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ON THE EXISTENCE OF REGULAR SOLUTIONS OF A BOUNDARY VALUE PROBLEM FOR A CLASS OF HIGHER ORDER OPERATOR-DIFFERENTIAL EQUATIONS IN WEIGHT SPACE

Abstract

In this paper the regular solvability of a boundary value problem for a higher order operator-differential equations in weight space, when leading part of the equation has multiple characteristic, has been investigated.

In separable Hilbert space H consider the operator-differential equation of $2n$-th order such that

$$\left(-\frac{d^2}{dt^2}+A^2\right)^n u(t)+\sum_{j=0}^{2n-1} A_{2n-j}u^{(j)}(t)=f(t), \quad t\in R_+=(0,+\infty), \qquad (1)$$

Here A is a positive definite self-adjoint operator, $A_j\ \left(j=\overline{0,2n-1}\right)$ are linear, generally speaking, unbounded operators.

Connect equation (1) with the initial boundary conditions

$$u^{(S_v)}(0)=0, \quad \nu=\overline{0,n-1}, \qquad (2)$$

where $0\le S_0<S_1<...<S_{n-1}\le 2n-1$, S_ν are integers.

Let $f(t)\in L_{2,\gamma}(R_+:H)$, $u(t)\in W_{2,\gamma}^{2n}(R_+:H)$, where the spaces $L_{2,\gamma}(R_+:H)$ and $W_{2,\gamma}^{2n}(R_+:H)$ are defined as [1]

$$L_{2,\gamma}(R_+:H)=\left\{f(t):\left(\int_0^\infty \|f(t)\|_H^2 e^{-2\gamma t}dt\right)^{1/2}=\|f\|_{L_{2,\gamma}}<\infty\right\},$$

$$W_{2,\gamma}^{2n}(R_+:H)=\Big\{u(t):u^{(2n)}\in L_{2,\gamma}(R_+:H),\ A^{2n}u\in L_{2,\gamma}(R_+:H),$$

$$\|u\|_{W_{2,\gamma}^{2n}(R_+:H)}=\left(\left\|u^{(2n)}\right\|^2_{L_{2,\gamma}(R_+:H)}+\left\|A^{2n}u\right\|^2_{L_{2,\gamma}(R_+:H)}\right)^{1/2}\Big\},$$

Then denote

$$W_{2,\gamma}^{2n}(R_+:H;\{S_\nu\})=\left\{u(t):u\in W_{2,\gamma}^{2n}(R_+:H),u^{(S_\nu)}(0)=0,\ \nu=\overline{0,n-1}\right\}.$$

Definition 1. *If the vector-function $u(t)\in W_{2,\gamma}^{2n}(R_+:H)$ satisfies equation (1) almost everywhere in R_+, it said to be the regular solution of equation (1).*

Definition 2. *If the regular solution of equation (1) $u(t)$ satisfies boundary conditions (2) in the sense of* $\lim\limits_{t\to 0}\left\|u^{(S_v)}(t)\right\|_{H_{2n-S_v-1/2}}=0$, *where H_α is Hilbert* *space scale generated by the operator A, i.e.* $H_\alpha = D(A^\alpha)$, $(x,y)_\alpha = (A^\alpha x, A^\alpha y)$ *and the inequality*

$$\|u\|_{W^{2n}_{2,\gamma}(R_+:H)} \leq const\, \|f\|_{L_{2,\gamma}(R_+:H)},$$

holds, then problem (1), (2) will be called regularly solvable.

Note, that in the papers [2,3] regular solvability of problem (1), (2) has been investigated in special cases, for example, at $n=1$ and $n=2$. In this paper we'll show some conditions providing regular solvability of problem (1), (2).

Firstly, we study the simple problem

$$P_0 u_0(t) = \left(-\frac{d^2}{dt^2} + A^2\right)^n u_0(t) = f(t), t \in R_+ = (0,+\infty)\, e^{\gamma t} \tag{3}$$

$$u_0^{(S_\nu)}(0) = 0, \qquad \nu = \overline{0, n-1} \tag{4}$$

where S_ν are integers and $0 \leq S_0 < S_1 < ... < S_{n-1} \leq 2n-1$. To investigate solvability of problem (3), (4) we denote by $\hat{f}(\xi)$ and $\hat{u}_0(\xi)$ the transformations of the vector-functions $f(t)$ and $u_0(t)$. After substitution $u_0(t) = \upsilon_0(t)\, e^{\gamma t}$ we have

$$\left(-\left(\frac{d}{dt}+\gamma\right)^2 + A^2\right)^n \upsilon_0(t)\, e^{\gamma t} = f(t)$$

or

$$\left(-\left(\frac{d}{dt}+\gamma\right)^2 + A^2\right)^n \upsilon_0(t) = f(t)\cdot e^{-\gamma t} = g(t). \tag{5}$$

Denote by $g(t) = f(t)\cdot e^{-\gamma t} \in L_2(R_+ : H)$. Then boundary conditions (4) will have the form

$$\left(\frac{d}{dt}+\gamma\right)^{(S_\nu)} \upsilon_{0/t=0} = 0 \tag{6}$$

It is obvious that regular solvability of problem (5), (6) in $L_2(R_+ : H)$ is equivalent to regular solvability of problem (3), (4) in $L_{2\gamma}(R_+ : H)$. Let's solve problem (5), (6) in $L_2(R_+ : H)$, i.e. $g(t) \in L_2(R_+ : H)$, $\upsilon_0(t) \in W_2^{2n}(R_+ : H)$. At first, we investigate the equation

$$\left(-\left(\frac{d}{dt}+\gamma\right)^2 + A^2\right)^n \upsilon_0(t) = g(t). \tag{7}$$

After Fourier transformation we find $\left(-(-i\xi+\gamma)^2 E + A^2\right)^n \upsilon_0(\xi) = \hat{g}(\xi)$ or

$$\hat{\upsilon}_0(\xi) = \left(-(-i\xi+\gamma)^2 E + A^2\right)^{-n} \hat{g}(\xi).$$

Show that the vector-function

$$\upsilon_0(t) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} \left(-(-i\xi+\gamma)^2 E + A^2\right)^{-n} \hat{g}(\xi)\, e^{i\xi t} d\xi, \qquad t \in R,$$

satisfies equation (7) almost everywhere in R and belongs to the space $W_2^{2n}(R:H)$. It is obvious that at $|\gamma| < \mu_0$, where μ_0 is a lower boundary of the operator spectrum A, $\upsilon_0(t) \in W_2^{2n}(R:H)$. For that using Plansherel's theorem, show that $\xi^{2n}\hat{\upsilon}(\xi) \in L_2(R:H)$ and $A^{2n}\hat{\upsilon}_0(\xi) \in L_2(R:H)$. Since

$$\left\|\xi^{2n}\hat{\upsilon}_0(\xi)\right\|_{L_2(R:H)} = \left\|\xi^{2n}\left(-(-i\xi+\gamma)^2 E + A^2\right)^{-n}\hat{g}(\xi)\right\|_{L_2(R:H)} \leq$$

$$\leq \sup_{\xi}\left\|\xi^{2n}\left(-(-i\xi+\gamma)^2 E + A^2\right)^{-n}\right\| \cdot \left\|\hat{g}(\xi)\right\|_{L_2(R:H)},$$

let's estimate the norm $\left\|\xi^{2n}\left(-(-i\xi+\gamma)^2 E + A^2\right)^{-n}\right\|$ at $\xi \in R$. Since at $\xi \in R$,

$$|\gamma| < \left\|\xi^{2n}\left(-\left(-i\xi+\gamma^2\right)E + A^2\right)^{-n}\right\| =$$

$$= \left\|\xi^{2n}\left(-\left(-\xi^2 - 2i\xi\gamma + \gamma^2\right)E + A^2\right)^{-n}\right\| =$$

$$= \left\|\left[\xi^2\left(\left(\xi^2 + 2i\xi\gamma - \gamma^2\right)E + A^2\right)^{-1}\right]^n\right\|,$$

then using spectral expansion of the operator A, we have:

$$\left\|\left[\xi^2\left(\left(\xi^2 + 2i\xi\gamma - \gamma^2\right)E + A^2\right)^{-1}\right]^n\right\| =$$

$$= \sup_{\mu\in\sigma(A)}\left|\left[\xi^2\left(\left(\xi^2 + 2i\xi\gamma - \gamma^2\right) + \mu^2\right)^{-1}\right]^n\right| \leq$$

$$\leq \sup_{\mu\geq\mu_0}\left(\frac{\xi^2}{\left|\left(\xi^2 + 2i\xi\gamma - \gamma^2 + \mu^2\right)\right|}\right)^n \leq \left(\frac{\xi^2}{\xi^2 + \mu^2 - \gamma^2}\right)^n \leq 1.$$

Thus

$$\left\|\upsilon_0^{(2n)}\right\|_{L_2} = \left\|\xi^{2n}\hat{\upsilon}_0(\xi)\right\|_{L_2} \leq \left\|g(t)\right\|_{L_2(R:H)}, \text{ i.e. } \upsilon_0^{(2n)} \in L_2(R:H).$$

On the other hand,

$$\left\|A^{2n}\hat{\upsilon}_0(\xi)\right\|_{L_2(R:H)} = \left\|A^2\left(-(-i\xi+\gamma)^2 E + A^2\right)^{-n}\hat{g}(\xi)\right\|_{L_2(R:H)} \leq$$

$$\leq \sup_{\xi}\left\|A^{2n}\left(-(-i\xi+\gamma)^2 E + A^2\right)^{-n}\right\| \cdot \left\|\hat{g}(\xi)\right\|_{L_2(R:H)}.$$

Therefore we estimate the norm $\left\|A^{2n}\left(-(-i\xi+\gamma)^2 E + A^2\right)^{-n}\right\|$ at any $\xi \in R$. Since at any $\xi \in R$ and $|\gamma| < \mu_0$,

$$\left\|A^{2n}\left(-(-i\xi+\gamma)^2 E + A^2\right)^{-n}\right\| = \sup_{\mu\geq\mu_0}\frac{\mu^{2n}}{\left|\left(\xi^2 + 2i\xi\gamma - \gamma^2 + \mu^2\right)\right|^n} \leq$$

$$\leq \sup_{\mu\geq\mu_0}\left(\frac{\mu^2}{\xi^2 - \gamma^2 + \mu^2}\right)^n \leq \sup_{\mu\geq\mu_0}\left(\frac{\mu^2}{\mu^2 - \gamma^2}\right)^n \leq \left(\frac{\mu^2}{\mu^2 - \gamma^2}\right)^n$$

and the function $\varphi(\mu)=\dfrac{\mu^2}{\mu^2-\gamma^2}$ monotonically decreases on $[\mu_0;+\infty)$,

$$\left\|A^{2n}\upsilon_0\right\|_{L_2}=\left\|A^{2n}\hat{\upsilon}_0(\xi)\right\|_{L_2}\le\left(\frac{\mu_0^2}{\mu_0^2-\gamma^2}\right)^n\|g(t)\|_{L_2},\ \text{i.e. } A^{2n}\upsilon_0\in L_2(R:H).$$

Denote by $\overline{\upsilon_0(t)}$ the contraction of the vector-function on the semi-axis $[0,+\infty)$, i.e. $\overline{\upsilon_0(t)}=\upsilon_0(t)/\,t\in[0,+\infty)$. Then from the trace theorem it follows that

$$\overline{\upsilon_0}^{(s_i)}(0)\in H_{2m-s_i-1/2},\ \ i=\overline{0,n-1}.$$

We'll search a solution of problem (5), (6) in the form of $\upsilon(t)=\overline{\upsilon_0(t)}+\sum\limits_{q=0}^{n-1}e^{\omega_q\cdot tA}\varphi_q$, where ω_q are the roots of the equation $\left(\omega_q^2+1\right)^n=0$ from the left half-plane, i.e. $\operatorname{Re}\omega_q<0$, and $\varphi_q\in H_{2n-1/2}$. Then to determine the vectors φ_q we have the following system of equations

$$0=\upsilon^{(s_v)}(0)=\overline{\upsilon_0}^{(s_v)}(0)+\sum_{q=0}^{n-1}A^{s_\nu}\omega_q^{s_\nu}\varphi_q,\quad \nu=\overline{0,1}$$

or

$$\sum_{q=0}^{n-1}A^{s_\nu}\omega_q^{s_\nu}\varphi_q=-\overline{\upsilon_0}^{(s_\nu)}(0)\,,\nu=\overline{0,n-1}.$$

Whence it follows that

$$\sum_{q=0}^{n-1}\omega_q^{s_\nu}\varphi_q=-A^{-s_\nu}\overline{\upsilon_0}^{(s_\nu)}(0)\,,\nu=\overline{0,n-1}.$$

Since $\det\left(\omega_q^{s_\nu}\right)_{\substack{\nu=0\\q=0}}^{n-1}\neq0$, we determine the vectors φ_q. From the inverse matrix expression $\left(\omega_q^{s_\nu}\right)_{\substack{\nu=0\\q=0}}^{n-1}$ and from $-A^{-s_\nu}\overline{\upsilon_0}^{(s_\nu)}(0)\in H_{2n-s_\nu-q-1/2}$ it follows that $\varphi_q\in H_{2n-1/2}$, i.e. we define φ_q. But when $\varphi_q\in H_{2n-1/2}$, it is obvious that $e^{\omega_q tA}\varphi_q\in W_2^{2n}(R_+:H)$. Thus the regular solution of problem (5), (6) has the form

$$\upsilon(t)=\overline{\upsilon_0}(t)+\sum_{q=0}^{n-1}e^{\omega_q tA}\varphi_q,$$

where φ_q is defined as above. Now, let's show that the homogeneous equation

$$\left(-\left(\frac{d}{dt}+\gamma^2\right)^2+A^2\right)\upsilon(t)=0\tag{8}$$

has nonzero solution from the space $W_2^{2n}(R_+:H;\{s_\nu\})$. General solution of equation (8) from $W_2^{2n}(R_+:H;)$ takes the form $\upsilon(t)=\sum\limits_{q=0}^{n-1}e^{\omega_q tA}\varphi_q$, where $\varphi_q\in H_{2n-1/2}$, $(\omega_q)^n=1$, $\operatorname{Re}\omega_q<0$, $q=\overline{0,n-1}$. Hence from the condition $\upsilon^{(s_\nu)}(0)=0$ we obtain

$\sum\limits_{q=0}^{n-1} A^{s_\nu}\omega_q^{s_\nu}\varphi_q = 0$ or $\sum\limits_{q=0}^{n-1} \omega_q^{s_\nu}\varphi_q = 0$. Since $\det\left(\omega_q^{s_\nu}\right) \neq 0$, then $\varphi_q = 0$. Hence we get that $\upsilon(t) = 0$. Thus, the operator $P_{0,\gamma}$ generated by equation (5) and by boundary conditions (6), one-to-one maps the space $W_2^{2n}(R_+ : H; \{s_\nu\})$ onto $L_2(R_+ : H)$. Since

$$\|P_{0,\gamma}\upsilon\|_{L_2(R_+:H)} = \left\|\left(-\left(\frac{d}{dt}+\gamma^2\right)^2 + A^2\right)\upsilon\right\|_{L_2(R_+:H)},$$

using a theorem on intermediate derivatives, we obtain

$$\|P_{0,\gamma}\upsilon\|_{L_2} \leq const\,\|\upsilon\|_{W_2^{2n}}.$$

Thus, the operator $P_{0,\gamma}$ is continuously maps the space $W_2^{2n}(R_+ : H; \{s_\nu\})$ onto $L_2(R_+ : H)$ Then by the Banach theorem on the inverse operator, $P_{0,\gamma}$ is an isomorphism. Thereby we proved

Theorem 1. *Let A be a self-adjoint positive definite operator, with $A \geq \mu_0 E\,(\mu_0 > 0)$. Then at $|\gamma| < \mu_0$ problem (5) has a unique regular solution.*

Theorem 2. *Let the conditions of theorem 1 be hold and the operators $B_j = A_j \cdot A^{-j}$ be bounded in H, then when the conditions*

$$\alpha(\gamma;\mu_0) = \sum_{j=0}^{2n-1} \left(C_j(\gamma;\mu_0)\right)\|B_j\| \leq 1$$

are fulfilled, problem (1), (2) is regular solvable. Here the numbers $C_j(\gamma;\mu_0)$ are defined in the following way

$$C_j(\gamma;\mu_0) = \sup_{0\neq u\in W_{2,\gamma}^{2n}(R_+:H;\{s_\nu\})} \left\|A^{2n-j}u^{(j)}\right\|_{L_{2,\gamma}(R_+:H)} \cdot \|P_0 u\|_{L_{2,\gamma}(R_+:H)}^{-1},$$

$$j = \overline{0, 2n-1}.$$

Proof. From theorem 1 it follows that the operator P_0 generated by main part of problem (1), (2) isomorphically maps the space $W_{2,\gamma}^{2n}(R_+ : H; \{s_v\})$ onto $L_2(R_+ : H)$. Then let's write problem (1), (2) in the form of $P_0u + P_1u = f$, where

$$P_0 u = \left(-\frac{d^2}{dt^2} + A^2\right)^n u,\ u \in W_{2,\gamma}^{2n}(R_+ : H; \{s_\nu\}),$$

$$P_1 u = \sum_{j=0}^{2n-1} A_{2n-j}u^{(j)},\ u \in W_{2,\gamma}^{2n}(R_+ : H; \{s_\nu\}).$$

Substituting $u = P_0^{-1}\omega$ we get $\omega + P_1P_0^{-1}\omega = f$ or $\left(E + P_1P_0^1\right)\omega = f$.

Since for any $w \in L_{2,\gamma}(R_+ : H)$

$$\|P_1P\omega\|_{L_{2,\gamma}} = \|P_1u\|_{L_{2,\gamma}} = \left\|\sum_{j=0}^{2n-1} A_{2n-j}u^{(j)}\right\| \leq$$

$$\leq \sum_{j=0}^{2n-1}\left\|A_{2n-j}A^{-(2n-j)}A^{2n-j}u^{(j)}\right\| \leq$$

$$\leq \sum_{j=0}^{2n-1} \left(C_j\left(\gamma;\mu_0\right)\right)\left\|B_{2n-j}\right\| \left\|\omega\right\|_{L_{2,\gamma}} = \alpha\left(\gamma;\mu_0\right)\left\|\omega\right\|_{L_{2,\gamma}},$$

then in view of the fact that $\alpha(\gamma;\mu_0) < 1$ the operator $E + P_1P_0^{-1}$ is reversible in $L_{2,\gamma}(R_+ : H)$. Then $\omega = \left(E + p_1p_0^{-1}\right)^{-1} f$, and $u = P_0^{-1}\left(E + P_1P_0^{-1}\right)^{-1} f$. Hence we have

$$\|u\|_{W_{2,\gamma}^{2n}(R_+:H)} \leq const \|f\|_{L_{2,\gamma}(R_+:H)}.$$

The theorem is proved.

References

[1]. Lions J.-L., Majenes E. *Nonhomogeneous boundary value problems and their applications.* Moscow, "Mir", 1971, 371p. (Russian)

[2]. Humbataliyev R.Z., Aliyev A.R. *On solvability of a boundary value problem in weight space.* Vestnik Bakinskii Universitet, 2, 2003, pp.46-56. (Russian)

[3]. Humbataliyev R.Z. *On regular solvability of boundary value problems for a class of operator-differential equations in weight space.* Mathematics Inter. Ecoener. Academ., 3, 2005, pp. 41-48.

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