Zaur A. GASIMOV

ON A GENERALIZATION OF A SYSTEM OF EXPONENTS IN WEIGHT SPACES

Abstract

In the paper the system of exponents with complex-valued coefficients, containing a unique function in Lebesque weight spaces of the functions $L_{p,\rho}$ *is considered. The basicity in this space is proved and at* $p=2$ *the case of Riesz basicity is considered.*

Let's consider the generalization of a system of exponents

$$1 \cup \left\{ A(t) e^{int};\ B(t) e^{-int} \right\}_{n\geq 1}, \tag{1}$$

where $A(t) \equiv |A(t)| e^{i\alpha(t)},\ B(t) \equiv |B(t)| e^{i\beta(t)}$ are complex-valued functions on the segment $[-\pi,\pi]$. Earlier we considered the basicity of the system

$$\left\{ A(t)^{int};\ B(t) e^{-i(n+1)t} \right\}_{n\geq 0}, \tag{2}$$

in the weight space $L_{p,\rho} \equiv L_{p,\rho}(-\pi,\pi),\ 1 < p < +\infty$, where the weight ρ is determined by the formula

$$\rho(t) = \prod_{i=1}^{l} \left\{ \sin \left| \frac{t-\tau_i}{2} \right| \right\} \beta_i,$$

$\{\tau_i\} \subset (-\pi,\pi),\ \{\beta_i\} \subset R$ are some sets.

It is easy to note that system (1) differs from system (2) only by one member. Similar idea belongs N.Levitan [1], who established that if a system of exponents $\left\{e^{i\lambda_k(t)}\right\}_{k\in Z}$ (Z is a set of integers) in complete in L_p to replace the final number $\{\lambda_k\}_{k\in M}$, by the others $\{\mu_k\}_{k\in M}$, where $\{\mu_k\}_{k\in M} \cap \{\lambda_k\}_{k\in Z\backslash M} = \{\varnothing\},\ \mu_k \neq \mu_l$ at $k \neq l$ then the system of exponents obtained after the substitution remains complete in $L_p,\ M \subset Z$ is a finite subset Z. Presence of variables of coefficients of system (2) doesn't allow to apply Levinson method. Therefore we'll separately study the basicity of system (1) in

$$L_{p,\rho} \equiv \left\{ f : \int_{-\pi}^{\pi} |f(t)|^p \rho(t)\, dt < +\infty \right\},$$

with the norm

$$\|f\|_{p,\rho} \equiv \left(\int_{-\pi}^{\pi} |f(t)|^p \rho dt \right)^{1/p}.$$

Let's make the following assumptions for the coefficients of system (1) and the weight ρ:

1) $|A(t)|^{\pm 1}; |B(t)|^{\pm 1} \in L_\infty$;

2) $\alpha(t),\ \beta(t)$ are piece-wise Holderian on the segment $[0,\pi]$ and

$\{\rho_i : 0 < \rho_1 < ... < \rho_r < \pi\}$ is a set of break points of the function $\beta(t)$;

3) The sets $T \equiv \{\tau_i\}_1^l$ and $S \equiv \{\rho_i\}_1^r$ are disjoint: $T \cap S = \{\varnothing\}$.

4) $A(t) \cdot A(-1) = B(t) \cdot B(-t)$, $\beta(t) \equiv -\alpha(t)$, $\forall t \in (0, \pi)$.

5) The following theorem is true

Theorem. *Let the following inequalities be fulfilled*

$$-1 < \beta_i < \frac{p}{q}, \; i = \overline{1, l};$$

$$-\frac{1}{p} < \frac{h_k}{\pi} < \frac{1}{q}, \; k = 1, r;$$

$$-\frac{\pi}{2p} < \beta(0) < \frac{\pi}{2q},$$

$$-\frac{\pi}{2q} < \beta(\pi) < \frac{\pi}{2p},$$

where $h_k = \beta(\rho_k + 0) - \beta(\rho_k - 0)$ *are jumps of the function* $\beta(t)$ *at the points* ρ_k, $k = \overline{1, r}$. *Then system (1) forms a basis in* $L_{p,\rho}$; *moreover if* $\exists n_0 : \beta_{n_0} > 0$ *then it is not Bessel in* $L_{2,\rho}$ *and if* $\exists i_0 : \beta_{i0} < 0$ *then it is not Hilbert in* $L_{2,\rho}$.

Before we prove the theorem we remind the definitions of Bessel and Hilbert property of systems in the Hilbert space H.

Let B be some Banach space, B^* be its conjugate space. The system $\{y_n\}_{n\in N} \subset B^*$ is called biorthogonal to the system $\{x_n\}_{n\in N} \subset B$ if $y_n(x_k) = \delta_{nk}$, $\forall n, k \in N$, where δ_{nk} is a Kroneker symbol, N is a set of natural numbers. Following the N.K.Bari paper [2], such system $\{x_n\}_{n\in N}$ we'll call B system.

Definition [1]. *B-system $\{x_n\}_{n\in N}$ in H we'll call a Bessel system if for* $\forall x \in H$

$$\sum_{n\in N} |y_n(x)|^2 < +\infty.$$

Definition [2]. *B-system $\{x_n\}_{n\in N}$ in H we'll call a Hilbert system, if for* $\forall \{c_n\}_{n\in N} \in l_2$, $\exists x \in H$ *that* $y_n(x) = c_n$, $\forall n \in N$.

Now we begin to prove the theorem. At the same time we'll consider the following system of exponents

$$\left\{A(t)\, e^{int}; B(t)\, e^{-ikt}\right\}_{n\geq 0, k\geq 1}. \tag{3}$$

As earlier we proved at fulfillment of conditions 1)-3) and the theorem, system (3) forms a basis in $L_{p,\rho}(-\pi, \pi)$, moreover the system $\left\{e_n^+(t);\; e_{n+1}^-(t)\right\}_{n\geq 0}$ biorthogonal to it has the form

$$e_n^{\pm}(t) \equiv \frac{h_n^{\pm}(t)}{\rho(t)},$$

where

$$h_n^+(t) = \frac{\sum\limits_{k=0}^{n} b_{n-k}^+ \cdot e^{-ikt}}{2\pi Z_0^+(e^{it})\, A(t)}, \; n \geq 0;$$

$$h_n^-(t) = -\frac{\sum\limits_{k=0}^{n} b_{n-k}^- \cdot e^{-ikt}}{2\pi Z_0^+(e^{it})\, A(t)}, \; n \geq 1.$$

In these expressions $Z_0^{\pm}\left(e^{it}\right)$ are not-tangential boundary values of the functions $Z_0^{\pm}(z)$ on a unique circle, and $Z_0^{\pm}(z)$ is canonical solution of the following homogeneous conjugation problem in the Hardy classes $H_p^{\pm}$.

$$Z_0^{+}\left(e^{it}\right)+\frac{B(t)}{A(t)}Z_0^{-}\left(e^{it}\right)=0 \text{ a.e. on } [-\pi,\pi], \tag{4}$$

i.e., we need to find a pair of the analytical functions $\left\{Z_0^{+}(z);Z_0^{-}(z)\right\}$, belonging correspondingly to the Hardy classes H_p^{+}, H_p^{-} respectively inside and outside of a unique circle, whose not tangential boundary values almost everywhere in a unique circle satisfy relation (4). Theory of such problems is well elaborated and shown in the monograph [3]. $\{b_n^{\pm}\}_{n\geq 0}$ are coefficients of the function $Z_0^{\pm}(z)$ at expansions in Taylor series in power of z, respectively, in zero and at the neighbourhood of the point at infinity. As is known [3] the functions $Z_0^{\pm}(z)$ are determined by the following expressions:

$$Z_0(z)=\begin{cases} X_1(z)\cdot Y_1(z), & |z|>1,\\ -X_2^{-1}(z)\cdot Y_2^{-1}(z), & |z|>1,\end{cases}$$

where

$$X_1(z)=\exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi}\ln|G(\rho)|\frac{e^{i\rho}+z}{e^{i\rho}-z}d\rho\right\},\ |z|<1,$$

$$X_2(z)=\exp\left\{-\frac{1}{4\pi}\int_{-\pi}^{\pi}\ln|G(\rho)|\frac{e^{i\rho}+z}{e^{i\rho}-z}d\rho\right\},\ |z|>1,$$

$$Y_1(z)=\exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi}\theta(\rho)\frac{e^{i\rho}+z}{e^{i\rho}-z}d\rho\right\},\ |z|<1,$$

$$Y_2(z)=\exp\left\{-\frac{1}{4\pi}\int_{-\pi}^{\pi}\theta(\rho)\frac{e^{i\rho}+z}{e^{i\rho}-z}d\rho\right\},\ |z|>1,$$

$$G(\rho)\equiv\frac{B(\rho)}{A(\rho)},\ \theta(\rho)\equiv\beta(\rho)-\alpha(\rho)=2\beta(\rho).$$

Note that the case $|z|<1$ corresponds the functions $Z_0^{+}(z)$, and the case $|z|>1$ to $Z_0^{-}(z)$. Sokhotskii-Plemel formulae [3] give:

$$\ln z_0^{+}\left(e^{it}\right)=\ln G(t)+\ln\left[-Z_0^{-}\left(e^{it}\right)\right]=\ln G(t)-\ln X_2\left(e^{it}\right)-\ln Y_2\left(e^{it}\right).$$

Again applying Sokhotskii-Plemel formula to the expressions $X_2(z)$ and $Y_2(z)$ we have:

$$\ln Z_0^{+}\left(e^{it}\right)=\frac{1}{2}G(t)+\frac{i}{4\pi}\int_{-\pi}^{\pi}\ln G(\rho)\,ctg\frac{t-\rho}{2}d\rho.$$

If we'll take into account the identity

$$ctg\frac{t-\rho}{2}=i+i\frac{2e^{i\rho}}{e^{it}-e^{i\rho}},$$

we'll get

$$\ln Z_0^+\left(e^{it}\right)=\frac{1}{2}\ln G\left(t\right)-\frac{1}{4\pi}\int\limits_{-\pi}^{\pi}\ln G\left(\rho\right)d\rho+\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\frac{\ln G\left(\rho\right)e^{i\rho}}{e^{i\rho}-e^{it}}d\rho.$$

From condition 4) it immediately follows that $G\left(-t\right)=G^{-1}\left(t\right),\ \forall t\in[0,\pi]$.

Let

$$I\left(t\right)=\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\frac{\ln G\left(\rho\right)e^{i\rho}}{e^{i\rho}-e^{it}}d\rho.$$

Thus, we have:

$$I\left(t\right)=\frac{1}{2\pi}\int\limits_{0}^{\pi}\frac{\ln G\left(\rho\right)e^{i\rho}}{e^{i\rho}-e^{it}}d\rho.$$

It is easy to establish that the following relations are true

$$\frac{1}{1-e^{i(t-\rho)}}-\frac{1}{1-e^{i(t+\rho)}}=-\frac{i}{2}\frac{\cos\frac{\rho}{2}}{\cos\frac{t}{2}}\left[\frac{1}{\sin\frac{\rho-t}{2}}+\frac{1}{\sin\frac{t+\rho}{2}}\right].$$

Taking into account the relation in the expression for $I\left(t\right)$ we'll obtain:

$$I\left(t\right)=-\frac{i}{4\pi}\int\limits_{0}^{\pi}\ln G\left(\rho\right)\frac{\cos\frac{\rho}{2}}{\cos\frac{t}{2}}\left[\frac{1}{\sin\frac{\rho-t}{2}}+\frac{1}{\sin\frac{t+\rho}{2}}\right]d\rho.$$

Thus $I\left(t\right)$ is an even function on $[-\pi,\pi]$ and finally for $Z_0^+\left(e^{it}\right)$ we have:

$$Z_0^+\left(e^{it}\right)=G^{\frac{1}{2}}\left(t\right)\cdot e^{I(t)},$$

It is evident that $b_0^+\neq 0$. Thus for $h_0^+\left(t\right)$ we have:

$$h_0^+\left(t\right)=\frac{b_0^+}{2\pi}\left[Z_0^+\left(e^{it}\right)A\left(t\right)\right]^{-1}=\frac{b_0^+}{2\pi}e^{-I(t)}\left[A\left(t\right)B\left(t\right)\right]^{-1/2}.$$

As a result we have:

$$C_0^+=\int\limits_{-\pi}^{\pi}e_0^+(t)dt=\int\limits_{-\pi}^{\pi}\frac{h_0^+(t)}{\rho(t)}dt=\frac{b_0^+}{2\pi}\int\limits_{-\pi}^{\pi}e^{-I(t)}\left|A(t)B(t)\right|^{-1/2}\rho^{-1}(t)dt\neq 0.$$

Denote

$$C_n^+=\int\limits_{-\pi}^{\pi}\overline{e_n^{\pm}(t)}dt,n\geq 1.$$

Consider the system $\left\{H_n^+(t);H_k^-\right\}_{n\geq 0,k\geq 1}$, determined by the formulae:

$$H_0^+(t)=\frac{1}{C_0^+}e_0^+(t);H_n^+(t)=e_n^+(t)-b_n^{\pm}e_0^+(t),n\geq 1;$$

where $b_n^{\pm} = \frac{C_n^{\pm}}{C_0^{+}}$.

It is easy to check that the system

$$H_0^{+} \cup \left\{ H_n^{+}; H_n^{-} \right\}_{n\geq 1}$$

is biorthogonally conjugate to system (1). Further we consider the partial sum

$$S_{N^{+},N^{-}} = \sum_{n=0}^{N^{+}} (H_n^{+}, \psi)_\rho \varphi_n^{+}(t) + \sum_{n=1}^{N^{-}} (H_n^{-}, \psi)_\rho \varphi_n^{-}(t),$$

where $\varphi_n^{+}(t) \equiv A(t)e^{int}; \varphi_n^{-}(t) \equiv B(t)e^{int}$ and $(f,g)_\rho = \int_{-\pi}^{\pi} g(t)\overline{f(t)}\rho(t)dt.$

After simple transformations it is easy to get the inequality:

$$\left\| S_{N^{+},N^{-}} - \psi \right\|_{p,\rho} \leq \left\| \sum_{n=0}^{N^{+}} (e_n^{+}, \psi)_\rho \varphi_n^{+} + \sum_{n=1}^{N^{-}} (e_n^{-}, \psi)_\rho \varphi_n^{-} - \psi \right\|_{p,\rho} +$$

$$+ \left\| \frac{1}{C_0^{+}} (e_0^{+}, \psi)_\rho - (e_0^{+}, \psi)_\rho \left[\sum_{n=1}^{N^{+}} b_n^{+} \varphi_n^{+} + \sum_{n=1}^{N^{-}} b_n^{-} \varphi_n^{-} \right] - (e_0^{+}, \psi)_\rho \varphi_0^{+}) \right\|_{p,\rho} \longrightarrow 0$$

as $N^{\pm} \longrightarrow \infty$.

Thus any function is expanded in series by system (1). From $\{H_n^{+}; H_n^{-}\} \subset L_{q,\rho}$ $\left(\frac{1}{p} + \frac{1}{q} = 1 \right)$ it follows that it is minimal in $L_{p,\rho}$ and consequently, such series is unique. As a result system (1) forms a basis in $L_{p,\rho}$.

Consider the case when $p = 2$. Let at some $n_0; \beta_{n0} > 0$. It is obvious that in this case there exists the function $\psi$from $L_{2,\rho}$, which doesn't belong to L_2. From the conditions of the theorem it follows that $\rho^{-1} \in L_2$. It is easy to note that $\{C_n^{\pm}\}$ are biorthogonal coefficients of the function ρ^{-1} by system (3). Therefore $\{C_n^{\pm}\} \in L_2$. For the choosen function ψ we'll find biorthogonal coefficients by system (1) in $L_{2,\rho}$:

$$\psi_n^{\pm} \equiv (H_n^{\pm}, \psi)_\rho = \int_{-\pi}^{\pi} \overline{H_n^{\pm}(t)} \psi(t)\rho(t)dt = \int_{-\pi}^{\pi} \overline{h_n^{\pm}(t)} \psi(t)dt - b_n^{\pm} \int_{-\pi}^{\pi} \overline{e_0^{\pm}(t)} \psi(t)dt.$$

Hence it immediately follows that $\left\{ \psi_n^{\pm} \right\}$ doesn't belong to l_2, since $\{b_n^{\pm}\} \subset l_2$.

And now let $\exists i_0 : \beta_{i0} < 0$. In this case there exists a function $\psi(t)$ from L_2 such that $\psi(t) \cdot \omega(t)$ doesn't belong to L_2. Let $\left\{ \psi_n^{\pm} \right\}$ be biorthogonal coefficients of the function $\psi(t)$ by system (2),.i.e.,

$$\psi_n^{\pm} = \int_{-\pi}^{\pi} \psi(t) \overline{h_n^{\pm}(t)} dt. \tag{5}$$

By virtue of the theorem system (2) forms the Riesz basis in L_2 and thus $\left\{ \psi_n^{\pm} \right\} \in l_2$. Let $\varphi(t) \equiv \psi(t) \cdot \omega(t)$. It is easy to note that

$$\psi_n^{\pm} = \int_{-\pi}^{\pi} \varphi(t) \overline{e_n^{\pm}(t)} dt, \tag{6}$$

i.e., $\{\psi_n^{\pm}\}$ are biorthogonal coefficients of the function $\varphi(t)$ by system (2) in $L_{2,\rho}$. It is evident that system $\{h_n^{\pm}(t)\}$ is complete in L_1. Therefore, passing from relation (6) to (5) we obtain that the function $\varphi(t) \in L_1$ satisfying (6) is unique. Allowing for the connection between the biorthogonal coefficients of the functions from $L_{2,\rho}$ by system (1) and (2), hence we immediately get that system (1) doesn't Hilbert in $L_{2,\rho}$.

The theorem is proved.

The author expresses his deep gratitude to doc. of phys.-math. sci.
B.T.Bilalov for the statement of the problem and attention to the work.

References

[1]. Levin B.Ya. *Distribution of roots of entire functions.* M.: 1956. (Russian)

[2]. Bari N.K. *Biorthogonal systems and basis in Hilbert space.* Uchen. zapiski MSU,1951, v.4, issue 148, pp.69-107. (Russian)

[3]. Danilyuk I.I. *Nonregular boundary-value problems on plane.* M.: "Nauka", 1975. (Russian)

Zaur A. Gasimov
Institute of Mathematics and Mechanics of NAS of Azerbaijan
9, F.Agayev str., AZ1141, Baku, Azerbaijan
Tel.: (99412) 439 47 20 (off.)

Received January 17, 2006; Revised Aprel 12, 2006.
Translated by Mamedova V.A.