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SOLVABILITY OF CAUCHY PROBLEM FOR QUASILINEAR EVOLUTION EQUATIONS OF PARABOLIC TYPE

Abstract

A unique existence of the Cauchy problem is studied for first and second order evolution equations.

As the second order differential equation is reduced to the equivalent first order parabolic type equation it has the properties of parabolic type equations.

The Cauchy problem for first order quasilinear differential - operator equations of parabolic type was studied in the papers of many authors (see ex. [1,2]). The present paper is devoted to studying such type equations of more general form.

1. Let's consider the Cauchyy problem for the first order quasilinear equation

$$u'(t) + Au(t) = f(t, u(t)), \tag{1}$$

$$u(0) = u_0, \tag{2}$$

in the Banach space E.

Under the solution of problem (1)-(2) we understand the function $u(t)$ continuous on $[0,T]$ for which there exist $u'(t)$, $Au(t)$ and they are continuous on $[0,T]$ and relations (1)-(2) are fulfilled.

In the paper [1] problem (1)-(2) is studied when the operator A generates an analytic sub-group of the class (C_0) and the right hand side is continuous by Holder.

In the paper [2] problem (1)-(2) is investigated in the case when the operator A is a generating operator of the class $(0, A)$. The right hand side $f\left(t, A^{-1}u\right)$ on $[0,T] \times S_0$ ($S_0 \subset E$ is a ball) has partial derivatives continuous in aggregate of variables that satisfy Lipschits condition.

Problem (1), (2) can be studied by methods in the theory of semigroups. For problem (1)-(2) we prove the following theorem on local solvability.

Theorem 1. *Let the following conditions be fulfilled:*

1^0). *For some $\eta > 0$, $\beta \in (0,1]$ a close operator A has a bounded inverse A^{-1} and satisfies the condition*

$$\|R(\lambda, -A)\| \le c|\lambda|^{-\beta}, \quad |\arg\lambda| \le \frac{\pi}{2} + \eta,$$

2^0). *The operator-function $f(t,u)$ satisfies the condition*

$$\|f(t + \Delta t, u_1) - f(t, u_2)\| \le C(R)\left[\omega(|\Delta t|) + \|A^\rho(u_1 - u_2)\|\right],$$

when

$$\|A^\rho u_i\| \le R, \qquad (i = 1,2), \quad \rho \in [0, \beta);$$

here $\omega(t)$ is a continuous and monotone increasing function determined on the interval $(0,\infty)$ that for some $p \in \left[\frac{1}{\beta}, \infty\right]$ and $\alpha \in \left(2 - \frac{1}{p} - \beta, 1\right]$ satisfies the condition:

$$\int_0^{T_0} \frac{\omega^p(t)}{t^{p\alpha}} dt < \infty;$$

3^0). $u_0 \in D(A^\sigma)$, *where* $\sigma \in (1+p-\beta, 1]$.
Then problem (1)-(2) has a unique solution on the interval $[0,t_0]$ $(t_0 \le T)$.

Proof. Let's consider the integral equation

$$V(t) = A^\rho e^{-tA} u_0 + \int_0^t A^\rho e^{-(t-s)A} f\left(s, A^{-\rho} V(s)\right) ds. \tag{3}$$

In this equation integrated addend allows the estimate

$$\left\| A^\rho e^{-tA} u_0 \right\| = \left\| A^{-(\sigma-\rho)} e^{-tA} A^\sigma u_0 \right\| \le C \left\| A^\sigma u_0 \right\|,$$

since here $\sigma - \rho > 1-\beta$, and for $\sigma-\rho > 1-\beta$ the operator-function $A^{-(\sigma-\rho)}e^{-tA}$ is continuous with respect to $t \in [0,T]$ (see [3]). Therefore the compressed mappings principle is applicable to equation (3). So, this equation has a unique continuous solution $V(t)$.

Let's show that the function $V(t)$ satisfies the condition:

$$\left\| V(t+\Delta t) - V(t) \right\| \le c\omega(|\Delta t|).$$

To this end we constitute its complement using equation (3).

$$V(t+\Delta t) - V(t) = A^\rho \left[e^{-(t+\Delta t)A} - e^{-tA} \right] u_0 +$$

$$+ \int_0^{\Delta t} A^\rho e^{-(t+\Delta t - s)A} f\left(s, A^{-\rho}V(s)\right) ds + \int_0^t A^\rho e^{-(t-s)A} \left[f\left(\rho + \Delta t, A^{-\rho} V(s+\Delta t)\right) - \right.$$

$$\left. - f\left(s, A^{-\rho}V(s)\right)\right] ds = J_1 + J_2 + J_3.$$

Estimate each addend

$$\|J_1\| = \left\| \int_0^{t+\Delta t} A^{1+\rho-\sigma} e^{-\xi A} A^\sigma u_0 d\xi \right\| \le \int_0^{t+\Delta t} c e^{-\eta\xi} \xi^{\beta+\sigma-p-2} \left\| A^\sigma u_0 \right\| d\xi \le$$

$$\le c t^{\beta+\sigma-p-2} \Delta t \left\| A^\sigma u_0 \right\|.$$

$$\|J_2\| = \left\| \int_0^{\Delta t} A^\rho e^{-(t+\Delta t-s)A} ds \right\| \le c \int_0^{\Delta t} c(R) (t+\Delta t - s)^{\beta-\rho-1} ds \le c t^{\beta-\rho-1} \Delta t.$$

$$\|J_3\| = \left\| \int_0^t A^\rho e^{-(t-s)A} \left[f\left(s+\Delta t, A^{-\rho}V(s+\Delta t)\right) - f\left(s, A^{-\rho}V(s)\right)\right] ds \right\| \le$$

$$\le c \int_0^t c(R) \left[\omega(\Delta t) + s^{-\mu} \left(s^\mu \left\| V(s+\Delta t) - V(s) \right\| \right) \cdot (t-s)^{\beta-\rho-1} ds \le \right.$$

$$\le c_1 t^{\beta-\rho} \omega(\Delta t) + c_2 t^{\beta-\rho-\mu}.$$

Here $\mu = \max\{2-\beta+\rho-\sigma, 1+\rho-\beta\}$.

Thus, we arrive at the inequality

$$\|V(t+\Delta t)-V(t)\| \leq c\left(t^{-\mu}\Delta t+t^{\beta-\rho}\omega(\Delta t)+\right.$$
$$\left.+t^{\beta-\rho-\mu}\max_{s}\left[s^{\mu}\|V(s+\Delta t)-V(s)\|\right]\right).$$

Hence

$$\max_{t}\left[t^{\mu}\|V(t+\Delta t)-V(t)\|\right] \leq c\left(\Delta t+t^{\beta-\rho}\omega(\Delta t)+\right.$$
$$\left.+t^{\beta-\rho-\mu}\max_{s}\left[s^{\mu}\|V(s+\Delta t)-V(s)\|\right]\right).$$

For small t_0 we get the inequality

$$\|V(t+\Delta t)-V(t)\| \leq ct^{-\mu}\omega(|\Delta t|).$$

Now, let's assume $u(t)=A^{-\rho}V(t)$. Then we have

$$u(t)=e^{-tA}u_0+\int_0^t e^{-(t-s)A}f\left(s,A^{-\rho}V(s)\right)ds. \tag{4}$$

Here the function $f(s,A^{-\rho}V(s))$ satisfies the condition

$$\left\|f\left(s+\Delta s,A^{-\rho}V(s+\Delta s)\right)-f\left(s,A^{-\rho}V(s)\right)\right\| \leq$$
$$\leq c\left[\omega(|\Delta s|)+\|V(s+\Delta s)-V(s)\|\right] \leq cs^{-\mu}\omega(|\Delta s|), \qquad \mu<1$$

So, theorem 3.2 of [3] is applicable to integral in (3) that implies $u(t)\in D(A)$ for $t>0$ and there exists derivative $u'(t)$ continuous for $t>0$.

The following relations are valid

$$u'(t)=-e^{-tA}Au_0+f\left(t,A^{-\rho}V(t)\right)-A\int_0^t e^{-(t-s)A}f\left(s,A^{-\rho}V(s)\right)ds,$$

$$Au(t)=Ae^{-tA}+A\int_0^t e^{-(t-s)A}f\left(s,A^{-\rho}V(s)\right)ds.$$

Hence, it follows that the function $u(t)$ satisfies equation (1). Uniqueness of the solution is proved as in the case of linear equation. The theorem is proved.

In sequel, second order quasilinear parabolic evolution equations will be reduced to the problem

$$u'(t)+Au(t)+Bu(t)=f(t,u(t)), \tag{5}$$

$$u(0)=u_0, \tag{6}$$

and this problem should be investigated.

Theorem 2. *Let conditions* $1^0)$, $2^0)$, $3^0)$ *of theorem 1 be fulfilled. Then, let the condition*

$4^0)$. $D(B)\supset D(A)$ *be fulfilled for any* $\varepsilon>0$, too

$$\|Bu\| \leq \varepsilon\|Au\|^{\beta}\|u\|^{1-\beta}+c(\varepsilon)\|u\|, \qquad u\in D(A).$$

Then problem (5)-(6) has a unique solution determined on the segment $[0,t_0] \subset [0,T]$ *that may be found by sequential approximations method.*

Proof. By lemma 1.1 from [3] the operator $A+B$ satisfies condition $1^0)$ of theorem 1. Then theorem 1 should be applied to problem (5)-(6).

II. Now, let's consider the Cauchy problem for a second order quasilinear evolution equation

$$u''(t) + Au'(t) + Bu(t) = f\left(t, u(t), u'(t)\right), \tag{7}$$

$$u(0) = u_0, \quad u'(0) = u_1 \tag{8}$$

in the Banach space E. Local solvability theorem is proved for problem (7)-(8) as well.

Theorem 3. *Let the following conditions be fulfilled:*

$5^0)$ *for some* $\eta > 0$, $\beta_1 \in (0,1]$ *and* $\beta_2 \in (0,1]$ *where* $\beta_1 + \beta_2 > 1$,

$$\|R(\lambda, -A)\| \le c(1+|\lambda|)^{-\beta_1}, \qquad |\arg \lambda| \le \frac{\pi}{2} + \eta,$$

$$\left\|R\left(\lambda, -BA^{-1}\right)\right\| \le (1+|\lambda|)^{-\beta_2}, \qquad |\arg \lambda| \le \frac{\pi}{2} + \eta, \quad |\lambda| \to \infty;$$

$6^0)$ $D\left(A^2\right) \subset D(B)$ *and for any* $\varepsilon > 0$

$$\|Bu\| \le \varepsilon \left\|A^2 u\right\|^{\beta_1+\beta_2-1} \|u\|^{2-\beta_1-\beta_2} + c(\varepsilon)\|u\|, \quad u \in D\left(A^2\right).$$

$7^0)$ *the operator-function* $f(t,u,\nu)$ *satisfies the condition*

$$\|f(t,u_1,\upsilon_1) - f(\tau,u_2,\upsilon_2)\| \;\le\; c(R)\left[\omega(t-\tau) + \|u_1-u_2\|_{E(A)} + \|\upsilon_1-\upsilon_2\|_{E(A^\alpha)}\right]$$

when

$$\|Au_i\| \le R, \quad \|A^\alpha \upsilon_i\| \le R, \quad (i=1,2) \quad \alpha \in [0, \beta_1+\beta_2-1);$$

$\omega(t)$ *is a continuous and monotone increasing function determined on the interval* $(0,\infty)$ *that for some* $p \in \left[\dfrac{1}{\beta_1+\beta_2-1}, \infty\right)$, $\gamma \in \left(3 - \dfrac{1}{p} - \beta_1 - \beta_2, 1\right]$ *satisfies the condition*

$$\int_0^{T_0} \frac{\omega^p(t)}{t^{p\gamma}} dt < \infty.$$

$8^0)$ $u_0 \in D(A^\sigma)$, $Au_0 \in D\left(\left(BA^{-1}\right)^\sigma\right)$; $u_1 \in D(A^\sigma)$ *where* $\sigma \in (2+\alpha-\beta_1-\beta_2, 1]$.

Then problem (7)-(8) has a unique solution determined on some segment $[0,t_0] \subset [0,T]$ *that may be found by sequential approximations method.*

Proof. By substituting $\upsilon_1^{(t)} = Au(t) + u'(t)$, $\upsilon_2(t) = u'(t)$ the issue on the existence of the solution of problem (7)-(8) from $C^1([0,t_0];\ E(A), E) \cap C^2([0,t_0];\ E(A)\cap E(B), E)$ is reduced to the issue on the existence of the solution from $C([0,t_0],\ E\times E) \cap C^1\left((0,t_0];\ E\left(BA^{-1}\right)\times E(A), E\times E\right)$ of the problem

$$V'(t) + \mathfrak{U}V(t) + \mathfrak{B}V(t) = F(t, V(t)), \tag{9}$$

$$V(0) = V_0, \tag{10}$$

where

$$\mathfrak{U} = \begin{pmatrix} BA^{-1} & 0 \\ BA^{-1} & A^{-1} \end{pmatrix}, \quad \mathfrak{B} = \begin{pmatrix} 0 & -BA^{-1} \\ 0 & -BA^{-1} \end{pmatrix},$$

$$V(t) = \begin{pmatrix} \upsilon_1(t) \\ \upsilon_2(t) \end{pmatrix}, \quad V_0 = \begin{pmatrix} Au_0 + u_1 \\ u_1 \end{pmatrix},$$

$$F(t, V(t)) = \begin{pmatrix} f\left(A^{-1}(\upsilon_1 - \upsilon_2), \upsilon_2\right) \\ f\left(A^{-1}(\upsilon_1 - \upsilon_2), \upsilon_2\right) \end{pmatrix}$$

It is easy to verify that $R(\lambda, -\mathfrak{U})$ is determined by the formula

$$R(\lambda, -\mathfrak{U}) = \begin{pmatrix} R\left(\lambda, -BA^{-1}\right) & 0 \\ R(\lambda, -A)\, BA^{-1} R\left(\lambda, -BA^{-1}\right) & R(\lambda, -A) \end{pmatrix} \tag{11}$$

Then by $5^0)$, $6^0)$ and (11) we get the estimate

$$\|R(\lambda, -\mathfrak{U})\| \le c(1 + |\lambda|)^{-(\beta_1+\beta_2-1)}, \qquad |\arg\lambda| \le \frac{\pi}{2} + \eta,$$

$|\lambda| \to \infty$, i.e. the operator $\mathfrak{U}$ satisfies the condition of theorem 1. Let $V = (V_1, V_2) \in D\left(BA^{-1}\right) \times D(A)$. Then by conditions $5^0)$ and $6^0)$ it is proved that (see [3])

$$\|\mathfrak{B}V\| \le \varepsilon \|\mathfrak{U}V\|^{\beta_1+\beta_2-1} \|V\|^{2-\beta_1-\beta_2} = c(\varepsilon)\|V\|,$$

i.e. the operator $\mathfrak{B}$ satisfies condition $4^0)$ of theorem 2. On the other hand, considering the formula for the semi-group generated by the operator $\mathfrak{U}$

$$e^{-t\mathfrak{U}} = \begin{pmatrix} e^{-tBA^{-1}} & 0 \\ c(t) & e^{-tA} \end{pmatrix}$$

where

$$c(t) = e^{-tBA^{-1}} - e^{-tA} - A\int_0^t e^{-(t-s)A} e^{-sBA^{-1}} ds,$$

we construct negative fractional powers of the operator $\mathfrak{U}$:

$$\mathfrak{U}^{-\alpha} = \begin{pmatrix} \left(BA^{-1}\right)^{-\alpha} & 0 \\ \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^\infty \tau^{\alpha-1} c(\tau)\, d\tau & A^{-\alpha} \end{pmatrix} \tag{12}$$

It follows from (12) and conditions of $7^0)$ that there exists the number R_1 such that for all $t, \tau \in [0, T]$, $V, V' \in E^2$ with $\|V\| \le R_1$, $\|V'\| \le R^1$ there exists the inequality

$$\left\|F\left(t, \mathfrak{U}^{-\alpha}V\right) - F\left(\tau, \mathfrak{U}^{-\alpha}V'\right)\right\| \le c\left[\omega(t-\tau) + \left\|V - V'\right\|\right].$$

Thus, theorem 2 is applicable to problem (9)-(10), whence it follows that problem (9)-(10) has a unique solution from $C([0, t_0]; E \times E) \cap \left(C^1(0, t_0]; E\left(BA^{-1}\right) \times E(A), E \times E\right)$. Hence, using the inverse substitution $u(t) = A^{-1}(V_1(t) - V_2(t))$, $u'(t) = \upsilon_2(t)$ we easily get affirmation of theorem 3.

In conclusion we note that problem (7)-(8) is investigated in the paper [4] when the operator A generates a sub-group of class $(1, A)_\infty$ and the operator f operates from $[0,T]\times E(A)\times E$ to E. In the paper [5] problem (7)-(8) is studied for complete evolution equation of the second order with factorized right hand side. Additionally assuming smoothness of nonlinear operator f the problem of regularity of solutions were studied.

III. Let's consider the initial boundary value problem with non-local boundary conditions:

$$\frac{\partial u(t,x)}{\partial t}-a_0(x)\frac{\partial^2 u(t,x)}{\partial x^2}+a_1(x)u(t,x)=f(t,x,u(t,x)) \tag{13}$$

$$L_1 u=\int_0^1 \varphi_1(x)u(t,x)\,dx=0, \tag{14}$$

$$L_2 u=\int_0^1 \varphi_2(x)u(t,x)\,dx=0 \tag{15}$$

$$u(0,x)=\varphi_0(x). \tag{16}$$

If we introduce the operator $A: L_2(0,1)\to L_2(0,1)$ determined by the formula

$$D(A)=\left\{u\in W_2^2(0,1):\ L_j u=0,\quad j=1,2\right\}$$

$$Au=-a_0(x)\ u''(x)+a_1 u(x).$$

and if we additionally assume $a_0(x)>M>0$, $x\in[0,1]$, $a_0'(0)=a_0'(1)=0$, $a_1(x)\in C[0,1]$, $\Delta_\varphi=\varphi_1(0)\varphi_2(1)-\varphi_2(0)\varphi_1(1)\neq 0$. $\varphi_j\in C^1[0,1]$, $j=1,2$, are linearly independent functions.

Then for the operator A we have: $\overline{D(A)}\neq L_2(0,1)$ and $\|R(\lambda,A)\|\leq C|\lambda|^{-\frac{3}{4}}$.

Applying theorem 1 we get the solvability (local) of problem (13)-(16).

References

[1]. P.E.Sobolevskii. Trudy Mosc. obsh., v.10, 1961. (Russian)

[2]. S.Ya.Yakubov. Soviet Doklady, v.156, No5, 1964. (Russian)

[3]. S.Ya.Yakubov. *Linear differential-operator equations and their applications.* Baku, 1985. (Russian)

[4]. S.Ya.Yakubov. Izv. AN Azerb., SSSR, ser. fiz. tech. i matem. nauk, 1966, No4. (Russian)

[5]. A.Favini and A.Venni. *Semigroup forum.* V.45, No2, 1992, p.135-147.

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