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ON CLOSE BASES FROM SUBSPACES

Abstract

Some problems of close criterian bases from subspaces of Banach spaces are considered in the paper. Basicity tests are given and some sufficient conditions of basicity of close systems from subspaces that contain the earlier known results consering systems of elements, are cited.

Introduction

By studying spectral properties of linear operators by means of Riesz projectors we have to study basis properties of systems from subspaces.

We can say that problems of bases from the elements of Banach spaces have been well studied and many monographs devoted to these problems [1,2]. Naturally, there are many methods to establish the basicity of concrete systems. Use of closeness in this or other sense is one of these methods. These problems were studied and treated for ordinary systems, for example, in the papers [1-6]. To the author's mind such problems for systems from subspaces have not been sufficiently researched. Some results are cited in the monograph [7] (Hilbert case) and [2] (Banach case).

It should be noted that in all these papers the closeness of subspaces is determined by means of appropriate projection or unit balls of these subspaces. We have defined the notion of generating operator of a subspace and closeness is given through it. This allows to get many analogies of theorems for systems.

The present work is devoted to the above-noted problems in Banach spaces. Some results obtained in the paper are new even in Hilbert case. Let's give some designation and notion that we'll need in future. Notice that some notion and facts connected with theorem 1 are in the book [2]. For explicitness we state the part concerning projectors.

1. Basic notion and facts

B is a Banach space with norm;
$\mathcal{L}(B)$ is algebra of bounded operators from B to B;
$\Pi(B) \subset \mathcal{L}(B)$ is a set of projectors;
$R(B) \subset \mathcal{L}(B)$ is a class of all finitedimensional operators;
$L[M]$ is a linear span of the set M;
$\bar{M}$ is closure of the set M in B;
Let $\{B_n\}_{n\in N} \subset B$ be a sequence.

Definition 1. *$\{B_n\}_{n\in N}$ is said to be complete in B, if $\overline{L\left[\{B_n\}_{n\in N}\right]} = B$.*

Definition 2. *$\{B_n\}_{n\in N}$ is said to be minimal in B, if*

$$\overline{B_n} \cap \overline{L\left[\{B_k\}_{k\neq n}\right]} = \{0\}, \quad \forall n \in N,$$

Definition 3. *$\{B_n\}_{n\in N}$ is said to be basis in B if for*

$$\forall x \in B, \quad \exists!\, \{x_n\}_{n\in N} : x_n \in \overline{B_n}, \quad \forall n \in N : x = \sum_n x_n.$$

It is obvious that completeness and minimality follows from basicity.

Let $\{B_n\}_{n\in N}$ be a basis in B.

Let's consider the space $\hat{B} = \left\{ \hat{x} = \{x_n\}_{n\in N} : x_n \in B_n,\ \forall n \in N \wedge \exists \sum\limits_n x_n \right\}$ with norm $\|\hat{x}\|_{\hat{B}} = \sup_m \left\| \sum\limits_l^m x_n \right\|$. By ordinary operations of addition and multiplication, $\hat{B}$ turns into normed space.

Let $\{\hat{x}_n\}_{n\in N}$ be a sequence fundamental in $\hat{B}$: $\|\hat{x}_m - \hat{x}_n\|_B \to 0,\ \ m,n \to \infty$. It is clear that $\left(\hat{x}_n = \{x_k^n\}_{k\in N}\right)$

$$\|x_n\| \le 2\,\|\hat{x}\|_{\hat{B}}\,, \forall n \in N$$

Thus

$$\|x_k^m - x_k^n\| \le 2\,\|\hat{x}_m - \hat{x}_n\|_{\hat{B}} \to 0, \quad m,n \to \infty$$

for $\forall k \in N$. Then $x_k^n \to x_k, \quad n \to \infty$. It follows from $\overline{B_n} = B_n$ that $x_k \in B_k$ for $\forall k \in N$. Let $\varepsilon > 0$ be an arbitrary number. Then $\exists n_0$:

$$\|\hat{x}_m - \hat{x}_n\|_{\hat{B}} < \varepsilon, \quad \forall m,n \ge n_0.$$

Thus

$$\left\| \sum_{k=1}^{l} \left(x_k^m - x_k^n\right) \right\| \le \varepsilon, \quad \forall m,n \ge n_0; \quad \forall l \in N, \tag{1}$$

Here, passing to limit as $n \to \infty$, we have:

$$\left\| \sum_{k=1}^{l} \left(x_k^m - x_k\right) \right\| \le \varepsilon, \quad \forall m \ge n_0; \quad \forall l \in N. \tag{2}$$

Let

$$S_l^m = \sum_{k=1}^{l} x_k^m,\ \ S_l = \sum_{k=1}^{l} x_k$$

We have:

$$\|S_{l+p} - S_l\| = \left\|S_{l+p} - S_{l+p}^m\right\| + \left\|S_{l+p}^m - S_l^m\right\| + \|S_l^m - S_l\|$$

It follows from (2)

$$\|S_{l+p} - S_l\| \le 2\varepsilon + \left\|S_{l+p}^m - S_l^m\right\|, \quad \forall m \ge n_0; \quad \forall l,p \in N.$$

Take $\delta > 0$ and $\varepsilon < \frac{\delta}{4}$. For each fixed $m \ge n_0(\varepsilon)$, by virtue of convergence of series $\sum\limits_{k=1}^{\infty} x_k^m$, we take $l_0 \in N$ so that for $\forall l \ge l_0$ and $\forall p \in N$ the inequality

$$\left\|S_{l+p}^m - S_l^m\right\| < \frac{\delta}{2}.$$

would hold.

As a result

$$\|S_{l+p} - S_l\| < \delta,\ \forall l \ge l_0,\ \forall p \in N.$$

So S_l converges in B, i.e. $\hat{x} = \{x_k\}_{k\in N} \in \hat{B}$. It follows from (2) that $\|\hat{x}_m - \hat{x}\|_{\hat{B}} < \varepsilon$, $\forall m \geq n_0$, i.e. $\|\hat{x}_m - \hat{x}\|_{\hat{B}} \to 0$, $m \to \infty$. Thus, $\hat{B}$ is a Banach space.

It is clear that to each $\hat{x} \in \hat{B}$ there corresponds $x \in B : x = A\hat{x} = \sum\limits_n x_n$, $\hat{x} = \{x_n\}\, n \in N$. This operator is linear and in a one-to-one manner maps $\hat{B}$ onto B. Moreover:

$$\|A\hat{x}\| = \|x\| = \left\|\sum_n x_n\right\| \leq \sup_m \left\|\sum_1^m x_n\right\| = \|\hat{x}\| . \tag{3}$$

Then, by Banach theorem $A^{-1} \in \mathcal{L}\left(B, \hat{B}\right)$, i.e.

$$\|\hat{x}\|_{\hat{B}} = \left\|A^{-1}x\right\|_{\hat{B}} \leq c\,\|x\| . \tag{4}$$

It follows from the uniqueness of expansion that for $\forall n \in N$, $\exists P_n : B \to B_n :$ $x_n = P_n x$. Obviously, P_n is a linear operator. But again it follows from uniqueness of expansion that if $x \in B_n$, then $P_k x = \delta_{nk} x$. Let $I_n : B_n \to B_n$ be a unit operator. Thus, $P_n^2 = P_n$, and moreover $P_n P_k = \delta_{nk} P_n$. Consequently, each basis $\{B_n\}_{n\in N}$ generates projectors $\left\{P_n : \mathcal{L}\left(P_n \subset B_n,\, KerP_n \supset \overline{L\left[\{B_k\}_{k\neq n}\right]}\right)\right\}$ that $x = \sum\limits_n P_n x$.

It is obvious that $P_n/_{B_k} = \delta_{nk} I_n$, where $P/_M$ is contraction of the operator P on M. Allowing for inequalities (3), (4) we have:

$$\|P_n x\| = \|x_n\| = \left\|\sum_1^n x_k - \sum_1^{n-1} x_k\right\| \leq 2\,\|\hat{x}\| \leq 2c\,\|x\| .$$

Thus, the projectors $\{P_n\}_{n\in N}$ are uniformly bounded.

The following theorem is true.

Theorem 1. *A sequence from subspaces $\{B_n\}_{n\in N} \subset B$ forms a basis in B if and only if there exists generating sequence of projectors $\{P_n\}$:*

1) *$\{B_n\}_{n\in N}$ is complete in B;*

2) *$P_i P_j = \delta_{ij} P_i$;*

3) *$\left\|\sum\limits_1^m P_n x\right\| \leq M\,\|x\|$, $\forall m \in N$.*

Definition 4. *A sequence of operators $\{T_n\}_{n\in N} \subset \mathcal{L}(B)$ will be said to be generating for $\{B_n\}_{n\in N} \subset B$ if $\overline{T_n(B)} = B_n$, $\forall n \in N$..*

Definition 5. *A sequence of operators $\{T_n\}_{n\in N} \subset \mathcal{L}(B)$ will be said to be qs-system (in the case of basicity qs-basis), if*

$$\sum_{n=1}^{\infty} \|T_n x\|^q \leq M^q\,\|x\|^q , \ \forall x \in B.$$

Definition 6. *A sequence $\left\{T_n^l\right\}_{n\in N} \subset \mathcal{L}(B)$, $l = 1, 2$ is said to be p-close, if*

$$\sum_{n=1}^{\infty} \left\|T_n^1 - T_n^2\right\|^p < +\infty,$$

and vps-close, if

$$\sum_{n=1}^{\infty} \left\| \left(T_n^1 - T_n^2\right) x \right\|^p v_n^p < M^p \left\|x\right\|^p , \ \forall x \in B,$$

where $v = \{v_n : v_n > 0, \ n \in N\}$ *is a sequence.*

Definition 7. $\{T_n\}_{n\in N} \subset \mathcal{L}(B)$ *is said to be vs-system, if*

$$\sum_{n=1}^{\infty} \left\|T_n\right\|^q v_n^{-q} < +\infty,$$

where $\{v_n : v_n > 0, \ n \in N\} \equiv v$ *is a sequence.*

2. Basic results

Lemma 1. *Let* $F \in \mathcal{L}(B)$ *be a Fredholm operator and* $\{B_n\}_{n\in N} \subset B$. *If* $\{FB_n\}_{n\in N}$ *is complete* B, *then* $\{B_n\}_{n\in N}$ *is also complete in* B *and* F *is boundedly inverse.*

Lemma 2. *Let a sequence of projectors* $\{P_n\}_{n\in N} \in \mathcal{L}(B)$ *form a qs-basis in* B, $\dim P_n(B) < +\infty$, $\forall n \in N$ and $\{T_n\}_{n\in N} \in \mathcal{L}(B)$ *p-close to* $\{P_n\}_{n\in N}$, *where* $1 \le p < +\infty$, $\frac{1}{p} + \frac{1}{q} = 1$. *Then the expression*

$$Fx \equiv \sum_{n\ge 1} T_n P_n x \tag{5}$$

generates a Fredholm operator.

Proof. Let's consider $(m > n)$:

$$\left\|\sum_{n}^{m} T_k P_k x\right\| \le \left\|\sum_{n}^{m} \left(T_k P_k - P_k\right) x\right\| + \left\|\sum_{n}^{m} P_k x\right\| = \left\|\sum_{n}^{m} \left(T_k - P_k\right) P_k x\right\| + \left\|\sum_{n}^{m} P_k x\right\| \le$$

$$\le \left(\sum_{n}^{m} \left\|P_k - T_k\right\|^p\right)^{1/p} \left(\sum_{n}^{m} \left\|P_k x\right\|^q\right)^{1/q} + \left\|\sum_{n}^{m} P_k x\right\| \to 0, \ n, m \to \infty$$

Thus, series (5) converges in B. Moreover $Fx = (I + T)x$, where

$$T = \sum_{n\ge 1} \left(T_n P_n - P_n\right) x, \ I \in \mathcal{L}(B)$$

is on identity operator. Let

$$F_m = \sum_{n=1}^{m} \left(T_n P_n\right) P_n x.$$

Obviously, F_m is a finite-dimensional operator for $\forall m \in N$. We have

$$\left\|(T - F_m x)\right\| = \left\|\sum_{m+1}^{m} \left(T_n - P_n\right) P_n x\right\| \le \left(\sum_{m+1}^{\infty} \left\|T_n - P_n\right\|^p\right)^{1/p} \left(\sum_{m+1}^{\infty} \left\|P_n x\right\|^q\right)^{1/q} \le$$

$$\le M \left(\sum_{m+1}^{\infty} \left\|P_n - T_n\right\|^p\right)^{1/p} \left\|x\right\|, \ \forall x \in B.$$

Thus

$$\|T-F_m\| \le M\left(\sum_{m+1}^{\infty}\|T_n-P_n\|^p\right)^{1/p} \to 0,\ m\to\infty.$$

So T is completely continuous and by virtue of that F is a Fredholm operators. The lemma is proved.

The following lemmas are proved similarly.

Lemma 3. *Let $\Pi(B)\supset\{P_n\}_{n\in N}$, $\dim P_n(B)<+\infty$, $\forall n\in N$ and $\{P_n\}_{n\in N}$ form qs-basis in B. If $\{T_n\}_{n\in N}\in L(B): \sum_{n=1}^{\infty}\|T_n-I\|^p<+\infty$, then expression (5) generates a Fredholm operator.*

In fact

$$\left\|\sum_n^m T_kP_kx\right\| \le \left\|\sum_n^m (T_kP_k-P_k)\,x\right\| + \left\|\sum_n^m P_kx\right\| =$$

$$= \left\|\sum_n^m (T_k-I)\,P_kx\right\| + \left\|\sum_n^m P_kx\right\| \le$$

$$\le \left(\sum_n^m \|P_k-I\|^p\right)^{1/p}\left(\sum_n^m \|P_kx\|^q\right)^{1/q} + \left\|\sum_n^m P_kx\right\|.$$

Further reasoning is conducted similar to the proof of lemma 2.

Lemma 4. *Let $\{P_n\}_{n\in N}\subset\Pi(B)$, $\{T_n\}_{n\in N}\subset\mathcal{L}(B): T_nP_n=T_n$ (or $T_n=P_nT_n$) $\forall n\in N$; $\dim P_n(B)<+\infty$, $\forall n\in N$. Then if $\{P_n\}_{n\in N}$ is a basis in B and $\sum_{n\ge1}\|T_n-P_n\|<+\infty$, then*

$$Fx=\sum_{n\ge1}T_nx$$

generates a Fredholm operator. This follows from the relation

$$\left\|\sum T_nx\right\| = \left\|\sum T_nP_nx\right\| \le \sum\|T_n-P_n\|\,\|x\| + \left\|\sum P_nx\right\|.$$

Lemma 5. *Let $\{P_n\}_{n\in N}\subset\Pi(B)$ be vp-basis in B, $\{T_n\}_{n\in N}\subset\mathcal{L}(B)$ is vqs-close to $\{P_n\}_{n\in N}$ and $\dim P_n(B)<+\infty$ or $\dim T_n(B)<+\infty$ for $\forall n\in N$. Then the expression $\left(\frac{1}{p}+\frac{1}{q}=1\right)$*

$$Fx=\sum_{n\ge1}P_nT_nx \tag{6}$$

generates a Fredholm operator.

Indeed,

$$\left\|\sum_n^m P_nT_nx\right\| \le \left\|\sum_n^m (P_kT_k-P_k)\,x\right\| + \left\|\sum_n^m P_kx\right\| =$$

$$= \left\|\sum_n^m P_k\,(T_k-P_k)\,x\right\| + \left\|\sum_n^m P_kx\right\| \le$$

$$\le \left(\sum_n^m \left\|v_k^{-1}P_k\right\|^p\right)^{1/p}\left(\sum_n^m \|v_k\,(T_k-P_k)\,x\|^q\right)^{1/q} + \left\|\sum_n^m P_kx\right\|.$$

Further reasoning is obvious.

Lemma 6. *Let* $\{P_n\}_{n\in N} \subset \Pi(B)$ *form vp-basis in* B, $\{T_n\}_{n\in N} \subset \mathcal{L}(B)$ *satisfy*

$$\left\| \sum_n^m v_k (T_k - I)\, x \right\|^q \leq M^q \|x\|^q ,$$

for $\forall x \in B$, $\forall m \in N$. *Then expression (6) generates a Fredholm operator.*

Before we give the main theorem, introduce one more definition.

Definition 8. *A sequence of operators* $\{T_n\}_{n\in N}$ *is said to be* dB_n*-invariant, if* $\dim T_n B_n = \dim B_n$, $\forall n \in N$.

Theorem 2. *Let* $\{P_n\}_{n\in N} \subset \Pi(B)$ *form ps-basis in* B, $\{T_n\}_{n\in N} \subset \mathcal{L}(B)$ *be q-close to* $\{P_n\}_{n\in N}$ $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$ *and* $\{P_n\}_{n\in N} \subset R(B)$ *or* $\{T_n\}_{n\in N} \subset R(B)$. *If* $\{T_n P_n\}_{n\in N}$ *is complete (minimal and* dB_n*-invariant), then it also forms a basis in* B *isomorphic to* $\{P_n\}_{n\in N}$.

Proof. Let's consider operator (5). By lemma 2 it is of Fredholm property. It is obvious that $FB_n = T_n B_n, \forall n \in N$, where $B_n = P_n(B)$. Thus, it suffices to prove basicity of $\{FB_n\}_{n\in N}$. If $\{T_n P_n\}_{n\in N}$ is complete in B, then by lemma 1 F is boundedly inverse and consequently $\{FB_n\}_{n\in N}$ is a basis in B. Let $\{FB_n\}_{n\in N}$ be minimal and dB_n-invariant for each n, take a basis $\{x_k^n\}_{k=1}^{k_n}$ in B_n. It follows from dB_n-invariance that $\{Fx_k^n\}_{k=1}^{k_n}$ is linearly independent for each $n \in N$. Let F be irreversible. Then $KerF \neq 0$. Take $x \neq 0 : x \in KerF$. It follows from $x \neq 0$ that $\exists n_0 \in N : P_{n_0} x \neq 0$. We have: $0 = Fx = \sum\limits_{n\geq 1} FP_n x$ or $FP_{n_0} x = -\sum\limits_{n\neq n_0} FP_n x$. Expand $P_{n_0} x$ in basis $\left\{x_k^{n_0}\right\} \subset B_{n_0} : P_{n_0} x = \sum\limits_{k=1}^{k_{n_0}} a_k x_k^{n_0}$. From minimality $\{FB_n\}_{n\in N}$ we get $FP_{n_0} x = 0$. On the other hand it follows from $P_{n_0} x \neq 0$ that $\exists k_0 : a_{k_0} \neq 0$. Then $Fx_{k_0}^{n_0} = -\frac{1}{a_{k_0}} \sum\limits_{k\neq k_0} a_k F x_k^{n_0}$ i.e. the system $\left\{Fx_{k_0}^{n_0}\right\}_{k=1}^{n_0}$ is linearly dependent. We get contradiction. As a result F is boundedly inverse.

The theorem is proved.

Remark. It is obvious that in a similar way using lemmas 1-6 we can give other sufficient conditions for basicity of close systems.

References

[1]. Zinger I. Bases in Banach Spaces I, SVBHNY, 1970, 672 p.

[2]. Zinger I. Bases in Banach Spaces II, Romania, SVBHNY, 1981, 880 p.

[3]. Milman V.D. *Geometrical theory of Banach spaces.* Uspekhi mat. nauk. P.I, 1970, v.XXV, issue 3(15), pp.113-174. (Russian)

[4]. Hochberg I.Ts., Markus A.S. *On stability of bases of Banach and Hilbert spaces.* Izv. AN Mold. SSR, 1962, No5, pp.17-35 (Russian)

[5]. BarryN.K. *Biorthogonal systems and bases in Hilbert space.* Ucheniye zapiski: MGU, 1951, v.4, issue 148, pp.69-107. (Russian)

[6]. Bilalov B.T. *Bases from exponents, cosines and sines being eigen-functions of differential operators.* Diff. Uravn., 2003, v.35, No5, pp.1-5. (Russian)

[7]. Hochberg I.Ts., Krein M.G. *Introduction to the theory of linear not self-adjoint operators.* M.: "Nauka", 1965. (Russian)

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