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n-TH REGULARIZED TRACE OF DIFFERENTIAL OPERATOR EQUATION

Abstract

The formula of the n-th regularized trace of the Sturm-Liouville operator equation on the final segment, when one of the boundary conditions contains a parameter, is obtained.

Let H be a separable Hilbert space. In Hilbert space $H_1 = L_2(H, [0, \pi])$ we consider the following two differential operators L_0 and L, generated, by the following expressions

$$l_0[y] = -y'' + Ay$$

$$l[y] = -y'' + Q(x)y$$

respectively, and with the boundary conditions

$$y(0) = 0,\ y'(\pi) + hy(\pi) = 0$$

where A is a self-adjoint, lower semi-bounded operator and inverse for completely continuous operator in H.

Assume that operator function $Q(x)$ is weakly measurable and satisfies the following conditions:

1. The operator function $Q(x)$ has $2n$-th weak derivative on the segment $[0, \pi]$ and

$$Q^{(2p-1)}(0) = Q^{(2p-1)}(\pi) = 0,\ \overline{1, n}$$

$$Q^{(2p)}(\pi) = 0,\ p = \overline{0, n-2}$$

2. $\|Q(x)\|_H \leq const$

3. $Q^{(l)}(x)$, $l = \overline{0, 2n}$ at each $x \in [0, \pi]$ are kernel self-adjoint operators in the space H

$$A^p Q^{(2(n-p))}(x),\ A^p Q^{(2(n-1-p))}(x) \in \sigma_1,\ x \in [0, \pi]$$

functions $\left\|A^p Q^{(2(n-p))}(x)\right\|_1, \left\|A^p Q^{(2(n-1-p))}(x)\right\|_1,\ p = \overline{0, n-1}$ are bounded on the segment $[0, \pi]$.

4. $\int\limits_0^\pi (Q(x)f, f)\,dx = 0$ for each $f \in H$.

The operator L_0 has discrete spectrum. Let $\mu_1 \leq \mu_2 \leq \ ...$ be eigen-values, $\psi_1(x)$, $\psi_2(x), ...$ be the corresponding orthonormal eigen vector-functions of this operator. We write out each eigen-value according to its multiplicity.

Since Q is a bounded operator in H_1 the operator L also will have discrete spectrum. Let $\lambda_1 \leq \lambda_2 \leq \ ...$ be eigen – values of the operator L.

We denote by $\gamma_1 \leq \gamma_2 \leq \ ...$ and $\varphi_1, \varphi_2, \ ...$ the eigen-values and orthonormal eigen elements of the operator A in H, respectively.

As is known [4] if $i \to \infty \gamma_i \sim a i^{\alpha} \left(0 < a < \infty,\ 2 < a < \infty\right)$, there exists a subsequence $\mu_{k_1} < \mu_{k_2} < \ ... \ < \mu_{k_m} < \ ...$ of the sequence$\mu_1 \leq \mu_2 \leq \ ... \ \mu_p \leq \ ...$ such that

$$\mu_p - \mu_{k_m} \geq \left(p^{\frac{2\alpha}{2+\alpha}} - k_m^{\ \frac{2\alpha}{2+\alpha}}\right), \quad p = k_m, k_m + 1, \ ... \ , \tag{1}$$

where d_0 is a positive number. Let R_λ^0 and R_λ be resolvents of the operators L_0 and L. Introduce the following notation

$$\mu_{(i)}^{(n)} = \sum_{s=k_{i-1}+1}^{k_i} \left\{ \mu_s^n + n \sum_{j=2}^{N} \frac{(-1)^j}{j} \operatorname*{Re}_{\lambda = \mu_{k_s}} \left[\lambda^{n-1} Sp \left(Q R_\lambda^0\right)^j\right] \right\},$$

$$\lambda_{(i)}^{(n)} = \sum_{s=k_{i-1}+1}^{k_i} \lambda_s^n, \quad k_0 = 0,$$

where $k_1 < k_2 < \ ...$ is some sequence of natural numbers, which satisfies the condition (1), $N > n + 1 + \frac{n+2}{\delta} \quad \left(\delta = \frac{2\alpha}{2+\alpha} - 1\right)$.

In the present paper the formula for the sum of series $\sum\limits_{i=1}^{\infty} \left(\lambda_{(i)}^{(n)} - \mu_{(i)}^{(n)}\right)$ is obtained. This sum doesn't depend on the choice of sequence $k_1, k_2, \ ...$ satisfying inequality (1). We'll call the sum of the series $\sum\limits_{i=1}^{\infty} \left(\lambda_{(i)}^{(n)} - \mu_{(i)}^{(n)}\right)$ n-th regularized trace of operator L.

Regularized traces for scalar operators were studied in [1], [2], [3] and by many other authors. For differential operators with on operator coefficient similar problems were considered, for example in [4], [5], [6]. In the present paper we consider an operator different from operator in [6] by boundary condition.

The following lemma is true.

Lemma 1. *let at* $i \to \infty,\ \gamma_i \sim a i^{\alpha} \left(0 < a < \infty,\ 2 < \alpha < \infty\right)$ *and the condition 2 be fulfilled. Then at large* m *the following equality holds:*

$$\sum_{i=1}^{\infty} \left(\lambda_{(i)}^{(n)} - \mu_{(i)}^{(n)}\right) = -\frac{n}{2\pi i} \int\limits_{|\lambda| = l_m} \lambda^{n-1} Sp \left[Q R_\lambda^0\right] d\lambda, \tag{2}$$

where

$$l_m = \frac{1}{2} \left[\mu_{k_m+1} + \mu_{k_m}\right]$$

and μ_{k_m} $(m = 1, 2, \ ...)$ *is a subsequence, of sequence* $\mu_1, \mu_2, \ ...$ *which satisfies the inequality (1).*

The proof of this lemma is similar to the one of lemma 1 from [6], and we don't cite it here.

The right hand sidle of (2) denote by M_m^1. Thus, we have

$$\sum_{i=1}^{\infty} \left(\lambda_{(i)}^{(n)} - \mu_{(i)}^{(n)}\right) = -M_m^1$$

Let us calculate $\lim\limits_{m \to \infty} M_m^1$.

Since QR^0_λ is a kernel operator and the eigen vectors $\psi_1(x), \psi_2(x), \ldots$ form an orthonormal basis in the space H_1, then from (2)

$$M^1_m = \frac{n}{2\pi i} \int\limits_{|\lambda|=l_m} \lambda^{n-1} \sum_{k=1}^{\infty} \left(QR^0_\lambda \psi_k, \psi_k\right)_1 d\lambda =$$

$$= \frac{n}{2\pi i} \int\limits_{|\lambda|=l_m} \sum_{k=1}^{\infty} \frac{\lambda^{n-1}}{\mu_k - \lambda} \left(Q\psi_k, \psi_k\right)_1 d\lambda = -\sum_{k=1}^{k_m} \mu_k^{n-1} \left(Q\psi_k, \psi_k\right)_1$$

Eigen-vectors and eigen-values of the operator L_0 are of the form

$$\psi(x) = \sqrt{\frac{4\alpha_{n_k}}{2\alpha_{n_k}\pi - \sin 2\alpha_{n_k}\pi}} \sin\left(\alpha_{n_k} x\right) \varphi_{j_k}, \;\; \mu_k = \alpha^2_{n_k} + \gamma_{j_k}$$

where α_k is the k-th positive root of the equation $\lambda \cos \lambda\pi + h \sin \lambda\pi = 0$.

Theorem 1. *Let the condition of lemma 1 hold. If the operator function $Q(x)$ satisfies conditions 1)-4) then the formula*

$$\lim_{m\to\infty} \sum_{i=1}^{\infty} \left(\lambda^{(n)}_{(i)} - \mu^{(n)}_{(i)}\right) = -\lim_{m\to\infty} M^1_m = n\frac{(-1)^{n-1}}{4^n} \left[Q^{(2(n-1))}(\pi) - Q^{(2(n-1))}(0)\right] +$$

$$+ \sum_{i=1}^{n-1} \left(-\frac{1}{4}\right)^{n-i} SpA^i Q^{(2(n-1-i))}(0). \tag{3}$$

is true.

For proving this theorem we need the following lemma.

Lemma 2. *If the operator function $Q(x)$ satisfies the conditions of theorem 1, then*

$$\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \left| \left(\alpha^2_k + \gamma_j\right)^{n-1} \int\limits_0^\pi \frac{2\alpha_k}{2\alpha_k\pi - \sin 2\alpha_k\pi} \cos\left(2\alpha_k x\right) \left(Q(x)\varphi_j, \varphi_j\right) \right| dx < \infty.$$

Proof. Assume

$$f_j(x) = \left(Q(x)\varphi_j, \varphi_j\right).$$

Then

$$\begin{aligned} &\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \left| \left(\alpha^2_k + \gamma_j\right)^{n-1} \int\limits_0^\pi \tfrac{2\alpha_k}{2\alpha_k\pi - \sin 2\alpha_k\pi} \cos\left(2\alpha_k x\right) \left(Q(x)\varphi_j, \varphi_j\right) \right| dx \le \\ &\le \sum_{i=0}^{n-1} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \left| C^i_{n-1} \alpha_k^{2(n-1-i)} \gamma^i_j \tfrac{2\alpha_k}{2\alpha_k\pi - \sin 2\alpha_k\pi} \int\limits_0^\pi \cos\left(2\alpha_k x\right) f_j(x) \right| dx \end{aligned} \tag{4}$$

Integrating by parts $2(n-i)$ times and taking into account, that

$$f^{(2l-1)}_j(\pi) = f^{(2l-1)}_j(0) = 0, \qquad l = \overline{1,n}$$

$$f^{(2l)}_j(\pi) = 0, \qquad l = \overline{0, n-2},$$

we get

$$\int_0^\pi \cos(2\alpha_k x) f_j(x)\,dx = \frac{(-1)^{n-i}}{(2\alpha_k)^{2(n-i)}} \int_0^\pi \cos(2\alpha_k x) f_j^{(2(n-i))}(x)\,dx.$$

Considering $\alpha_k = \frac{1}{2} + k + O\left(\frac{1}{k}\right)$ and condition $\left\| A^i Q^{(2(n-i))}(x) \right\|_1 \le const$, at $i = \overline{0, n-1}$ (condition 3), from (4) and the last equality we get

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \left| \left(\alpha_k^2 + \gamma_j\right)^{n-1} \int_0^\pi \frac{2\alpha_k}{2\alpha_k\pi - \sin 2\alpha_k\pi} \cos(2\alpha_k x) f_j(x) \right| dx \le$$

$$\le \frac{1}{4\pi} \sum_{i=0}^{n-1}\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} C_{n-1}^i \alpha_k^{-2} \gamma_j^i \left(1 + O\left(\frac{1}{k}\right)\right) \times$$

$$\times \left| \int_0^\pi \cos(2\alpha_k x) f_j^{(2(n-i))}(x)\,dx \right| \le const \sum_{i=0}^{n-1}\sum_{k=1}^{\infty} \int_0^\pi \gamma_j^i \left| f_j^{(2(n-i))}(x) \right| dx \le$$

$$\le const \sum_{i=0}^{n-1} \int_0^\pi \left\| A^i Q^{(2(n-i))}(x) \right\|_1 dx \le const$$

The lemma is proved.

Let us turn to the proof of theorem 1.

From the preceding lemma we get

$$\lim_{m\to\infty} M_m^1 = n \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \int_0^\pi \frac{2\alpha_k \left(\alpha_k^2 + \gamma_j\right)^{n-1}}{2\alpha_k\pi - \sin 2\alpha_k\pi} \cos(2\alpha_k x) f_j(x)\,dx.$$

Let's compute the value of the repeated series on the right hand side of the equality

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \int_0^\pi \frac{2\alpha_k \left(\alpha_k^2 + \gamma_j\right)^{n-1}}{2\alpha_k\pi - \sin 2\alpha_k\pi} \cos(2\alpha_k x) f_j(x)\,dx =$$

$$= \sum_{i=0}^{n-1} C_{n-1}^i \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \alpha_k^{2(n-1-i)} \gamma_j^i \int_0^\pi \frac{2\alpha_k}{2\alpha_k\pi - \sin 2\alpha_k\pi} \cos 2\alpha_k x f_j(x)\,dx \qquad (5)$$

Applying integration by parts $2(n-1-i)$ times to the right-hand side of (5), and taking into account condition 1), we get

$$\lim_{m\to\infty} M_m^1 = n \sum_{i=0}^{n-1} C_{n-1}^i \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \gamma_j^i \left(-\frac{1}{4}\right)^{n-1-i} \times$$

$$\times \int_0^\pi \frac{2\alpha_k}{2\alpha_k\pi - \sin 2\alpha_k\pi} \cos(2\alpha_k x) f_j^{(2(n-1-i))}(x)\,dx \qquad (6)$$

Let's compute the value of the series

$$\sum_{k=1}^{\infty}\int_{0}^{\pi}\frac{2\alpha_k}{2\alpha_k\pi-\sin 2\alpha_k\pi}\cos\left(2\alpha_k x\right)f_j^{(2(n-1-i))}\left(x\right)dx$$

We'll investigate the asymptotic behavior of the function

$$L_N\left(x\right)=\sum_{k=1}^{N}\frac{2\alpha_k}{2\alpha_k\pi-\sin 2\alpha_k\pi}\cos 2\alpha_k x.$$

Consider the following complex function

$$\frac{z\cos 2zx}{\sin z\pi\left(z\cos z\pi+h\sin z\pi\right)},$$

which has the poles at the points α_k, k. Residues at these points are equal, respectively, to

$$\frac{-2\alpha_k}{2\alpha_k\pi-\sin 2\alpha_k\pi}\cos 2\alpha_k x,\qquad\frac{\cos 2kx}{\pi}.$$

As a contour of integration we take rectangle with the tops at $+iB$, A_N+iB where B tends to infinity later on, and $A_N=N+\frac{1}{4}$. In case of N enough large

$$\alpha_N<A_N<\alpha_{N+1},\qquad N<A_N<N+1.$$

Thus, we obtain the formula

$$\frac{1}{2\pi i}\lim_{\beta\to\infty}\int_{A_N-iB}^{A_N+iB}\frac{z\cos 2zx}{\sin z\pi\left(z\cos z\pi+h\sin z\pi\right)}=K_N\left(x\right)-L_N\left(x\right)\tag{7}$$

where

$$K_N\left(x\right)=\sum_{k=1}^{N}\frac{\cos 2kx}{\pi}$$

As $N\to\infty$

$$\frac{1}{2\pi i}\lim_{B\to\infty}\int_{A_N-iB}^{A_N+iB}\frac{z\cos 2zx}{\sin z\pi\left(z\cos z\pi+h\sin z\pi\right)}\sim\frac{1}{2\pi}\frac{\cos 2xA_N}{\cos\frac{x}{2}}.\tag{8}$$

On the other hand

$$\lim_{N\to\infty}\int_{0}^{\pi}\frac{1}{2\pi}\frac{\cos 2xA_N}{\cos\frac{x}{2}}f_j^{(2(n-1-i))}\left(x\right)dx=\frac{1}{2}f_j^{(2(n-1-i))}\left(\pi\right)\tag{9}$$

It has been shown earlier, that ([6])

$$\lim_{N\to\infty}n\sum_{i=0}^{n-1}\int_{0}^{\pi}K_N\left(x\right)f_j^{(2(n-1-i))}\left(x\right)dx=$$

$$= n \sum \frac{(-1)^{n-i} C_{n-1}^{i}}{4^{n-i}} \left[SpA^{i} Q^{(2(n-1-i))}(0) + SpA^{i} Q^{(2(n-1-i))}(\pi) \right] \qquad (10)$$

Considering (7), (8), (9), (10) in (6) we get

$$\lim_{m \to \infty} M_m^1 = n \frac{(-1)^{n-1}}{4^n} \left[Q^{(2(n-1))}(0) - Q^{(2(n-1))}(\pi) \right] -$$

$$- \sum_{i=0}^{n-1} SpA^{i} Q^{(2(n-1-i))}(0) C_{n-1}^{i} \left(-\frac{1}{4} \right)^{n-i}$$

whence it follows that (3) is valid.

The theorem is proved.

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