Rovshan A. BANDALIEV, Marifa M. ABBASOVA

ON AN INEQUALITY AND $p(x)$-MEAN CONTINUITY IN THE VARIABLE LEBESGUE SPACE WITH MIXED NORM

Abstract

The main purpose of this paper is to prove the analog of generalized Minkowski type integral inequality and to applied of these inequality to the boundedness of Hardy operator in weighted Lebesgue spaces with variable.

The first systematic study of spaces with variable exponent(called modular spaces) is due to Nakano [1]. Somewhat later, a more explicit version of such spaces, namely modular function spaces, were investigated by Musielak and others Polish mathematicians (see [2]). In particular, the Lebesgue space with variable exponent were an object of interest during the last two decades(see [3]-[5]). The first investigation of these spaces being undertaken in [3]. The study of these spaces has been stimulated by problems of elasticity, fluid dynamics, calculus of variations and differential equations with non-standard growth conditions.

Let R^n n-dimensional Euclidean space of points $x=(x_1,...,x_n)$ and $\Omega\subset R^n$ be a measurable set in the sense of Lebesgue. Suppose that $\mathbf{p}(x)=\big(p_1(x_1,...,x_n),$ $p_2(0,x_2,...,x_n),...,p_n(0,...,0,x_n)\big)$-is a vector function defined on R^n with measurable components $p_i(x^{(i)})$, such that $1\le p_i(x^{(i)})<\infty$ and $x^{(i)}=(0,...,0,x_i,...,x_n)$ $(i=1,...,n)$. Further in this paper all sets and functions are supposed measurable in the sense of Lebesgue and $x^{(1)}=x$, $x^{(n)}=x_n$.

By $L_{p_1(x),x_1}(R^n)$ we denote the space of all measurable functions on R^n such that

$$\int\limits_R |f(x)|^{p_1(x)}\,dx_1<\infty,$$

where $x^{(2)}\in R^{n-1}$ is a fixed point.

The expression

$$\|f\|_{L_{p_1(x),x_1}(R)}=\|f\|_{p_1,\ x_1}=\inf\left\{\lambda>0:\int\limits_R\left|\frac{f(x)}{\lambda}\right|^{p_1(x)}dx_1\le 1\right\}$$

is the norm in $L_{p_1(x),x_1}(R)$ (see.[1]). It is obvious that $\|f\|_{p_1,\ x_1}=\|f\|_{p_1,\ x_1}(x^{(2)})$.

Further, by $L_{(p_1(x),p_2(x^2)),x_1,x_2}(R^n)$ we denote the space of all measurable functions on R^n such that

$$\int\limits_R\left(\|f\|_{p_1,\ x_1}(x^{(2)})\right)^{p_2(x^{(2)})}dx_2\ <\infty$$

where $x^{(3)}\in R^{n-2}$ is a fixed point. The functional

$$\|f\|_{L_{(p_1(x),p_2(x^2)),x_1,x_2}(R^n)}=\Big\|\|f\|_{p_1,x_1}\Big\|_{p_2,x_2}$$

$$= \inf \left\{ \mu > 0 : \int\limits_R \left(\frac{\|f\|_{p_1,\ x_1} (x^{(2)})}{\mu} \right)^{p_2(x^{(2)})} dx_2 \le 1 \right\}$$

defines the norm in $L_{(p_1(x), p_2(x^2)), x_1, x_2}(R^n)$.
Finally, by the induction we have

Definition. *By* $L_{\mathbf{p}(x)}(R^n) = L_{\left(p_1(x),\ p_2\left(x^{(2)}\right), ..., p_n(x_n)\right)}(R^n)$ *we denote the space of measurable functions* f *on* R^n *such that*

$$\rho_{\mathbf{p}}(f) = \int\limits_R \left(\left\| ... \left\| \|f\|_{p_1, x_1} \right\|_{p_2, x_2} ... \right\|_{p_{n-1}, x_{n-1}} (x_n) \right)^{p_n(x_n)} dx_n < \infty.$$

The functional

$$\|f\|_{L_{\mathbf{p}(x)}(R^n)} = \|f\|_{L_{\left(p_1(x),\ p_2\left(x^{(2)}\right), ...,\ p_n(x_n)\right)}(R^n)} =$$

$$= \left\| ... \left\| \|f\|_{p_1, x_1} \right\|_{p_2, x_2} ... \right\|_{p_n, x_n} = \|f\|_{\mathbf{p}} =$$

$$= \inf \left\{ \nu > 0 : \int\limits_R \left(\frac{\left\| ... \left\| \|f\|_{p_1,\ x_1} \right\|_{p_2,\ x_2} ... \right\|_{p_{n-1},\ x_{n-1}} (x_n)}{\nu} \right)^{p_n(x_n)} dx_n \le 1 \right\}$$

defines a norm in $L_{\mathbf{p}(x)}(R^n)$ *(see [6]).*

Suppose that $\Omega \subset R^n$ is a measurable set, and f is a measurable function defined in Ω. In this case the norm in the space $L_{\mathbf{p}(x)}(\Omega)$ with mixed norm is defined as

$$\|f\|_{L_{\mathbf{p}(x)}(\Omega)} = \|f \chi_\Omega\|_{L_{\mathbf{p}(x)}(R^n)} = \|f\|_{\mathbf{p}},$$

where $\chi_\Omega(x)$ is a characteristic function of a set Ω.

Corollary 1. *Suppose that* $\mathbf{p}(x) = (p_1, ..., p_n) = \mathbf{p}$, *i.e., if* $p_i(x^{(i)}) = p_i = const$, $i = 1, ..., n$, *then* $L_{\mathbf{p}(x)}(R^n)$ *oincides with the* $L_{\mathbf{p}}(R^n)$ *space of mixed norm.*

Proof. Indeed, by the definition, we have

$$\|f\|_{p_1,\ x_1} = \inf \left\{ \lambda > 0 : \int\limits_R \left| \frac{f(x)}{\lambda} \right|^{p_1} dx_1 \le 1 \right\} =$$

$$= \inf \left\{ \lambda > 0 : \lambda \ge \left(\int\limits_R |f(x)|^{p_1} dx_1 \right)^{1/p_1} \right\} = \left(\int\limits_R |f(x)|^{p_1} dx_1 \right)^{1/p_1}.$$

where $x^{(2)} \in R^{n-1}$ is a fixed point. Further, if a point $x^{(3)} \in R^{n-2}$ is fixed, then

$$\left\| \|f\|_{p_1, x_1} \right\|_{p_2, x_2} = \inf \left\{ \mu > 0 : \int\limits_R \frac{\left(\int\limits_R |f(x)|^{p_1} dx_1 \right)^{p_2/p_1}}{\mu^{p_2}} dx_2 \le 1 \right\} =$$

$$= \inf \left\{ \mu > 0 : \mu \geq \left(\int\limits_R \left(\int\limits_R |f(x)|^{p_1} dx_1 \right)^{p_2/p_1} dx_2 \right)^{1/p_2} \right\} =$$

$$= \left(\int\limits_R \left(\int\limits_R |f(x)|^{p_1} dx_1 \right)^{p_2/p_1} dx_2 \right)^{1/p_2} .$$

Finally, we get

$$\|f\|_{\mathbf{p}} = \left\| ... \left\| \|f\|_{p_1,x_1} \right\|_{p_2,x_2} ... \right\|_{p_n,x_n} =$$

$$= \inf \left\{ \nu > 0 : \int\limits_R \frac{\int\limits_R \left[... \left(\int\limits_R \left(\int\limits_R |f(x)|^{p_1} dx_1 \right)^{p_2/p_1} dx_2 \right)^{p_3/p_2} ... dx_{n-1} \right]^{p_n/p_{n-1}}}{\nu^{p_n}} dx_n \leq 1 \right\} =$$

$$= \left\{ \int\limits_R \left[... \left(\int\limits_R \left(\int\limits_R |f(x)|^{p_1} dx_1 \right)^{p_2/p_1} dx_2 \right)^{p_3/p_2} ... \right]^{p_n/p_{n-1}} dx_n \right\}^{1/p_n} .$$

Corollary 1 is proved.

Remark 1. *Note that for $p_i(x^{(i)}) = p_i = const$ $(i = 1, ..., n)$ the main properties of $L_{\mathbf{p}(x)}(R^n) = L_{(p_1(x), p_2(x^{(2)}), ..., p_n(x_n))}(R^n)$ space, first were studied in [1].*

Corollary 2. *Let $p_1(x_1, ...x_n) = p_2(x_2, ..., x_n) = ... = p_n(x_n) = p(x_n)$, i.e.,if $\mathbf{p}(x) = (p(x_n), ..., p(x_n))$, then $L_{\mathbf{p}(x)}(R^n) = L_{p(x_n)}(R^n)$.*

Proof. By the definition, we have

$$\|f\|_{p_1,\, x_1} = \inf \left\{ \lambda > 0 : \lambda \geq \left(\int\limits_R |f(x)|^{p(x_n)} dx_1 \right)^{1/p(x_n)} \right\} =$$

$$= \left(\int\limits_R |f(x)|^{p(x_n)} dx_1 \right)^{1/p(x_n)} ,$$

where a point $x^{(2)} \in R^{n-1}$ is fixed.

Analogously, at the fixed point $x^{(3)} \in R^{n-2}$, we get

$$\left\| \|f\|_{p_1,x_1} \right\|_{p_2,x_2} = \inf \left\{ \mu > 0 :\ \mu \geq \left(\int\limits_R \left(\int\limits_R |f(x)|^{p(x_n)} dx_1 \right) dx_2 \right)^{1/p(x_n)} \right\} =$$

$$= \left(\int\limits_{R^2} |f(x)|^{p(x_n)} dx_1 dx_2 \right)^{1/p(x_n)} .$$

Therefore, we get

$$\|f\|_{L_{\mathbf{p}(x)}(R^n)} = \left\|\ldots\left\|\|f\|_{p,x_1}\right\|_{p,x_2}\ldots\right\|_{p,x_n} =$$

$$= \inf\left\{\nu>0: \int\limits_R \left(\frac{\int\limits_{R^{n-1}} |f(x)|^{p(x_n)}\,dx_1dx_2...dx_{n-1}}{\nu^{p(x_n)}}\right) dx_n \le 1\right\} =$$

$$= \inf\left\{\nu>0: \int\limits_R \left(\int\limits_{R^{n-1}} \left|\frac{f(x)}{\nu}\right|^{p(x_n)} dx_1dx_2...dx_{n-1}\right)\, dx_n \le 1\right\}$$

$$= \inf\left\{\nu>0: \int\limits_{R^n} \left|\frac{f(x)}{\nu}\right|^{p(x_n)} dx \le 1\right\} = \|f\|_{L_{p(x_n)}(R^n)}\,.$$

Corollary 2 is proved.

Remark 2. *Note that for $n=1$, $x\in[0,1]$ and for $n>1$, $x\in\Omega\subset R^n$ main properties of $L_{\mathbf{p}(x)}(R^n) = L_{(p(x_n),p(x_n),...,p(x_n))}(R^n) = L_{p(x_n)}(R^n)$ space were investigated in [2] and in [3], respectively.*

Lemma 1. (Generalized Holder inequality) *Let $\mathbf{1}\le\mathbf{p}(x)<\infty$, $\mathbf{p}'(x) = \left(p_1'(x), p_2'(x^{(2)}),...,p_n'(x_n)\right)$ and $f_1\in L_{\mathbf{p}(x)}(R^n)$, $f_2\in L_{\mathbf{p}'(x)}(R^n)$ and $\dfrac{1}{\mathbf{p}(x)}+\dfrac{1}{\mathbf{p}'(x)}=1$, i.e., $\dfrac{1}{p_i(x^{(i)})}+\dfrac{1}{p_i'(x^{(i)})}=1$ and $i=1,...,n$. Then $f_1f_2\in L_1(R^n)$ and the inequality*

$$\int\limits_{R^n} |f_1(x)f_2(x)|\ dx \le \prod_{i=1}^n \left(\frac{1}{\underline{p}_i}+\frac{1}{\underline{p}_i'}\right)\|f_1\|_{\mathbf{p}}\,\|f_2\|_{\mathbf{p}'} \tag{1}$$

is valid.

Proof of the Lemma. We assume that $\|f_i\|_p\ne0$ and $i=1,2$, otherwise the inequality (1) is trivial. Let $\mathbf{1}<\mathbf{p}(x)<\infty$. It is known that, for any fixed point $x\in R^n$ the inequality

$$ab \le \frac{r_1(x)}{p_1(x)}\,a^{p_1(x)/r_1(x)} + \frac{r_2(x)}{p_2(x)}\,\frac{b^{p_2(x)}/r_2(x)}{,}\quad a,b\ge0$$

is valid. Taking in the last inequality $a=\dfrac{|f_1(x|}{\|f_1\|_{p_1,x_1}}$ and $b=\dfrac{|f_2(x)|}{\|f_2\|_{p_1',x_1}}$ we have

$$\frac{|f_1(x)\ f_2(x)|}{\|f_1\|_{p_1,x_1}\|f_2\|_{p_1',x_1}} \le \frac{1}{p_1(x)}\left|\frac{f_1(x)}{\|f_1\|_{p_1,x_1}}\right|^{p_1(x)} + \frac{1}{p_1'(x)}\left|\frac{f_2(x)}{\|f_2\|_{p_1',x_1}}\right|^{p_1'(x)} \le$$

$$\le \frac{1}{\underline{p}_1}\left|\frac{f_1(x)}{\|f_1\|_{p_1,x_1}}\right|^{p_1(x)} + \frac{1}{\underline{p}_1'}\left|\frac{f_2(x)}{\|f_2\|_{p_1',x_1}}\right|^{p_1'(x)}. \tag{2}$$

Integrating inequality (2) by variable x_1 and using the definition for a fixed point $x^{(2)} \in R^{n-1}$, we have

$$\int\limits_R |f_1(x)f_2(x)|\ dx_1 \le \left(\frac{1}{\underline{p}_1}+\frac{1}{\underline{p}'_1}\right)\|f_1\|_{p_1,x_1}\|f_2\|_{p'_1,x_1}. \tag{3}$$

Since

$$\int\limits_{R^n} |f_1(x)\ f_2(x)|\ dx = \int\limits_R ... \int\limits_R \left(\int\limits_R |f_1(x)\ f_2(x)|\ dx_1\right) dx_2...dx_n,$$

then apply the inequality (3) by each variables individually, we obtain the inequality (1).

Note that in the cases $\mathbf{p}(x)=\mathbf{1}$ and $\mathbf{p}'(x)=\infty$ the inequality (1) is trivial.

Theorem 1 is proved.

If $p_i(x^{(i)}) = p_i = const,\ i=1,...,n$, then from the inequality (1) we get the classical Holder's inequality.

In the case $\mathbf{p}(x)=(p(x_n),...,p(x_n))$ Lemma 1 was proved in [3].

In [4] it was proved the following

Lemma 2. *Let $1\le\alpha(x)\le\beta(x)\le\overline{\beta}<\infty$, $x\in\Omega\subset R^n$. Then*

$$\|f\|_{\beta}^{\underline{\alpha}} \le \|f^{\alpha}\|_{\beta/\alpha} \le \|f\|_{\beta}^{\overline{\alpha}}, \quad if\ \|f\|_{\beta}\ge 1,$$

$$\|f\|_{\beta}^{\overline{\alpha}} \le \|f^{\alpha}\|_{\beta/\alpha} \le \|f\|_{\beta}^{\underline{\alpha}}, \quad if\ \|f\|_{\beta}\le 1,$$

where $f^{\beta}=|f(x)|^{\beta(x)}$ and $\overline{\beta}=\sup\limits_{x\in\Omega}\beta(x)$. If $\alpha(x)$ and $\beta(x)$ are continuous on Ω, there exists a point $x_0\in\Omega$ such that $\|f^{\alpha}\|_{\beta/\alpha}=\|f\|_{\beta}^{\alpha(x_0)}$.

If $\alpha(x)=\beta(x)$ and $\beta\in C(\Omega)$, then Lemma 1 implies that

$$\|f\|_{\beta}=\left(\int\limits_{\Omega}|f(x)|^{\beta(x)}\,dx\right)^{1/\beta(x_0)},$$

where a point $x_0\in\Omega$ depends only on f.

Now we consider the case $\mathbf{p}(x,y)=(p(x),q(y))$, where $x\in\Omega_1\subset R^n$ and $y\in\Omega_2\subset R^m$.

It is valid

Theorem 1. *Let $1<\underline{p}\le p(x)\le\underline{q}\le\overline{q}<\infty$. If $p(x)\in C(\Omega_1)$ then the inequality*

$$\|\|f\|_{p,\Omega_1}\|_{q,\Omega_2} \le \left(\frac{\overline{p}}{\underline{q}}+\frac{\overline{q}-\underline{p}}{\overline{q}}\right)^{\frac{2}{\underline{p}}} \|\|f\|_{q,\Omega_2}\|_{p,\Omega_1}$$

is valid, where $\underline{p}=\inf\limits_{\Omega_1}p(x)$, $\overline{p}=\sup\limits_{\Omega_1}p(x)$, $\underline{q}=\inf\limits_{\Omega_2}q(x)$, $\overline{q}=\sup\limits_{\Omega_2}q(x)$ and $f:\Omega_1\times\Omega_2\to R$-is any measurable function and $C(\Omega_1)$ is the space of continuous functions in Ω_1.

Proof of Theorem 1. It is proved in [5] that, if $1 \le r(y) \le \overline{r} < \infty$ and $y \in \Omega_2$ then for any measurable function $g : \Omega_1 \times \Omega_2 \to R$ we have

$$\left\| \int_{\Omega_1} g(x, \cdot)\, dx \right\|_{r(\cdot),\Omega_2} \le \left(\frac{1}{\underline{r}} + \frac{1}{\underline{r}'} \right)^2 \int_{\Omega_1} \|g(x, \cdot)\|_{r(\cdot),\Omega_2}\, dx. \qquad (1)$$

By replacing a function $g(x, y)$ by $|f(x, y)|^{p(x)}$ in (1) and taking a norm with respect to variable y on function $r(x, y) = \dfrac{q(y)}{p(x)}$ (at a fixed point $x \in R^n$) we have

$$\left\| \int_{\Omega_1} |f(x, \cdot)|^{p(x)}\, dx \right\|_{\frac{q(\cdot)}{p(x)},\Omega_2} \le \left(\frac{\overline{p}}{\underline{q}} + \frac{\overline{q} - \underline{p}}{\overline{q}} \right)^2 \int_{\Omega_1} \left\| [f(x, \cdot)]^{p(x)} \right\|_{\frac{q(\cdot)}{p(x)},\Omega_2} dx. \qquad (2)$$

By definition we get

$$\left\| [f(x, \cdot)]^{p(x)} \right\|_{\frac{q(\cdot)}{p(x)},\Omega_2} = \inf \left\{ \lambda > 0 : \quad \int_{\Omega_2} \left| \frac{[f(x, y)]^{p(x)}}{\lambda} \right|^{q(y)/p(x)} dy \le 1 \right\} =$$

$$= \inf \left\{ \lambda > 0 : \quad \int_{\Omega_2} \left| \frac{f(x, y)}{\lambda^{1/p(x)}} \right|^{q(y)} dy \le 1 \right\}.$$

Putting $\mu = \lambda^{1/p(x)}$, we have

$$\inf \left\{ \lambda > 0 : \quad \int_{\Omega_2} \left| \frac{f(x, y)}{\lambda^{1/p(x)}} \right|^{q(y)} dy \le 1 \right\} =$$

$$= \inf \left\{ \mu^{p(x)} > 0 : \quad \int_{\Omega_2} \left| \frac{f(x, y)}{\mu} \right|^{q(y)} dy \le 1 \right\} = \|f(x, \cdot)\|^{p(x)}_{q(\cdot),\Omega_2}.$$

Therefore

$$\left\| [f(x, \cdot)]^{p(x)} \right\|_{\frac{q(\cdot)}{p(x)},\Omega_2} = \|f(x, \cdot)\|^{p(x)}_{q(\cdot),\Omega_2}. \qquad (3)$$

Further, we have

$$\left\| \int_{\Omega_1} |f(x, \cdot)|^{p(x)}\, dx \right\|_{\frac{q(\cdot)}{p(x)},\Omega_2} =$$

$$= \inf \left\{ \nu > 0 : \quad \int_{\Omega_2} \left(\frac{\int_{\Omega_1} |f(x, y)|^{p(x)}\, dx}{\nu} \right)^{q(y)/p(x)} dy \le 1 \right\} =$$

$$= \inf \left\{ \nu > 0 : \quad \int\limits_{\Omega_2} \left(\frac{\left(\int\limits_{\Omega_1} |f(x,y)|^{p(x)} \, dx \right)^{1/p(x)}}{\nu^{1/p(x)}} \right)^{q(y)} dy \le 1 \right\}.$$

Suppose $\delta = \nu^{1/p(x)}$. Then

$$\inf \left\{ \nu > 0 : \quad \int\limits_{\Omega_2} \left(\frac{\left(\int\limits_{\Omega_1} |f(x,y)|^{p(x)} \, dx \right)^{1/p(x)}}{\nu^{1/p(x)}} \right)^{q(y)} dy \le 1 \right\} =$$

$$= \inf \left\{ \delta^{p(x)} > 0 : \quad \int\limits_{\Omega_2} \left(\frac{\left(\int\limits_{\Omega_1} |f(x,y)|^{p(x)} \, dx \right)^{1/p(x)}}{\delta} \right)^{q(y)} dy \le 1 \right\} =$$

$$= \left\| \left(\int\limits_{\Omega_1} |f(x,\cdot)|^{p(x)} \, dx \right)^{1/p(x)} \right\|_{q(\cdot),\Omega_2}^{p(x)}.$$

Therefore

$$\left\| \int\limits_{\Omega_1} |f(x,\cdot)|^{p(x)} \, dx \right\|_{\frac{q(\cdot)}{p(x)},\Omega_2} = \left\| \left(\int\limits_{\Omega_1} |f(x,\cdot)|^{p(x)} \, dx \right)^{1/p(x)} \right\|_{q(\cdot),\Omega_2}^{p(x)}. \qquad (4)$$

Substituting the relations (3) and (4) in (2) we have

$$\left\| \left(\int\limits_{\Omega_1} |f(x,\cdot)|^{p(x)} \, dx \right)^{1/p(x)} \right\|_{q(\cdot),\Omega_2}^{p(x)} \le \left(\frac{\overline{p}}{\underline{q}} + \frac{\overline{q} - \underline{p}}{\overline{q}} \right)^2 \int\limits_{\Omega_1} \|f(x,\cdot)\|_{q(\cdot),\Omega_2}^{p(x)} \, dx. \qquad (5)$$

By virtue of Lemma 2, if $p(x) \in C(\Omega_1)$, then there exist points $x_1, x_2 \in \Omega_1$ such that

$$\|f(\cdot,y)\|_{p(\cdot),\Omega_1} = \left(\int\limits_{\Omega_1} |f(x,y)|^{p(x)} \, dx \right)^{\frac{1}{p(x_1)}} \quad \| \|f\|_{q,\Omega_2} \|_{p,\Omega_1} =$$

$$= \left(\int\limits_{\Omega_1} \|f(x,\cdot)\|_{q(\cdot),\Omega_2}^{p(x)} \, dx \right)^{\frac{1}{p(x_2)}},$$

where the points x_1, x_2 depend on the function f and $\|f\|_{q,\Omega_2}$, respectively. Taking into account the last relation in the inequality (5), we have

$$\left\|\|f\|_{p,\Omega_1}\right\|_{q(\cdot),\Omega_2}^{p(x_1)} \le \left(\frac{\overline{p}}{\underline{q}} + \frac{\overline{q}-\underline{p}}{\overline{q}}\right)^2 \left\|\|f\|_{q,\Omega_2}\right\|_{p,\Omega_1}^{p(x_2)}.$$

Finally, we get

$$\left\|\|f\|_{p,\Omega_1}\right\|_{q,\Omega_2} \le \left(\frac{\overline{p}}{\underline{q}} + \frac{\overline{q}-\underline{p}}{\overline{q}}\right)^{\frac{2}{\underline{p}}} \left\|\|f\|_{q,\Omega_2}\right\|_{p,\Omega_1}^{\frac{p(x_2)}{p(x_1)}} \qquad (6).$$

Let $h(x,y)\neq 0$, for a.e. $(x,y)\in\Omega_1\times\Omega_2$ and $h\in L_{(q(y),p(x))}(\Omega_2\times\Omega_1)$. Suppose $f(x,y)=\dfrac{h(x,y)}{\left\|\|h\|_{q,\Omega_2}\right\|_{p,\Omega_1}}$. It is clear that $\left\|\|h\|_{q,\Omega_2}\right\|_{p,\Omega_1}=1$. Substituting this in inequality (6), we have

$$\left\|\|h\|_{p,\Omega_1}\right\|_{q,\Omega_2} \le \left(\frac{\overline{p}}{\underline{q}} + \frac{\overline{q}-\underline{p}}{\overline{q}}\right)^{\frac{2}{\underline{p}}} \left\|\|h\|_{q,\Omega_2}\right\|_{p,\Omega_1}.$$

Theorem 1 is proved.

Remark 3. *Note that in the case $p(x)=1$, Theorem 1 is the analog of generalized Minkowski type inequality and was proved in [4].*

A function $f\in L_{\mathbf{p}(x)}(\Omega)$ is called $p(x)$-mean continuous, if for each $\varepsilon>0$ there is $\delta=\delta(\varepsilon)>0$ such that $\|f(\cdot+h)-f(\cdot)\|_{L_{\mathbf{p}(\cdot)}(R^n)}<\varepsilon$ for $|h|<\delta$, where $|h|=\left(\sum\limits_{i=1}^{n}h_i^2\right)^{1/2}$ and $h=(h_1,\cdots,h_n)$ (see [3]). In this case assume that $f(x)=0$ for $x\in R^n\setminus\Omega$.

Theorem 2. *Let $\Omega\subset R^n$ be an open set, $1\le p(x)\le\overline{P}<\infty$, $x\in R^n$, $p\in C(R^n)$.*

If condition

$$A=\sup_{|h|<\frac{1}{2}}\|f(\cdot+h)\|_{p(\cdot),R^n}<\infty, \qquad (7)$$

is satisfied, then condition (7) is necessary and sufficient for mean continuity in $L_{p(x)}(\Omega)$.

Proof of Theorem 2. We prove that the condition (7) is necessary. Let $f\in L_{p(x)}(\Omega)$ be $p(x)$-mean continuous. Then we have

$$\left|\|f(\cdot+h)\|_{p(\cdot),R^n}-\|f\|_{p,R^n}\right|\le\|f(\cdot+h)-f\|_{p(\cdot),R^n}\to 0.$$

It implies that $\lim\limits_{h\to 0}\|f(\cdot+h)\|_{p(\cdot),R^n}=\|f\|_{p,R^n}$. Therefore

$$\forall\varepsilon>0\quad\delta>0\quad|h|<\delta\implies\|f\|_{p,R^n}-\varepsilon<\|f(\cdot+h)\|_{p(\cdot),R^n}<\|f\|_{p,R^n}+\varepsilon.$$

Now for $|h|<\delta\le\frac{1}{2}$, the inequality (7) holds.

Now we prove sufficiency of condition (7). Suppose that the inequality (7) is hold. Then we have

$$\|f(\cdot+h)-f\|_{p(\cdot),R^n}\le\|f(\cdot+h)\|_{p(\cdot),R^n}+\|f\|_{p,R^n}\le$$

$$\leq 2\, max\, \{A, \|f\|_{p,\,R^n}\} \quad \text{for} \quad |h| < \frac{1}{2}.$$

It is proved in [3] under the conditions of theorem 2 the space $C_0^\infty(\Omega)$ is dense in $L_{p(x)}(\Omega)$, where $C_0^\infty(\Omega)$ is the space of infinitely differentiable and finite in Ω function. Therefore for any given $\varepsilon > 0$ there exists $\varphi \in C_0^\infty(\Omega)$ such that $\|f - \varphi\|_{p,\Omega} < \varepsilon$. We have

$$\|f(\cdot+h)-f\|_{p(\cdot),R^n} \leq \|f-\varphi\|_{p,R^n} + \|f(\cdot+h)-\varphi(\cdot+h)\|_{p(\cdot),R^n} + \|\varphi(\cdot+h)-\varphi\|_{p(\cdot),R^n} <$$

$$< \varepsilon + \|f(\cdot+h) - \varphi(\cdot+h)\|_{p(\cdot),R^n} + \|\varphi(\cdot+h) - \varphi\|_{p(\cdot),R^n}.$$

Estimate the $\|f(\cdot+h) - \varphi(\cdot+h)\|_{p(\cdot),R^n}$. We have

$$\|f(\cdot+h) - \varphi(\cdot+h)\|_{p(\cdot),R^n} \leq \|f(\cdot+h)\|_{p(\cdot),R^n} + \|\varphi(\cdot+h)\|_{p(\cdot),R^n}.$$

By virtue of condition (7) $\int\limits_{R^n} |f(x)|^{p(x-h)}\,dx \leq A$, for $h \to 0$ and $\int\limits_{R^n} |\varphi(x)|^{p(x-h)}\,dx \leq$ $\leq\ max\left\{M^{\overline{p}},\, M^{\underline{p}}\right\}\,|F|$, where $M = \sup\limits_{x\in F} |\varphi(x)|$, $F = supp\,\varphi$ and $|F|$-denote the Lebesgue measure of a set F. It is obviously that

$$\int\limits_{R^n} |f(x+h) - \varphi(x+h)|^{p(x)}\,dx = \int\limits_{R^n} |f(x) - \varphi(x)|^{p(x-h)}\,dx.$$

Since $p \in C(R^n)$, we have

$$\lim_{h\to 0} \int\limits_{R^n} |f(x) - \varphi(x)|^{p(x-h)}\,dx \leq C \int\limits_{R^n} |f(x) - \varphi(x)|^{p(x)}\,dx.$$

Therefore $\|f(\cdot+h) - \varphi(\cdot+h)\|_{p(\cdot),R^n} < \varepsilon$, for $h \to 0$. It is known that F is a bounded closed set and let $F \subset B[0,r_1] \subset B[0,r_2]$, where $B[0,r_i]$ are bounded closed balls with centers at the origin. Therefore a function φ is uniformly continuous in $B[0,r_2]$. Let $|h| < \delta < r_2 - r_1$ and $x \in B[0,r_1]$. Then $|x+h| \leq |x| + |h| < r_1 + (r_2 - r_1) = r_2$, i.e., $x + h \in B[0,r_2]$. Since a function φ is uniformly continuous in $B[0,r_2]$, we have

$$\forall\varepsilon > 0\ \exists\delta(\varepsilon) > 0\ \forall x', x'' \in B[0,r_2]\ \ |x' - x''| < \delta(\varepsilon) \Longrightarrow |\varphi(x') - \varphi(x'')| < \varepsilon.$$

Let $\varepsilon < 1$, $|h| < \delta$ and $\delta > 0$ be small. Then we have

$$\int\limits_{R^n} |\varphi(x+h) - \varphi(x)|^{p(x)}\,dx = \int\limits_{B[0,r_1]} |\varphi(x+h) - \varphi(x)|^{p(x)}\,dx \leq$$

$$\leq \int\limits_{B[0,r_2]} |\varphi(x+h) - \varphi(x)|^{p(x)}\,dx \leq C\, r_2^n\, \varepsilon \to 0, \quad \varepsilon \to 0.$$

Therefore

$$\|f(\cdot+h) - f\|_{p(\cdot),R^n} \leq 2\,\varepsilon + C\, r_2^n\, \varepsilon = 2\,\varepsilon(1 + r_2^n) \to 0.$$

Theorem 2 is proved.

In the case of Sobolev spaces with variable exponent some result of $p(x)$-mean continuity was proved in [6]

Note that the space $L_{\mathbf{p}(x)}\,(R^n)$ is Banach function space and studied in [7].

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Rovshan A. Bandaliev
Institute of Mathematics and Mechanics of NAS of Azerbaijan
9, F.Agayev str., AZ1141, Baku, Azerbaijan
Tel.: (99412) 439 47 20 (off.)
Email: bandalievr@rambler.ru

Marifa M. Abbasova
"NORD" Scientific Production Association
1, R.Shahsuvarov str., AZ1089, Hovsan vil., Baku, Azerbaijan
Tel.: (99412) 439 47 20 (off.)

Received July 12, 2006; Revised October 17, 2006.