,t)$ is estimated in the same way as the function $G_1^{(*)}(x,t)$, with the difference, that integrand function in expression $G_1^{(*)}(x,t)$ at zero has one zero more, than integrand function in expression $G_1^{(*)}(x,t)$. Therefore at large $|x|$ and t we have

$$|G_0^{(*)}(x,t)| \leq C(n,\omega)|x|^{1-\frac{n}{2}}t^{-\gamma_3}, \tag{35}$$

where $\gamma_3 = \min\left\{\frac{n}{2}, \frac{n+3}{4}\right\} = \gamma_2 + 1.$

Prove the following lemma, that is necessary to estimate solution of Cauchy problem (2), (3).

Denote by $H^{(m)}_{(|x|^{\alpha_1},R_n)}$ Sobolev weight space, i.e. space of functions from $L_2(R_n)$, for which

$$\|\varphi(x)\|_{H^{(m)}_{(|x|^{\alpha_1},R_n)}} = \sum_{|v|=0}^{m}\left\{\int_{R_n}|x|^{2\alpha_1}\left|\frac{\partial^v}{\partial x^v}\varphi(x)\right|^2 dx\right\}^{\frac{1}{2}} < +\infty$$

Lemma 3. *Let $\varphi(x) \in L_2(|x|^{\alpha_1}, R_n)$ and*

$$I(x) = \int_{R_n}|x-\xi|^{1-\frac{n}{2}}\varphi(\xi)\,d\xi.$$

Then at $|x| \geq 1$ for $I(x)$ there satisfies estimation

$$|I(x)| \leq \frac{2^{\frac{\alpha_1}{2}+1}\pi^{\frac{n}{4}}}{\Gamma^{\frac{1}{2}}\left(\frac{n}{2}\right)}\left(1+|x|^2\right)^{\frac{\alpha_1}{2}}\|\varphi(x)\|_{L_2(|x|^{\alpha_1},R_n)}. \tag{36}$$

Proof. In expression $I(x)$ pass to spherical coordinates. So, we get

$$I(x) = \int_{|x|}^{\infty}|x-\xi|^{\frac{n}{2}}\left\{\int_{\Omega_1^{(x)}}\varphi(\theta,|\xi|)\,d\theta\right\}d|\xi|, \tag{37}$$

where $\Omega_1^{(x)}$ is a surface of a unit sphere with centre at the point x. Applying Cauchy Bunyakovskii inequality to (37), we get

$$|I(x)| \equiv \left\{mes\Omega_1^{(x)}\right\}^{\frac{1}{2}}\left\{\int_{|x|}^{\infty}|x-\xi|^n\left(1+|\xi|^2\right)^{-\alpha_1}d|\xi|\right\}^{\frac{1}{2}} \times$$

$$\times\left\{\int_{|x|}^{\infty}\left(1+|\xi|^2\right)^{\alpha_1}\left[\int_{\Omega_1^{(x)}}\varphi^2(\theta,|\xi|)\,d\theta\right]d|\xi|\right\}^{\frac{1}{2}}. \tag{38}$$

Consider at $|x| \geq 1$ integral in (38)

$$I_1(x) \equiv \int_{|x|}^{\infty}\left(1+|\xi|^2\right)^{\alpha_1}\int_{\Omega_1^{(x)}}\varphi(\theta,|\xi|)\,d\theta d|\xi| \leq \int_{|x|}^{\infty}\left(1+|\xi|^2\right)^{\alpha_1}|\xi|^{n-1}\times$$

$$\times\left[\int_{\Omega_1^{(x)}}\varphi^2(\theta,|\xi|)\,d\theta\right]d|\xi| = \int_{R_n}\left(1+|\xi|^2\right)^{\alpha_1}\varphi^2(\xi)\,d\xi. \tag{39}$$

By virtue of conditions of lemma, the last integral converges. Now consider

$$I_2(x)\equiv\left\{\int\limits_{|x|}^{\infty}|x-\tau|^n\left(1+|\tau|^2\right)^{-\alpha_1}d|\tau|\right\}.$$

By Petre inequality [8] (p.16)

$$\left(1+|x-y|^2\right)^{-1}\le 2\frac{1+|x|^2}{1+|y|^2}.$$

Substituting $x-y=\tau$ and then applying Petre inequality in $I_2(x)$, we get

$$I_2(x)\le 2^{\frac{\alpha_1}{2}}\left(1+|x|^2\right)^{\frac{\alpha_1}{2}}\left\{\int\limits_0^{\infty}\frac{|y|^n d|y|}{\left(1+|y|^2\right)^{\alpha_1}}\right\}^{\frac{1}{2}}. \qquad (40)$$

Choosing $\alpha_1=\frac{n}{2}+1$ we get, that integral in (40) converges. Since $mes\Omega_1^{(x)}=\frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}$ from (38) and (40) we get

$$|I(x)|\le 2^{\frac{\alpha_1}{2}+1}\frac{\pi^{n/4}}{\Gamma\left(\frac{n}{2}\right)}\left(1+|x|^2\right)^{\frac{\alpha_1}{2}}\|\varphi(x)\|_{L_2(|x|^{\alpha_1},R_n)}.$$

Lemma 3 is proved.

Remark. Since $I(x)$ is a continuous function of x, so estimation (36) holds for all $x\in R_n$.

Study now behaviour of solution of Cauchy problem (2), (3) as $t\to\infty$. From formula (8) it follows, that $u_j(x,t)$ $(j=0,1)$ means solution of Cauchy problem (2), (3) for homogeneous equation, whose only initial data $\varphi_j(x)$ differs from zero, and $u_f(x,t)$ is solution of Cauchy problem (2), (3) for which $\varphi_0(x)=\varphi_1(x)=0$, $f(x,t)\not\equiv 0$.

Theorem 1. *Let $\varphi_0(x)\in H^{(m_0)}_{(|x|^{\alpha_1},R_n)}$. Then for solution $u_0(x,t)$ of Cauchy problem (2), (3) at $x\in R$ and large t the following estimation*

$$\left|D_t^{\mu_0}D_{x_j}^{\nu_j}u_0(x,t)\right|\le C\left(1+|x|^2\right)^{\frac{\alpha_1}{2}}t^{-(\gamma_3+\mu_0)}\|\varphi_0(x)\|_{H^{(m_0)}(|x|^{\alpha_1},R_n)}, \qquad (41)$$

$$\mu_0,\nu_j=0,1,2;\quad m_0=2(\mu+\mu_0+2),$$

holds, where γ_3 was defined above.

Proof. Using estimation (35) from expression $u_0(x,t)$ and from (12), (13), we get

$$|u_0(x,t)|\le\int\limits_{R_n}\left|G_0^{(*)}(x-\xi,t)\right|\left|(1-\Delta)^{n+1}\varphi_0(\xi)\right|d\xi\le$$

$$\le C(n,\omega)t^{-\gamma_3}\int\limits_{R_n}|x-\xi|^{1-\frac{n}{2}}\left|(1-\Delta)^{n+1}\varphi_0(\xi)\right|d\xi. \qquad (42)$$

Applying lemma 3 to the right hand side of inequality (42), we get

$$|u_0(x,t)|\le C(n,\omega)\left(1+|x|^2\right)^{\frac{\alpha_1}{2}}t^{-\gamma_3}\left\|(1-\Delta)^{\mu+1}\varphi_0(\xi)\right\|_{L_2(|x|^{\alpha_1},R_n)}\le$$

$$\leq C\left(n,\omega\right)\left(1+\left|x\right|^{2}\right)^{\frac{\alpha_{1}}{2}}t^{-\gamma_{3}}\left\|\varphi_{0}\left(x\right)\right\|_{H^{(2(\mu+1))}_{(|x|^{\alpha_{1}},R_{n})}}.$$

Now consider $D_t^{\mu_0}D_{x_j}^{\nu_j}u_0\left(x,t\right)$, where μ_0, $\nu_j=0,1,2$, for which, by virtue of conditions on the functions $\varphi_0\left(\xi\right)$ the following representation

$$D_t^{\mu_0}D_{x_j}^{\nu_j}u_0\left(x,t\right)=\int\limits_{R_n}D_t^{\mu_0}D_{x_j}^{\nu}G_0\left(x-\xi,t\right)\varphi_0\left(\xi\right)d\xi,\quad j=1,2,...,n \tag{43}$$

holds.

Taking into account a condition on the function $\varphi_0\left(\xi\right)$ and properties of differentiation of convolution, we get

$$D_t^{\mu_0}D_{x_j}^{\nu_j}u_0\left(x,t\right)=\int\limits_{R_n}D_t^{\mu_0}G_0\left(x-\xi,t\right)D_{\xi_j}^{\nu}\varphi_0\left(\xi\right)d\xi,\quad \nu_j=0,1,2.$$

From expression $G_0\left(x,t\right)$ of (9) it follows, that

$$D_t^{\mu_0}G_0\left(x,t\right)=\frac{1}{\left(2\pi\right)^n}\int\limits_{R_n}\frac{\lambda_2\left(s\right)\lambda_1^{\mu_0}\left(s\right)e^{t\lambda_1(s)}-\lambda_1\left(s\right)\lambda_2^{\mu_0}\left(s\right)e^{t\lambda_2(s)}}{\lambda_2\left(s\right)-\lambda_1\left(s\right)}e^{-i(x,s)}ds\equiv$$

$$\equiv\frac{1}{\left(2\pi\right)^n}\int\limits_{R_n}D_t^{\mu_0}Q_0\left(x,t\right)e^{-i(x,s)}ds. \tag{44}$$

Integral in (44) converges in distributions [3]. As above, representing (42) in the form

$$D_t^{\mu_0}D_{x_j}^{\nu_j}u_0\left(x,t\right)=\left(-1\right)^{\mu+\mu_0}\int\limits_{R_n}D_t^{\mu_0}G_0^*\left(x-\xi,t\right)\left(1-\Delta\right)^{(\mu+\mu_0+1)}D_{\xi_j}^{\nu_j}\varphi_0\left(\xi\right)d\xi, \tag{45}$$

for $D_t^{\mu_0}G_0^*\left(x,t\right)$ we get representation

$$D_t^{\mu_0}G_0^*\left(x,t\right)=\frac{1}{\left(2\pi\right)^n}\int\limits_{R_n}D_t^{\mu_0}Q_0\left(x,t\right)\left(1+\left|s\right|^2\right)^{-\mu-\mu_0-1}e^{-i(x,s)}ds. \tag{46}$$

Passing in (46) to spherical coordinates, we get

$$D_t^{\mu_0}G_0^*\left(x,t\right)=\left(2\pi\right)^{\frac{n}{2}-1}\left|x\right|^{1-\frac{n}{2}}\times$$

$$\times\int\limits_0^\infty\rho^{\frac{n}{2}}J_{\frac{n}{2}-1}\left(\left|x\right|\rho\right)D_t^{\mu_0}Q_0\left(x,t\right)\left(1+\left|s\right|^2\right)^{-\mu-\mu_0-1}d\rho.$$

We make estimation of $D_t^{\mu_0}G_0^*\left(x,t\right)$ as $t\to+\infty$ just as for $G_0^*\left(x,t\right)$. Asymptotic of $D_t^{\mu_0}G_0^*\left(x,t\right)$ as $t\to+\infty$, by Watson lemma [7] (p.57), is defined by the order of zero of integrand function, at $\rho=0$ and order of zero of $D_t^{\mu_0}G_0^*\left(x,t\right)$ is $2\mu_0$ ($\mu_0=0,1,2$) unit more than of $Q_0\left(s,t\right)$.

Therefore decreasing degree as $t\to+\infty$ of $D_t^{\mu_0}G_0^*\left(x,t\right)$ and thereby $D_t^{\mu}u_0\left(x,t\right)$ is μ_0 unit more than of $u_0\left(x,t\right)$. Using (44), we get

$$\left|D_t^{\mu_0}D_{x_j}^{\nu_j}u_0\left(x,t\right)\right|\leq C\left(\omega\right)t^{-(\gamma_3+\mu_0)}\left(1+\left|x\right|^2\right)^{\frac{\alpha_1}{2}}\times$$

$$\times\left\|\left(1-\Delta\right)^{(\mu+\mu_0+1)}D_{\xi_j}^{\nu}\varphi_0\left(\xi\right)\right\|_{L_2(|x|^{\alpha_1},R_n)},\quad \mu_0,\ \ \nu_j=0,1,2$$

or $\left|D_t^{\mu_0} D_{x_j}^{\nu_j} u_0(x,t)\right| \leq C(\omega)\left(1+|x|^2\right)^{\frac{\alpha_1}{2}} t^{-(\gamma_3+\mu_0)} \|\varphi_0(\xi)\|_{H^{(2(\mu+1))}_{(|x|^{\alpha_1},R_n)}}$, where $m_0 = 2(\mu+\mu_0+2)$.

The theorem is proved.

Theorem 2. *Let $\varphi_1(x) \in H^{(2(\mu+1))}_{(|x|^{\alpha_1},R_n)}$. Then for solution $u_1(x,t)$ of Cauchy problem (2), (3) at $x \in R_n$ and large t the estimation*

$$\left|D_t^{\mu_0} D_{x_j}^{\nu_j} u_0(x,t)\right| \leq C(n,\omega)\left(1+|x|^2\right)^{\frac{\alpha_1}{2}} t^{-(\gamma_2+\mu_0)} \|\varphi_1(\xi)\|_{H^{(2(\mu+1))}_{(|x|^{\alpha_1},R_n)}},$$

holds, where $m_1 = 2(\mu+\mu_0+2)$, and γ_2 was defined above.

Proof. Using representation $u_1(x,t)$ from (12) and estimation of Green function $G_1(x,t)$ from (32), we get

$$|u_1(x,t)| \leq C(n,\omega)\, t^{-\gamma_2} \int_R |x-\xi|^{\frac{1-n}{2}} \left|(1-\Delta)^n \varphi_1(\xi)\right| d\xi. \tag{47}$$

Further, applying lemma 3 to integral on the right hand side of (47) for $u_1(x,t)$ we get the following estimation

$$|u_1(x,t)| \leq C(n,\omega)\, t^{-\gamma_2} \left(1+|x|^2\right)^{\frac{\alpha_1}{2}} \left|\left|(1-\Delta)^{\mu} \varphi_1(\xi)\right|\right|_{L_2(|x|^{\alpha_1},R_n)}$$

or

$$|u_1(x,t)| \leq C(n,\omega)\, t^{-\gamma_2} \left(1+|x|^2\right)^{\frac{\alpha_1}{2}} \left|\left|\varphi_1(\xi)\right|\right|_{H^{2\mu}_{(|x|^{\alpha_1},R_n)}}.$$

Analogously as $t \to +\infty$ we get

$$\left|D_t^{\mu_0} D_{x_j}^{\nu_j} u_1(x,t)\right| \leq C(n,\omega)\left(1+|x|^2\right)^{\frac{\alpha_1}{2}} t^{-(\gamma_2+\mu_0)} \|\varphi_1(\xi)\|_{H^{(m_1)}_{(|x|^{\alpha_1},R_n)}},$$

$$m_1 = 2(\mu+\mu_0+1).$$

The theorem is proved.

Theorem 3. *Let $f(x,t)$ be continuous by t, $f(x,t) \in H^{2(\mu+3)}_{(|x|^{\alpha_1},R_n)}$ and*

$$\|f(x,t)\|_{H^{2(\mu+3)}_{(|x|^{\alpha_1},R_n)}} \leq Ct^{\beta}. \tag{48}$$

Then for solution $u_f(x,t)$ of Cauchy problem (2), (3) at $x \in R_n$ and large t the estimation

$$\left|D_t^{\mu_0} D_{x_j}^{\nu_j} u_f(x,t)\right| \leq C(n,\omega)\left(1+|x|^2\right)^{\frac{\alpha_1}{2}} t^{1+\gamma_{4+\mu_0}},$$

holds, where $\gamma_{4+\mu_0} = \max\{-(\gamma_2+\mu_0), \beta\}$, $\mu_0 = 0,1,2$; β is an arbitrary real number.

Proof. Using form $u_f(x,t)$ from (8), represent it in the next form of

$$u_f(x,t) = (-1)^{\mu} \int_0^t d\tau \int_{R_n} G_1^{(*)}(x-\xi, t-\tau)(1-\Delta)^{\mu} f(\xi,\tau)\, d\xi. \tag{49}$$

Estimating by module and using at that estimation (32) we get

$$|u_f(x,t)| \le C(n,\omega) \int_0^t |t-\tau|^{-\gamma_2} \left\{ \int_{R_n} |x-\xi|^{\frac{1-n}{2}} |(1-\Delta)^{\mu} f(\xi,\tau)| \, d\xi \right\} d\tau.$$

Using lemma 3 from the last inequality and from the conditions of theorem 3 we get

$$|u_f(x,t)| \le C(n,\omega) \left(1+|x|^2\right)^{\frac{\alpha_1}{2}} \int_0^t |t-\tau|^{-\gamma_2} \|(1-\Delta)^n f(\xi,\tau)\|_{L_2(|\xi|^{\alpha_1},R_n)} \, d\tau \le$$

$$\le C(n,\omega) \left(1+|x|^2\right)^{\frac{\alpha_1}{2}} \int_0^t |t-\tau|^{-\gamma_2} \tau^{\beta} d\tau.$$

Applying lemma 2, we get

$$|u_f(x,t)| \le C(n,\omega) \left(1+|x|^2\right)^{\frac{\alpha_1}{2}} t^{t+\gamma_4},$$

where $\gamma_4 = \max\{-\gamma_2, \beta\}$.

Now estimate derivatives $u_f(x,t)$. Since $G_1(x,0)=0$, so

$$D_t D_{x_f}^{\nu_j} u_f(x,t) = \int_0^t d\tau \int_{R_n} D_t G_t(x-\xi, t-\tau) \, D_{\xi_j}^{\nu_j} f(\xi,\tau) \, d\xi. \tag{50}$$

We will make estimation of internal integral in (50) in the same way as estimation of $u_0(x,t)$. Therefore we get

$$\left|D_t D_{x_f}^{\nu_j} u_f(x,t)\right| \le C(n,\omega) \left(1+|x|^2\right)^{\frac{\alpha_1}{2}} \int_0^t (1+|t-\tau|)^{-\gamma_3} \|f(\xi,\tau)\|_{H^{2(\mu+2)}_{(|x|^{\alpha_1},R_n)}} \, d\tau.$$

Taking into account the condition on the function $f(x,t)$ and lemma 3 from (50), at $x \in R_n$ and $t \to +\infty$, we get

$$\left|D_t D_{x_f}^{\nu_j} u_f(x,t)\right| \le C(n,\omega) \left(1+|x|^2\right)^{\frac{\alpha_1}{2}} t^{1+\gamma_5},$$

where $\gamma_5 = \max\{-\gamma_3, \beta\} = \max\{-(\gamma_2+1), \beta\}$.

Now consider $D_t^2 D_{x_f}^{\nu_j} u_f(x,t)$. From initial conditions for the function $G_1(x,t)$ and from (49) we get

$$D_t^2 D_{x_f}^{\nu_j} u_f(x,t) = f(x,t) + \int_0^t d\tau \int_{R_n} D_t^2 G_1(x-\xi, t-\tau) f(\xi,\tau) \, d\tau.$$

Estimating the last equality we get

$$\left|D_t^2 D_{x_f}^{\nu_j} u_f(x,t)\right| \le \|f(x,t)\|_{H^{2(\mu+2)}_{(|x|^{\alpha_1},R_n)}} + C(n,\omega) \left(1+|x|^2\right)^{\frac{\alpha_1}{2}} \times$$

$$\int_0^t (1+|t-\tau|)^{-(\gamma_3+1)} \left\|(1-\Delta)^{n+2} D_{\xi_j}^{\nu_j} f(x,\tau)\right\|_{L_2(|x|^{\alpha_1},R_n)} \tag{51}$$

or

$$\left|D_t^2 D_{x_f}^{\nu_j} u_f(x,t)\right| \le \|f(x,t)\|_{H^{2(\mu+2)}_{(|x|^{\alpha_1},R_n)}} + C(n,\omega)\left(1+|x|^2\right)^{\frac{\alpha_1}{2}} \times$$

$$\times \int_0^t (1+|t-\tau|)^{-(\gamma_3+1)} \|f(x,\tau)\|_{H^{2(\mu+3)}_{(|x|^{\alpha_1},R_n)}} .$$

Using condition (48) on the function $f(x,t)$ and applying lemma 3, we get

$$\left|D_t^2 D_{x_f}^{\nu_j} u_f(x,t)\right| \le C(n,\omega)\left(1+|x|^2\right)^{\frac{\alpha_1}{2}} t^{1+\gamma_6}$$

where $\gamma_6 = \max\{-(\gamma_3+1),\beta\} = \max\{-(\gamma_2+2),\beta\}$.

The theorem is proved.

Thus, it follows from theorem 1-3, that solution of Cauchy problem (2), (3) for homogeneous equation decreases powerlike and it is obtained the decrease speed of Cauchy problem solution, which depends on dimension of space variable, and behaviour of solution of Cauchy problem as $t \to +\infty$ for nonhomogeneous equation depends also on behaviour of right hand side of equation at large values of time.

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