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# WEIGHTED INEQUALITY FOR SOME SUBLINEAR OPERATORS IN LEBESGUE SPACES, ASSOCIATED WITH THE LAPLACE-BESSEL DIFFERENTIAL OPERATORS

**Abstract**

*In this paper, the author establish some general theorem for the boundedness of sublinear operators, associated with the Laplace-Bessel differential operator* $\Delta_{B_n} = \sum\limits_{k=1}^{n} \frac{\partial^2}{\partial x_k^2} + B_n$, $B_n = \frac{\partial^2}{\partial x_n^2} + \frac{\gamma}{x_n}\frac{\partial}{\partial x_n}$, $\gamma > 0$, *on a weighted Lebesgue space. The conditions of these theorem are satisfied by many important operators in analysis. Sufficient condition on weighted function* $\omega$ *is given so that certain sublinear operator is bounded on the weighted Lebesgue spaces* $L_{p,\omega,\gamma}(\mathbb{R}_+^n)$.

## 1. Introduction

The singular integral operators that have been considered by Mihlin [10] and Calderon and Zygmund [5] are playing an important role in the theory Harmonic Analysis and in particular, in the theory partial differential equations. Klyuchantsev [8] and Kipriyanov and Klyuchantsev [9] have firstly introduced and investigated by the boundedness in $L_p$-spaces of multidimensional singular integrals, generated by the Laplace-Bessel differential operator $\Delta_{B_n} = \sum\limits_{k=1}^{n} \frac{\partial^2}{\partial x_k^2} + B_n$, $B_n = \frac{\partial^2}{\partial x_n^2} + \frac{\gamma}{x_n}\frac{\partial}{\partial x_n}$, $\gamma > 0$ ($B_n$ singular integrals). Aliev and Gadjiev [3] and Gadjiev and Guliyev [4] have studied the boundedness of $B_n$ singular integrals in weighted $L_p$-spaces with radial and general weights consequently. The maximal functions, singular integrals, potentials and related topics associated with the Laplace-Bessel differential operator $\Delta_{B_n}$ which is known as an important differential operator in analysis and its applications, have been the research areas many mathematicans such as K. Stempak [15], I. Kipriyanov and M. Klyuchantsev [8, 9], L. Lyakhov [12, 13], A.D. Gadjiev and I.A. Aliev [2, 3], V.S. Guliyev [6, 7] and others.

In the paper, we shall prove the boundedness of some sublinear operators, generated by the $B_n$ Bessel differential operators on a weighted $L_p$ spaces. Sufficient conditions on weighted function $\omega$ is given so that certain sublinear operator is bounded from the weighted Lebesgue spaces $L_{p,\omega,\gamma}(\mathbb{R}_+^n)$ into $L_{p,\omega,\gamma}(\mathbb{R}_+^n)$. We point

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out that the condition (1) (see below). The condition (1) is satisfied by many interesting operators in harmonic analysis, such as the $B_n$ singular integrals (for example, see [8, 9]), $B_n$ Hardy–Littlewood maximal operators (see also, [6, 7] and [15]) and so on.

## 2. Notations and Background

Suppose that $\mathbb{R}^n$ is the $n$-dimensional Euclidean space, $x = (x_1, \ldots, x_n)$, $\xi = (\xi_1, \ldots, \xi_n)$ are vectors in $\mathbb{R}^n$, $(x, \xi) = x_1\xi_1 + \ldots + x_n\xi_n$, $|x| = (x, x)^{1/2}$. Let $\mathbb{R}^n_+ = \{x = (x_1, \ldots x_n) : \ x_n > 0\}$, $\gamma > 0$, $E(x, r) = \{y \in R^n_+ : \ |x - y| < r\}$, $\Sigma_+ = \{x \in \mathbb{R}^n_+ : \ |x| = 1\}$.

For measurable set $E \subset \mathbb{R}^n_+$ let $|E|_\gamma = \int\limits_E x_n^\gamma dx$, then $|E(0, r)|_\gamma = \omega(n, \gamma) r^{n+\gamma}$, where $\omega(n, \gamma) = |E(0, 1)|_\gamma$.

An almost everywhere positive and locally integrable function $\omega : \ \mathbb{R}^n_+ \to \mathbb{R}$ will be called a weight. We shall denote by $L_{p,\omega,\gamma}(\mathbb{R}^n_+)$ the set of all measurable function $f$ on $\mathbb{R}^n_+$ such that the norm

$$\|f\|_{L_{p,\omega,\gamma}(\mathbb{R}^n_+)} \equiv \|f\|_{p,\omega,\gamma;\mathbb{R}^n_+} = \left( \int\limits_{\mathbb{R}^n_+} |f(x)|^p \omega(x) x_n^\gamma dx \right)^{1/p}, \qquad 1 \le p < \infty$$

is finite. For $\omega = 1$ the space $L_{p,\omega,\gamma}(\mathbb{R}^n_+)$ is denoted by $L_{p,\gamma}(\mathbb{R}^n_+)$, and the norm $\|f\|_{L_{p,\omega,\gamma}(\mathbb{R}^n_+)}$ by $\|f\|_{L_{p,\gamma}(\mathbb{R}^n_+)}$.

The operator of generalized shift ($B_n$ shift operator) is defined by the following way (see [8], [11]):

$$T^y f(x) = C_\gamma \int\limits_0^\pi f\left(x' - y', \sqrt{x_n^2 - 2x_n y_n \cos\alpha_n + y_n^2}\right) \sin^{\gamma-1}\alpha d\alpha,$$

where $C_\gamma = \pi^{-\frac{1}{2}} \Gamma\left(\gamma + \frac{1}{2}\right) \Gamma^{-1}(\gamma)$.

Note that this shift operator is closely connected with $B_n$ Bessel's singular differential operators (see [8], [11]).

**Definition 1.** *A function $K$ defined on $\mathbb{R}^n_+$, is said to be $B_n$ singular kernel in the space $\mathbb{R}^n_+$ if*

*i) $K \in C^\infty(\mathbb{R}^n_+)$ ;*

*ii) $K(rx) = r^{-n-\gamma} K(x)$ for each $r > 0$, $x \in \mathbb{R}^n_+$;*

*iii) $\int\limits_{\Sigma_+} K(x) x_n^\gamma d\sigma(x) = 0$ , where $d\sigma$ is the element of area of the $\Sigma_+$.*

**Definition 2.** *The weight function* $\omega$ *belongs to the class* $A_{p,\gamma}(\mathbb{R}^n_+)$ *for* $1 < p < \infty$, *if*

$$\sup_{x\in\mathbb{R}^n_+, r>0} |E_+(x,r)|_\gamma^{-1} \int\limits_{E_+(x,r)} \omega(y) y_n^\gamma dy \left( |E_+(x,r)|_\gamma^{-1} \int\limits_{E_+(x,r)} \omega^{-\frac{1}{p-1}}(y) y_n^\gamma dy \right)^{p-1} < \infty$$

*and* $\omega$ *belongs to* $A_{1,\gamma}(\mathbb{R}^n_+)$, *if there exists a positive constant* $C$ *such that for any* $x \in \mathbb{R}^n_+$ *and* $r > 0$

$$|E_+(x,r)|_\gamma^{-1} \int\limits_{E_+(x,r)} w^{-\frac{1}{p-1}}(y) y_n^\gamma dy \le C \operatorname*{ess\,inf}_{y\in E_+(x,r)} \omega(y).$$

The properties of the class $A_{p,\gamma}(\mathbb{R}^n_+)$ are analogous to those of the B.Muckenhoupt classes. In particular, if $w \in A_{p,\gamma}(\mathbb{R}^n_+)$, then $w \in A_{p-\varepsilon,\gamma}(\mathbb{R}^n_+)$ for a certain sufficiently small $\varepsilon > 0$ and $w \in A_{p_1,\gamma}(\mathbb{R}^n_+)$ for any $p_1 > p$.

Note that, $|x|^\alpha \in A_{p,\gamma}(\mathbb{R}^n_+)$, $1 < p < \infty$, if and only if $-(n+\gamma) < \alpha < (n+\gamma)(p-1)$ and $|x|^\alpha \in A_{1,\gamma}(\mathbb{R}^n_+)$, if and only if $-(n+\gamma) < \alpha \le 0$.

First, we establish the boundedness in weighted $L_p$ spaces for a large class of sublinear operators, generated by the $B_n$ Bessel differential operators.

**Theorem 1.** *Let* $T$ *be a sublinear operator such that, for any* $f \in L_{1,\gamma}(\mathbb{R}^n_+)$ *with compact support and* $x \notin suppf$

$$|Tf(x)| \le c_0 \int\limits_{\mathbb{R}^n_+} T^y|x|^{-n-\gamma}\ |f(y)|\ y_n^\gamma dy, \tag{1}$$

*where* $c_0$ *is independent of* $f$ *and* $x$. *Let* $\omega$ *be a positive function for which there exists a constant* $c_1 > 0$ *such that*

$$\sup_{2^{k-2}\le|x|<2^{k+1}} \omega(x) \le c_1 \inf_{2^{k-2}\le|x|<2^{k+1}} \omega(x), \quad k \in \mathbb{Z}. \tag{2}$$

*Then the following statement hold:*

*(a) If* $T$ *is of strong type* $L_{p,\gamma}(\mathbb{R}^n_+)$, $p \in (1,\infty)$, *a.e. there exists a constant* $c_2$, *independent of* $f$, *such that for all* $f \in L_{p,\gamma}(\mathbb{R}^n_+)$

$$\int\limits_{\mathbb{R}^n_+} |Tf(x)|^p x_n^\gamma dx \le c_2 \int\limits_{\mathbb{R}^n_+} |f(x)|^p x_n^\gamma dx$$

*and* $\omega \in A_{p,\gamma}(\mathbb{R}^n_+)$, *then* $T$ *is of strong type* $L_{p,\omega,\gamma}(\mathbb{R}^n_+)$, *a.e. there exists a constant* $c_3$, *independent of* $f$, *such that for all* $f \in L_{p,\omega,\gamma}(\mathbb{R}^n_+)$

$$\int\limits_{\mathbb{R}^n_+} |Tf(x)|^p \omega(x) x_n^\gamma dx \le c_2 \int\limits_{\mathbb{R}^n_+} |f(x)|^p \omega(x) x_n^\gamma dx.$$

*(b) If $T$ is of weak type $L_{p,\gamma}(\mathbb{R}^n_+)$, $p\in[1,\infty)$, a.e. there exists a constant $c_4$, independent of $f$, such that for all $f\in L_{p,\gamma}(\mathbb{R}^n_+)$*

$$\int\limits_{\{x\in\mathbb{R}^n:\,|Tf(x)|>\lambda\}} x_n^\gamma dx \le \frac{c_3}{\lambda^p}\int\limits_{\mathbb{R}^n_+}|f(x)|^p x_n^\gamma dx$$

*and $\omega\in A_{p,\gamma}(\mathbb{R}^n_+)$, then $T$ is of weak type $L_{p,\omega,\gamma}(\mathbb{R}^n_+)$, a.e. there exists a constant $c_5$, independent of $f$, such that for all $f\in L_{p,\omega,\gamma}(\mathbb{R}^n_+)$*

$$\int\limits_{\{x\in\mathbb{R}^n:\,|Tf(x)|>\lambda\}} \omega(x)x_n^\gamma dx \le \frac{c_3}{\lambda^p}\int\limits_{\mathbb{R}^n_+}|f(x)|^p\omega(x) x_n^\gamma dx.$$

**Proof.** We proof this theorem along the same line as the proof of Theorem 1. in [14]. Throughout this paper, for $k\in\mathbb{Z}$ we define

$$E_k=\{x\in\mathbb{R}^n_+:\ 2^{k-1}\le|x|<2^k\}$$

and

$$E_k^*=\{x\in\mathbb{R}^n_+:\ 2^{k-2}\le|x|<2^{k+1}\}.$$

If $\omega$ satisfies (2) and we set

$$m_k=\inf\{\omega(x):x\in E_k^*\},$$

then

$$\omega(x)\sim m_k \quad \text{for every} \quad x\in E_k^*.$$

Here the expression $A\sim B$ means, as usual, that there are constants $\tau_0$, $\tau_1$ (independent of the main parameters involved) such that $\tau_0\le A/B\le\tau_1$. We will only prove part $(b)$, since the proof of part $(a)$ is similar.

Given $f\in L_{p,\omega,\gamma}(\mathbb{R}^n_+)$, we write

$$|Tf(x)|=\sum_{k\in Z}|Tf(x)|\chi_{E_k}(x)\le\sum_{k\in Z}|Tf_{k,1}(x)|\,\chi_{E_k}(x)+$$

$$+\sum_{k\in Z}|Tf_{k,2}(x)|\,\chi_{E_k}(x)\equiv T_1f(x)+T_2f(x),$$

where $\chi_{E_k}$ is the characteristic function of the set $E_k$, $f_{k,1}=f\chi_{E_k^*}$ and $f_{k,2}=f-f_{k,1}$.

By the weak type $L_{p,\gamma}(\mathbb{R}^n_+)$ boundedness of $T$ and (1), on $T_1$, we have

$$\omega(\{x\in\mathbb{R}^n_+:|T_1f(x)|>\lambda\})=\sum_{k\in\mathbb{Z}}\omega(\{x\in E_k:|T_1f(x)|>\lambda\})$$

$$\sim\sum_{k\in\mathbb{Z}}m_k|\{x\in E_k:|T_1f(x)|>\lambda\}|\le\sum_{k\in\mathbb{Z}}\frac{c_4}{\lambda^p}\int\limits_{E_k^*}|f(x)|^p m_k x_n^\gamma dx$$

$$\sim \sum_{k\in\mathbb{Z}} \frac{c_5}{\lambda^p} \int\limits_{E_k^*} |f(x)|^p \omega(x)\ x_n^\gamma dx.$$

On $T_2$, we first note that

$$\frac{1}{4}(|x|+|y|) \le |x-y| \le |x|+|y|, \quad x\in E_k, \quad \text{and} \quad y\notin E_k^*,$$

and by (1), we obtain

$$T_2 f(x) \le c_0 \sum_{k\in\mathbb{Z}} \left( \int\limits_{\mathbb{R}_+^n} T^y |x|^{-n-\gamma}\ |f_{k,2}(y)| y_n^\gamma dy \right) \chi_{E_k}(x) \le$$

$$\le c_0 \sum_{k\in\mathbb{Z}} \left( \int\limits_{\mathbb{R}_+^n\setminus E_k^*} |x-y|^{-n-\gamma} |f(y)|\ y_n^\gamma dy \right) \chi_{E_k}(x) \le$$

$$\le 4^{n+\gamma} c_0 \sum_{k\in\mathbb{Z}} \int\limits_{\mathbb{R}_+^n} (|x|+|y|)^{-n-\gamma} |f(y)|\ y_n^\gamma dy \le$$

$$\le 4^{n+\gamma} c_0 |x|^{-n-\gamma} \int\limits_{\{y\in\mathbb{R}_+^n : |y|\le|x|\}} |f(y)|\ y_n^\gamma dy +$$

$$+4^{n+\gamma} c_0 \int\limits_{\{y\in\mathbb{R}_+^n : |y|>|x|\}} |f(y)|\ |y|^{-n-\gamma}\ y_n^\gamma dy \equiv$$

$$\equiv 4^{n+\gamma} c_0 (A_1 f(x) + A_2 f(x)).$$

Let

$$M_\mu f(x) = \sup_{r>0} \mu(E(x,r))^{-1} \int\limits_{E(x,r)} |f(y)| d\mu(y).$$

Here $E(x,r) = \{y\in\mathbb{R}_+^n\ :\ |x-y|<r\}$.

We have

$$A_1 f(x) \le |x|^{-n-\gamma} \int\limits_{\{y\in\mathbb{R}_+^n : |x-y|\le 2|x|\}} |f(y)|\ y_n^\gamma dy \le 2^{n+\gamma} M_\mu f(x).$$

It is well known that the maximal function $M_\mu$ is weak type (1, 1) and is bounded on $L_p(X, d\mu)$ for $1<p<\infty$ (see [1]). Here we are concerned with the maximal operator defined by $d\mu(x) = x_n^\gamma dx$. It is clear that this measure satisfies the doubling condition

$$\mu(E(x,2r)) \le c_6 \mu(E(x,r))$$

with a constant $c_6$ independent of $x$ and $r>0$.

Therefore $A_1$ satisfies the conclusion of the theorem. By a duality argument, $A_2$ satisfies the same conclusion if $p \in (1,\infty)$. It remains to show that $A_2$ is of weak type $L_{1,\omega,\gamma}(\mathbb{R}^n_+)$, if $\omega \in A_{1,\gamma}(\mathbb{R}^n_+)$. Given $\lambda > 0$, let

$$R \equiv R_\lambda = \sup \left\{ r > 0 : \int\limits_{\{y \in \mathbb{R}^n_+ : |y| \geq r\}} |f(y)||y|^{-n-\gamma}\, y_n^\gamma dy > \lambda / c_0 \right\}.$$

Then

$$\omega(\{x \in \mathbb{R}^n_+ : |A_2 f(x)| > \lambda\}) = \omega(\{x \in \mathbb{R}^n_+ : |x| \leq R\}) \leq$$

$$\leq \frac{c_0}{\lambda} \int\limits_{|y| \geq R} |f(y)||y|^{-n-\gamma}\, y_n^\gamma dy\ \omega(\{x \in \mathbb{R}^n_+ : |x| \leq R\}) \leq$$

$$\leq \frac{c_0}{\lambda} \int\limits_{|y| \geq R} |f(y)||y|^{-n-\gamma}\ \omega(\{x \in \mathbb{R}^n_+ : |x| \leq |y|\})\ y_n^\gamma dy \leq$$

$$\leq \frac{c}{\lambda} \int\limits_{|y| \geq R} |f(y)| \inf_{|x| \leq |y|} \omega(x)\ y_n^\gamma dy \leq \frac{c}{\lambda} \int\limits_{\mathbb{R}^n_+} |f(y)|\ \omega(y)\ y_n^\gamma dy.$$

This finishes the proof of Theorem 1.

Let $K$ is a $B_n$ singular kernel and $T$ be the $B_n$ singular integral operator

$$Tf(x) = p.v. \int\limits_{\mathbb{R}^n_+} T^y K(x) f(y) y_n^\gamma dy.$$

Then $T$ satisfies the condition (1). Thus, we have

**Corollary 2.** *Let $p \in (1,\infty)$, $T$ be the $B_n$ singular integral operator. Moreover, let $\omega(x)$ be weight function on $\mathbb{R}^n_+$ satisfies condition (2) and $\omega \in A_{p,\gamma}(\mathbb{R}^n_+)$, then $T$ is of strong type $L_{p,\omega,\gamma}(\mathbb{R}^n_+)$.*

**Corollary 3.** *Let $p \in [1,\infty)$, $T$ be the $B_n$ singular integral operator. Moreover, let $\omega(x)$ be weight function on $\mathbb{R}^n_+$ satisfies condition (2) and $\omega \in A_{p,\gamma}(\mathbb{R}^n_+)$, then $T$ is of weak type $L_{p,\omega,\gamma}(\mathbb{R}^n_+)$.*

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