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ON OPERATOR CALCULUS AND SPECTRAL REPRESENTATION IN BANACH SPACES

Abstract

In this paper the problems of the operational calculus of some class operators and their spectral representability in Banach spaces are considered.

Let B be a Banach space and $T \in \mathcal{L}(B)$ be an operator, where $\mathcal{L}(B)$ is algebra of the bounded operators from B to B. The operational calculus of such operators for analytical functions in he neighborhood of the spectrum $\sigma(T)$ of operator T is well known. This theory is explicitly elucidated in the known monographs, for ex. in [1-5]. One of the methods of establishment of spectral representability of operators is the use of operator calculus. The third part of a series of the books of Danford N., Schwarz J.T. [3] are devoted to this problem and such operators are called scalar type spectral operators. If the spectrum of the operator is on a Jordan curve, then we can increase the class of such operators depending on the behavior of resolvent of the considered operator. Concerning the given problems see [3].

In the offered work we shall try to give an approach to introduction of operational calculus of some classes of operators and their spectral representability in a Banach space, when the spectrum lies on a curve.

1. Necessary concepts and designation. Let l be an open, rectifiable Jordan curve on complex surface $\mathbf{C}$, whereinto the direction is fixed and the ends do not belong to l. Denote by l^+ and l^- the right and left hand sides of l, respectively. Denote by $\mathcal{A}_l^+ \left(\mathcal{A}_l^-\right)$ the class of functions analytical in some domain $D^+ \subset \mathbf{C} \backslash l^+ \left(D^- \subset \mathbf{C} \backslash l^-\right)$ of functions f , $l^+ \subset \overline{D^+} \left(l^- \subset \overline{D^-}\right)$, that are continuous up to $l^+ \left(l^-\right)$, where $\overline{D}$ is closure of the domain D.

Fig.1.

It is obviously, that $\mathcal{A}_l^+$ is an algebra with respect to usual operations of addition and multiplication of functions.

Denote by $D^+ \left(D^-\right)$ a doamin which is in the left (right) side from l and has $l^+ \left(l^-\right)$ as a part of its boundary. Sequence of rectifiable Jordan curves $\{l_n^+\}_{n=0} \subset D^+ \left(\{l_n^+\}_{n\geq 0} \subset D^-\right)$ is said to be admissible, if they have identical ends with l and converge to $l^+ \left(l^-\right)$ in the sense, that any point $z \in int\left(l_0^+ \cup l^+\right) \left(z \in int\left(l_0^- \cup l^-\right)\right)$

belongs to all domains $int\,(l_n^+ \cup l^+)$ $(int\,(l_n^- \cup l^-))$ beginning with a number $n \in N$. Without loss of generality, we consider, that $l_{n+1}^+ \subset \overline{int\,(l_n^+ \cup l^+)}$, $\forall n_0 \geq 0$. If D^+ is the domain of analyticity for a function $f \in \mathcal{A}_l^+$, we shall denote it by $D_f^{\pm}$. If in domain D^+ resolvent of operator T is analytical, then we shall denote it by $D_T^{\pm}$. It is clear that $D_f^{\pm} \cap D_T^{\pm}$ is the domain D^+.

Denote by $\mathcal{A}_l^+ \times \mathcal{A}_l^-$ Cartesian products $\mathcal{A}_l^+$ and $\mathcal{A}_l^-$. Define component wise multiplication

$(f^+, f^-)\,(g^+, g^-) = (f^+ \cdot g^+, f^- \cdot g^-)$, for $\forall\,(f^+, f^-)\,,\ \ (g^+, g^-) \in \mathcal{A}_l^+ \times \mathcal{A}_l^-$.

Thus, we turn $\mathcal{A}_l^+ \times \mathcal{A}_l^-$ into algebra. Denote by $\mathcal{A}_l$ sub-algebra, consisting of diagonal elements $\mathcal{A}_l^+ \times \mathcal{A}_l^-$, i.e. $\mathcal{A}_l \overset{def}{\equiv} \left\{(f, f) : f \in \mathcal{A}_l^+ \cap \mathcal{A}_l^-\right\}$. Denote by $\mathcal{A}$ the sub-algebra, consisting of functions analytical in the neighborhood of curve l. It is clear, that $\mathcal{A} \subset \mathcal{A}_l \subset \mathcal{A}_l^+ \times \mathcal{A}_l^-$.

Analogously, we define algebra $\mathcal{L}\,(B) \times \mathcal{L}\,(B)$ with corresponding multiplication.

Definition 1. *Let operator function T_λ be analytical in a domain D^+. We'll say that $T_\lambda \in E_p^+\,(l^+)\,(T_\lambda \in E_p\,(l^-))$ if for arbitrary admissible sequence of Jordan curves $\{l_n^+\}_{n\geq 0} \subset D^+ \left(\{l_n^-\}_{n\geq 0} \subset D^-\right)$ there holds*

$$\sup_n \int_{l_n^+} \|T_\lambda\|^p\,|d\lambda| \leq M^+ < +\infty,$$

$$\left(\sup_n \int_{l_n^-} \|T_\lambda\|^p\,|d\lambda| \leq M^- < +\infty\right)$$

where $M^+\,(M^-)$ is an absolute constant.

Denote by $C\,(l)$ Banach space of continuous functions on compact l with sup-norm. $\mathcal{L}\,(X;Y)$ denotes Banach space of linear continuous operators from X to Y with operator norms, where X, Y are Banach spaces.

2. Auxiliary statements. Let's reduce some statements, which are necessary for the proof of the main theorems.

The following statements are true.

Statement 1. *Let $T \in \mathcal{L}\,(C\,(l)\,; \mathcal{L}\,(B))$, where $B = B^{**}$ is a reflexive Banach space. Then there exists a unique family of operators $\{T_t\}_{t\in l} \subset \mathcal{L}\,(B)$, that*

$$Tf = \int_l f\,(t)\,dT_t,\ \forall f \in C\,(l)\,, \tag{1}$$

where equality in (1) is understood in the sense, that

$$x^*\,(Tf)\,y = \int_l f\,(t)\,dx^*\,(T_t y)\,,\quad \forall x^* \in B^*,\ \forall y \in B,$$

where function $x^*\,(T_t y)$ has a restricted variation on l.

This statement is the corollary of the theorem of the paper [1].

Statement 2. *Let B be a Banach space, $\mathcal{L}(B)\times\mathcal{L}^{**}(B)$ and $T\in\mathcal{L}(L_p(l);\mathcal{L}(B))$, $1\le p<+\infty$. Then there exists a unique family of operators $\{T_t\}_{t\in l}\subset\mathcal{L}(B)$, that*

$$Tf=\int\limits_{l} f(\lambda)\,T_\lambda d\lambda,\quad \forall f\in L_p(l),$$

where

$$\int\limits_{l}\|T_\lambda\|^q\,|d\lambda|\le\|T\|^q,\quad \frac{1}{p}+\frac{1}{q}=1,$$

i.e.

$$T_\lambda\in L_q(l;\mathcal{L}(B)).$$

Proof. Let's take $\forall x^*\in\mathcal{L}^*(B)$ and consider the functional

$$l_{x^*}(f)=x^*(Tf).$$

We have

$$|l_{x^*}(f)|\le\|x^*\|\,\|Tf\|\le\|T\|\,\|x^*\|\,\|f\|_p,\tag{2}$$

where

$$\|f\|_p\overset{def}{\equiv}\left(\int\limits_{l}|f(t)|^p\,|dt|\right)^{1/p}.$$

Consequently $l_{x^*}\in L_p^*(l)=L_q(l)$. Then there exists a unique function $g_{x^*}\in L_q(l)$ that

$$l_{x^*}(f)=\int\limits_{l} f(\lambda)\,g_{x^*}(\lambda)\,d\lambda,$$

From the uniqueness of the representation it follows, that

$$g_{ax^*+by^*}=ag_{x^*}+bg_{y^*},$$

where $a,b\in\mathbf{C}$ are arbitrary complex numbers and $x^*,y^*\in\mathcal{L}^*(B)$

Denote by

$$\omega_{x^*}(\lambda)=\int\limits_{a}^{\lambda} g_{x^*}(\tau)\,d\tau,$$

where a is the starting point of the curve l, $\lambda\in l$ and integration is carried out from a to λ by l. It is clear, that

$$\omega_{ax^*+by^*}=a\omega_{x^*}+b\omega_{y^*}.$$

It is clear, that for each fixed $\lambda\in l$, $\omega_{x^*}(\lambda)$ is a linear functional with respect to x^* on $\mathcal{L}^*(B)$ and the following inequality is true

$$|\omega_{x^*}(\lambda)|\le C\,\|g_{x^*}\|_q=C\,\|l_{x^*}\|,$$

where C is an absolute constant. From (2) it follows, that

$$\|l_{x^*}\| \leq \|T\| \, \|x^*\| \, .$$

Thus, $\omega_{x^*}(\lambda) \in \mathcal{L}^{**}(B) = \mathcal{L}(B)$ for each fixed λ.

We introduce designation

$$\frac{d\omega_{x^*}(\lambda)}{d\lambda} = T_\lambda x^* = g_{x^*}(\lambda) \text{ a.e. with respect to } \lambda,$$

and have

$$x^*(Tf) = \int_l f(\lambda) \, T_\lambda x^* d\lambda, \quad \forall x^* \in L^*(B)$$

i.e.

$$Tf = \int_l f(\lambda) \, T_\lambda d\lambda, \quad \forall f \in L_p(l) \, .$$

On the other hand

$$|T_\lambda x^*|^q = |g_{x^*}(\lambda)|^q \text{ a.e. with respect to } \lambda \in l \text{ and}$$

$$\int_l |T_\lambda x^*|^q \, |d\lambda| = \|g_{x^*}\|_q^q = \|l_{x^*}\|^q \leq \|l\|^q \, \|x^*\|^q \leq \|T\|^q \, \|x^*\|^q$$

Consequently, a.e. with respect to $\lambda \in l$ we have

$$\|T_\lambda\|^q = \sup_{\|x^*\|=1} \|T_\lambda x^*\|^q$$

Thus

$$\int_l \|T_\lambda\|^q \, |d\lambda| = \int_l \sup_{\|x^*\|=1} \|T_\lambda x^*\|^q \, |d\lambda| \leq \|T\|^q$$

The uniqueness of the representation follows directly from the uniqueness of the function g_{x^*}.

This completes the proof.

By the analogical scheme the following one is proved.

Statement 3. *Let $B = B^{**}$ be a reflexive Banach space and*

$T \in \mathcal{L}(L_p(l); \mathcal{L}(B))$. Then all statements of statement 2 are true.

Proof. Take $\forall f \in L_p(l)$. Let $x^* \in B^*$, $y \in B$ be arbitrary elements. Let's consider the functional

$$l_{x^*,y}(f) = x^*((Tf) \, y) \, .$$

We have

$$|l_{x^*,y}(f)| \leq \|x^*\| \, \|(Tf) \, y\| \leq \|x^*\| \, \|Tf\| \, \|y\| \leq \|T\| \, \|x^*\| \, \|y\| \, \|f\|_p \, .$$

So, $l_{x^*,y} \in L_p^*(l) = L_q(l)$ and

$$\|l_{x^*,y}\| \leq \|T\| \, \|x^*\| \, \|y\| \, . \tag{3}$$

Then $\exists! g_{x^*,y} \in L_q(l)$

$$l_{x^*,y}(f) = \int_l f g_{x^*,y} d\lambda, \quad \forall f \in L_p.$$

Denoting

$$\omega_{x^*,y}(\lambda) = \int_a^\lambda g_{x^*,y}(\tau)\, d\tau,$$

It is easy to show, that for fixed y and λ, $\omega_{x^*,y}(\lambda)$ linearly depends on x^* and moreover,

$$|\omega_{x^*,y}(\lambda)| \le C \left\| g_{x^*,y} \right\|_q = C \left\| l_{x^*,y} \right\|.$$

It follows from (3)

$$|\omega_{x^*,y}(\lambda)| \le C \|T\| \, \|y\| \, \|x^*\|.$$

Thus, for fixed y and λ:

$$\omega_{x^*,y}(\lambda) \in B^{**} = B.$$

Identifying element $(Tf)\, y \in B$ with element $(Tf)\, y \in B^{**}$ we have representation:

$$(Tf)\, y = \int_l f(\lambda)\, \varphi_y(\lambda)\, d\lambda, \quad \forall f \in L_p(l), \tag{4}$$

where

$$\varphi_y(\lambda)\, x \overset{def}{\equiv} \frac{d\omega_{x^*,y}(\lambda)}{d\lambda}.$$

It is not difficult to note, that

$$\psi_\lambda y \overset{def}{=} \int_a^\lambda \varphi_y(\tau)\, d\tau$$

at fixed λ linearly depends on $y \in B$.

On the other hand

$$\|\psi_\lambda y\| = \sup_{\|x^*\|=1} |x^*(\psi_\lambda y)|, \quad x^* \in B.$$

Identifying $T_\lambda y \in B$ with element $T_\lambda y \in B^{**}$, we have:

$$\|\psi_\lambda y\| = \sup_{\|x^*\|=1} |(\psi_\lambda y)\, x^*|.$$

Thus

$$|(\psi_\lambda y)\, x^*| = \left| \int_a^\lambda \varphi_y(\tau)\,(x^*)\, d\tau \right| = |\omega_{x^*,y}(\lambda)| \le C \left\| l_{x^*,y} \right\| \le \|T\| \, \|y\| \, \|x^*\|$$

So

$$\|\psi_\lambda y\| \le \|T\| \, \|y\|,$$

As a result $\psi_\lambda \in \mathcal{L}(B)$ for each $\lambda \in l$ and

$$\|\psi_\lambda\| \leq \|T\|, \quad \forall \lambda \in l.$$

Introducing designation

$$T_\lambda = \frac{d\psi_\lambda}{d\lambda}$$

we shall get the required representation. On the other hand it is obvious, that

$$x^*(T_\lambda y) = g_{x^*,y}(\lambda) \text{ a.e. with respect to } \lambda$$

and hence

$$\|T_\lambda\|^q = \sup_{\substack{\|x^*\|=1\\ \|y\|=1}} |x^*(T_\lambda y)|^q$$

$$\int_l \|T_\lambda\|^q |d\lambda| \leq \int_l \sup_{\substack{\|x^*\|=1\\ \|y\|=1}} |g_{x^*,y}(\lambda)|^q |d\lambda| \leq \sup_{\substack{\|x^*\|=1\\ \|y\|=1}} \|l_{x^*,y}\| \leq \|T\|.$$

The uniqueness again follows from the uniqueness of the function $g_{x^*,y}(\lambda)$. Statement 2 is proved.

Remark. In all statements the representations of operators are understood in the sense, that corresponding equalities hold for $\forall x^* \in B^*$ (or $x^* \in \mathcal{L}(B)$) and $\forall y \in B$. The topologies in B and B^* are agreed as usual.

3. Establishment of operator calculus. Let's take $\forall f^+ \in \mathcal{A}_l^+$. Let T be an operator from $\mathcal{L}(B)$ with analytical in D_T^+ resolvent $R_\lambda(T)$ and $D^+ = D_f^+ \cap D_T^+$. We take arbitrary admissible sequence of curves $\{l_n^+\}_{n\geq 1} \subset D^+$ and consider integrals $\{I_n^+\}$:

$$I_n^+ = -\frac{1}{2\pi i} \int_{l_n^+} f^+(\lambda) R_\lambda(T) d\lambda, \quad n \geq 1. \tag{5}$$

From analyticity of subintegral function it follows, that integrals (5) are independent of admissible curves $\{l_n^+\}$, they depend only on their general points of beginnings and ends: $I_n^+ = I_m^+$, $\forall n, m \in N$. Denote this general value by $f_{l^+}^+(T)$. It is clear, that $f_{l^+}^+(T) \in \mathcal{L}(B)$. In complete analogy we define $f_{l^-}^-(T)$. Assume, that $R_\lambda(T) \in E_1^+(l^+)$. We have:

$$2\pi \left\|T_n^+\right\| \leq \int_{l_n^+} \left|f^+(\lambda)\right| \|R_\lambda(T)\| |d\lambda| \leq$$

$$\leq \max_{l_n^+} \left|f^+(\lambda)\right| \int_{l_n^+} \|R_\lambda(T)\| |d\lambda| \leq M^+ \max_{l_n^+} \left|f^+(\lambda)\right|$$

Thus

$$\left\|f_{l^+}^+(T)\right\| = \left\|\lim_n I_n^+\right\| \leq M^+ \lim_n \max_{l_n^+} \left|f^+(\lambda)\right| =$$

$$= M^+ \max_{l^+} \left|f^+(\lambda)\right|, \quad \forall f \in \mathcal{A}_l^+ \tag{6}$$

Denote by $C^{+}(l^{+})$ a manifold, consisting of boundary values of functions from $\mathcal{A}_l^{+}$. As $\mathcal{A}_l^{+}$ contains all polynomials, then from the classical Runge theorem it follows, that $C^{+}(l^{+})$ densely in $C(l^{+})$ class of continuous functions on l^{+} with sup-norm. It is easy to see, that there is one-to-one correspondence between $C^{+}(l^{+})$ and $\mathcal{A}_l^{+}$.

From evaluation (6) and statement 1 it follows, that there exists an operator-valued function of the set E_λ^{+} with boundary variation in the sense that for $\forall x^* \in B^*$, $\forall y \in B$, the function $x^*\left(E_\lambda^{+} y\right)$ has bounded variation and

$$f_{l^+}^{+}(T) = -\frac{1}{2\pi i}\int\limits_{l^+} f^{+}(\lambda)\, dE_\lambda^{+}. \qquad (7)$$

From density of $C^{+}(l^{+})$ in $C(l^{+})$ it follows, that representation (7) is unique.

If $R_\lambda(T) \in E_1^{-}(l^{-})$, the operator $f_l^{-}(T)$, corresponding to function $f^{-}(\lambda) \in \mathcal{A}_l^{-}$ has analogical representation

$$f_l^{-}(T) = -\frac{1}{2\pi i}\int\limits_{l^-} f^{-}(\lambda)\, dE_\lambda^{-}.$$

Consequently, for each function $F = (f^{+}, f^{-}) \in \mathcal{A}_l^{+} \times \mathcal{A}_l^{-}$ we can associate the operator $F_l(T) = \left(f_{l^+}^{+}(T), f_{l^-}^{-}(T)\right)$. Denote this correspondence by $\omega : \mathcal{A}_l^{+} \times \mathcal{A}_l^{-} \to \mathcal{L}(B) \times \mathcal{L}(B)$, $\omega(F) = F_l(T)$ is a linear mapping.

Now we take arbitrary sequences of admissible curves $\{l_n^{\pm}\}_{n\geq 1} \subset D^{\pm}$. Let $L_n = l_n^{+} \cup l_n^{-}$. We will assume direction in L_n as positive, i.e. interior domain $int\,(l_n^{+} \cup l_n^{-})$ remains from the left by passing in L_n. Let's consider

$$J_n = -\frac{1}{2\pi i}\int\limits_{L_n} f(\lambda)\, R_\lambda(T)\, d\lambda, \quad \forall n \geq 1,$$

where

$$f(\lambda) = \begin{cases} f^{+}(\lambda), & \lambda \in l_n^{+} \\ f^{-}(\lambda), & \lambda \in l_n^{-}, \end{cases}$$

and $(f^{+}, f^{-}) \subset \mathcal{A}_l^{+} \times \mathcal{A}_l^{-}$ is an arbitrary element. It is clear, that (remaining the previous designation)

$$J_n = f_{l^+}^{+}(T) + f_l^{-}(T). \qquad (8)$$

Let $\Gamma = l^{+} \cup l^{-}$, i.e. Γ is the curve of l, twice passable, at first from a to b in l^{-}, then from b to a in l^{+}. Then we can identify $\mathcal{A}_l^{+} \times \mathcal{A}_l^{-}$ with a manifold in $C(\Gamma; b)$, where $C(\Gamma; b)$ is a class of piecewise continuos in Γ functions on Γ with sup-norm, which can have at first order break at the point b.

Using representation for $f_{l^\pm}^{\pm}(T)$, from (8) we have:

$$f(T) \overset{def}{\equiv} \int\limits_{l^+} f^{+}(\lambda)\, dE_\lambda^{+} + \int\limits_{l^-} f^{-}(\lambda)\, dE_\lambda^{-}.$$

Introducing operator of sets E_λ by formula $E_\lambda = E_{\alpha\cap l^+}^{+} + E_{\alpha\cap l^-}^{-}$ for Borel sub-sets Γ we have representation:

$$f(t) = \int\limits_{\Gamma} f(\lambda)\, dE_\lambda, \qquad (9)$$

for $\forall f \in \mathcal{A}_l^+ \times \mathcal{A}_l^-$. But again, if take into attention, that class of functions $(p^+(\lambda), p^-(\lambda))$ is complete in $C^+(l^+) \times C^-(l^-)$, where $p^{\pm}(\lambda) \in C^{\pm}(l^{\pm})$ are arbitrary polynomials, we will get a uniqueness of representation (9). Thus, to any function $f \in \mathcal{A}_l^+ \times \mathcal{A}_l^-$ we will associate the operator $f(T) \in \mathcal{L}(B)$. It is easy to see, that such correspondence is linear.

Later, assume, that there is isomorphism between $\mathcal{A}_l^+$ and $\mathcal{A}_l^-$, i.e. there exists $\mathcal{F} : \mathcal{A}_l^+ \leftrightarrow \mathcal{A}_l^-$, where $\mathcal{F}$ is restrictedly reversible. Let $\omega^{\pm} : \mathcal{A}_l^{\pm} \to \mathcal{L}(B)$ be the mapping defined above:

$$\omega \pm \left(f^{\pm}\right) = f_{l^{\pm}}^{\pm}(T).$$

We shall get

$$\omega^-\left(f^-\right) = \omega^-\left(F^{-1}f^+\right), \quad \forall f^+ \in A_l^+$$

Thus, any operator ω^- on $\mathcal{A}_l^-$ defines some operator on $\mathcal{A}_l^+$ and vice versa. It is easy to show, that correspondence $u : \omega^- \leftrightarrow \omega^+$ is isomorphism. As a result

$$\omega^+\left(f^+\right) = u\left(\omega^-\left(\mathcal{F}f^{-1}\right)\right) = \int_{l^-} \left[\mathcal{F}^{-1}f^-\right](\lambda)\, duE_\lambda^- = \int_{l^+} f^+(\lambda)\, duE_\lambda^-.$$

Taking into account, that

$$\omega^+\left(f^+\right) = \int_{l^+} f^+(\lambda)\, duE_\lambda^+,$$

from the uniqueness of representation we have

$$E_\lambda^+ = uE_\lambda^-.$$

Now we consider sub-algebra $\mathcal{A}$. Considering $\mathcal{A}$ as sub-algebra of space $C(l)$ and repeating the previous reasoning we have that there exists a unique function of sets E_λ in l

$$\int_{l^+} f(\lambda)\, dE_\lambda^+ + \int_{l^+} f(\lambda)\, dE_\lambda^- = \int_l f(\lambda)\, dE_\lambda.$$

Thus

$$\int_l f(\lambda)\, d\left(E_\lambda^+ + E_\lambda^-\right) = \int_l f(\lambda)\, dE_\lambda.$$

From the uniqueness of the representation we have:

$$E_\lambda = E_\lambda^+ + E_\lambda^-.$$

As usual, now we prove, that correspondence $\omega : \mathcal{A} \to \mathcal{L}(T)$ is homomorphism.

Denote by $\chi_\alpha(\lambda)$ a characteristic function of set $\alpha \subset l$. Let α_i, $i = \overline{1,n}$ be connected components of l, moreover $\overline{\alpha}_i \cap \overline{\alpha}_j = \varnothing$, $i \neq j$. It is obviously, that in this case for the function $\chi_{\cup\alpha_i(\lambda)}$ there exists a sequence $\{f_n\} \subset \mathcal{A} : f_n(\lambda) \to f_n(\lambda) \to \chi_{\cup\alpha_i(\lambda)}$ and $|f_n(\lambda)| \leq M < +\infty$, $\forall \lambda \in l$.

By definition of the integral

$$E_\alpha = \int_l \chi_\alpha(\lambda)\, dE_\lambda$$

$$E_\alpha^* = \int_l \chi_\alpha(\lambda)\, dE_\lambda^*,$$

for arbitrary E_λ of a measurable set $\alpha \subset l$. It is clear, that from $f_n \to \chi_\alpha$ it follows, that $f_n^2 \to \chi_\alpha$. Let B be a reflexible space. Take $\forall x \in B$ and $\forall y^* \in B^*$. By definition of conjugate operator we have:

$$[f_n^*(T)\, y^*]\,[f_n(T)\, x] = y^*\,[f_n(T)\,[f_n(T)\, x]] = y^*\left[[f_n(T)]^2\, x\right].$$

From correlation

$$f_n^2(T) = f_n(T)\, f_n(T) = \omega(f_n)\,\omega(f_n) = \omega\left(f_n^2\right)$$

we will get

$$y^*\left[[f_n(T)]^2\, x\right] = y^*\left[\left[f_n^2\right](T)\, x\right]$$

Passing to limit as $n \to \infty$ in integrands

$$f_n(T) = \int_l f_n(\lambda)\, dE_\lambda, \quad f_n^2(T) = \int_l f_n^2(\lambda)\, dE_\lambda$$

we will get

$$f_n(T) \to E_\lambda, \quad f_n^2(T) \to E_\alpha$$

consequently

$$y^*\left[E_\alpha^2 x\right] = y^*\left[E_\alpha x\right], \quad \forall x \in B, \quad \forall y^* \in B^*.$$

Thus

$$E_\alpha^2 = E_\alpha, \quad \forall \alpha \subset l,$$

where α consists of non-crossing half-intervals.

Denumerable additively of measure E_α is proved in a standard way.

Now take $\forall \alpha,\ \beta \subset l$, where α, β is a union of connected components l of the above considered form. Establishing sequences

$$\left\{f_n^\alpha; f_n^\beta\right\} \subset \mathcal{A} : f_n^\alpha \to \chi_\alpha, \ f_n^\beta \to \chi_\beta,$$

where

$$\left|f_n^{\alpha,\beta}(\lambda)\right| \le M < +\infty, \quad \forall \lambda \in l,$$

we have

$$f_n^\alpha(T)\, f_m^\beta(T) = \omega(f_n^\alpha)\,\omega\left(f_m^\beta\right) = \omega\left(f_n^\alpha f_m^\beta\right) = \left[f_n^\alpha f_m^\beta\right](T) \ .$$

It is obvious, that $f_n^\alpha f_m^\beta \to \chi_{\alpha\cap\beta}$. Passing to limit as $n, m \to \infty$, from this equality we have:

$$E_\alpha E_\beta = E_{\alpha\cap\beta}.$$

Hence, it follows, that the family E_λ is commuting.

Let O_r be a circle of radius r with centre in zero. Take arbitrary function $f \in \mathcal{A}_l$ which is analytical in $\mathbf{C}\backslash l$. Assuming, that $\sigma(T) \subset l$, expanding $R_\lambda(T)$ at $|\lambda| > \|T\|$ in power λ in series

$$R_\lambda(T) = -\frac{1}{\lambda}\left(I + \frac{T}{\lambda} + ...\right),$$

we have

$$f(t) = \int_{L_0} f(\lambda) R_\lambda(T)\, d\lambda = -\left[\int_{O_r} \frac{f(\lambda)}{\lambda} d\lambda I + \int_{O_r} \frac{f(\lambda)}{\lambda^2} d\lambda T + ...\right]$$

Taking $f(\lambda) \equiv 1$ and $f(\lambda) \equiv \lambda$ we have:

$$1(T) = I, \quad \lambda(T) = T.$$

Thus,

$$I = \int_l 1 dE_\lambda,$$

$$T = \int_l \lambda dE_\lambda$$

Summarizing we have the following theorem

Theorem 1. *Let l be a Jordan curve with above-mentioned characteristics, $T \in \mathcal{L}(B)$ with resolvent $R_\lambda(T) \in E_1^+(l^+) \cap E_1^-(l^-)$, where $B = B^{**}$. Then there exists a unique class of projects $\{E_\lambda\}_{\lambda \in \Gamma} \subset \mathcal{L}(B)$, such that for $\forall f \in C(\Gamma; b)$ it holds representation*

$$f(T) = \int_\Gamma f(\lambda)\, dE_\lambda,$$

where if $\sigma(T) \subset l$, then

$$I = \int_\Gamma f(\lambda)\, dE_\lambda, \quad T = \int_\Gamma \lambda dE_\lambda\ .$$

Considering sub-algebra $\mathcal{A}$ as independent algebra we get the similar theorem:

Theorem 2. *Let all the conditions of Theorem 1 be fulfilled. Then there exists a unique class of projects $\{E_\lambda\}_{\lambda \in l} \subset \mathcal{L}(B)$, where for $\forall f \in \mathcal{A}$ operator $f(T)$ has the representation*

$$f(T) = \int_l f(\lambda)\, dE_\lambda,$$

when, if $\sigma(T) \subset l$, then

$$I = \int_l 1 dE_\lambda, \quad T = \int_l \lambda dE_\lambda.$$

Now assume, that $T \in \mathcal{L}(B)$ is an operator with resolvent $R_\lambda(T) \in E_q^+(l^+)$, where $q \in (1, +\infty)$, $\frac{1}{p} + \frac{1}{q} = 1$. For $\forall f^+ \in E_p^+(l^+)$ defining operator $f_{l^+}^+(T)$ by expression (5) we have

$$\left\| f_{l^+}^+(T) \right\| \leq M^+ \left\| f^+ \right\|_{L_p(l^+)}.$$

Thus, operator $\omega^+ : E_p^+(l^+) \to \mathcal{L}(B)$ defined by formula $\omega^+(f^+) = f_{l^+}^+(T)$ is restricted. Noting that $E_p^+(l^+)$ is dense in $L_p(l^+)$ and continuing operator ω^+ by continuity on all $L_p(l^+)$ and remaining previous designation we shall get $\omega^+ \in \mathcal{L}(L_p(l^+), \mathcal{L}(H))$. Then from statements it follows, that there exists a unique family at operators $\left\{T_\lambda^+\right\}_{\lambda \in l} \in L_q(l, \mathcal{L}(B))$, such that

$$f_{l^+}^+(T) = \int\limits_{l^+} f^+(\lambda) T_\lambda^+ d\lambda, \quad \forall f^+ \in L_p\left(l^+\right).$$

We get analogical representation for $f_{l^-}^-(T)$:

$$f_{l^-}^-(T) = \int\limits_{l^-} f^-(\lambda) T_\lambda^- d\lambda, \quad \forall f^- \in L_p\left(l^-\right).$$

Defining operator $f(T) = f_{l^+}^+(T) + f_{l^-}^-(T)$ for $\forall (f^+, f^-) \in \mathcal{A}_l^+ \times \mathcal{A}_l^-$, identifying $L_p(l^+) \times L_p(l^-)$ and $L_p(\Gamma)$, we get the following theorem.

Theorem 3. *Let $B = B^{**}$, l be a Jordan curve with above mentioned characteristics, $T \in \mathcal{L}(B)$ be an operator with resolvent $R_\lambda(T) \in E_q^+(l^+) \cap E_q^-(l^-)$. Then there exists a unique family of operators $\{T_\lambda\}_{\lambda \in \Gamma} \in L_p(\Gamma, \mathcal{L}(B))$ such that*

$$f(T) = \int\limits_\Gamma f(\lambda) T_\lambda d\lambda, \quad \forall f \in L_p(\Gamma).$$

where

$$E_\alpha = \int\limits_\Gamma T_\lambda d\lambda, \quad \forall \alpha \subset \Gamma \text{ is measurable}$$

are projects in $\mathcal{L}(B)$ and

$$I = \int\limits_\Gamma T_\lambda d\lambda, \quad T = \int\limits_\Gamma \lambda T_\lambda d\lambda.$$

In order to establish that E_α is a project for $\forall$ of Borel set $\alpha \subset \Gamma$, we can represent $f(T)$ in the form

$$f(T) = \int\limits_\Gamma f(\lambda)\, dE_\lambda,$$

and use the fact, that $C(\Gamma)$ is dense in $L_p(\Gamma)$.

Starting from functions $f \in E_p^+(l^+) \cap E_p^-(l^-) \overset{def}{\equiv} E_p(l)$, analogously to previous reasoning we have the theorem:

Theorem 4. *Let all conditions of theorem 3 be fulfilled. Then there exists a unique class of operators* $\{T_\lambda\}_{\lambda\in l} \in L_p(l, \mathcal{L}(B))$ *such that*

$$f(T) = \int_l f(\lambda)\, T_\lambda d\lambda, \quad \forall f \in L_p(l),$$

where operators

$$E_\alpha = \int_\alpha T_\lambda d\lambda,$$

are the projects in $\mathcal{L}(B)$ *for any Borel set* $\alpha \subset l$.

4. A class of curves $\Gamma_\lambda^\pm$ and corresponding operational calculus. We'll say that $l \in \Gamma_\lambda^\pm$ and if there exists a sequence of picewise smooth curves $\{l_n^\pm\}_{n\geq 0} \subset D^\pm$, having identical ends and beginnings with l, such that the conditions are fulfilled:

1) $\overline{D_{0,n}^\pm} \subset D_{0,n+1}^\pm$, $\forall n \geq 1$, where $D_{0,n}^\pm = int\left(l_0^\pm \cup l_n^\pm\right)$;

2) for $\forall z \in D_{0,n}^\pm$, $\exists n_z \in N : z \in D_{0,n}^\pm$, $\forall n \geq n_z$;

3) there exists analyticals in $D^\pm$ functions $v_n^\pm(\lambda)$ such that there exists a uniform limit $\lim\limits_{n\to\infty} v_n^\pm(\lambda) \equiv \lambda$ in $D^\pm$, moreover $v_n^\pm : l_n^\pm \leftrightarrow l^\pm$ one-to-one.

It is easy to see that the class $\Gamma_\lambda^\pm$ is sufficiently wide. Since if in some direction any straight line intersects l maximum at one point, it is easy to establish corresponding sequence. Let $\Gamma_\lambda = \Gamma_\lambda^+ \cap \Gamma_\lambda^-$.

And now let $l \in \Gamma_\lambda^+$ and $f^+ \in A_l^+$. Consider

$$f_{l^+}^+(T) \overset{def}{\equiv} -\frac{1}{2\pi i} \lim_{n\to\infty} \int\limits_{l_n^+} f^+\left(v_n^+(\lambda)\right) R_\lambda(T)\, d\lambda\,. \tag{10}$$

Let's show that this definition is correct. Thus, from analyticity of integrand function in $D_{0,n}^+$ we have:

$$I_n^+ \overset{def}{\equiv} \int\limits_{l_n^+} f^+\left(v_n^+(\lambda)\right) R_\lambda(T)\, d\lambda = \int\limits_{l_0^+} f^+\left(v_n^+(\lambda)\right) R_\lambda(T)\, d\lambda.$$

From $dist\left(l_0^+, \sigma(T)\right) > 0$ it follows, that

$$\|R_\lambda(T)\| \leq C, \ \forall \lambda \in l_0^+.$$

Thus, we have:

$$\left\|I_n^+ - I_m^+\right\| \leq \int\limits_{l_0^+} \left|f^+\left(v_n^+(\lambda)\right) - f^+\left(v_m^+(\lambda)\right)\right| \cdot \|R_\lambda(T)\| \cdot |d\lambda| \leq$$

$$\leq C\left|l_0^+\right| \max_{l_0^+} \left|f^+\left(v_n^+(\lambda)\right) - f^+\left(v_m^+(\lambda)\right)\right| \to 0, \quad \text{as} \quad n, m \to \infty.$$

Hence, the sequence $\{I_n^+\}$ is fundamental in $\mathcal{L}(B)$. Analogously we define

$$f_{l^-}^-(T) = -\frac{1}{2\pi i} \lim_{n\to\infty} \int\limits_{l_n^-} f^-\left(v_n^-(\lambda)\right) R_\lambda(T)\, d\lambda \tag{11}$$

for $f^- \in A_l^-$.

On the other hand from correlation

$$\left\| \int_{l_n^+} \left[f^+ \left(v_n^+ (\lambda) \right) - f^+ (\lambda) \right] R_\lambda (T) \, d\lambda \right\| = \left\| \int_{l_0^+} \left[f^+ \left(v_n^+ (\lambda) \right) - f^+ (\lambda) \right] R_\lambda (T) \, d\lambda \right\| \leq$$

$$\leq C \max_{l_0^+} \left| f^+ \left(v_n^+ (\lambda) \right) - f^+ \left(v_m^+ (\lambda) \right) \right| \to 0,$$

as $n \to \infty$, it follows, that definition (10) of operator $f_{l^+}^+ (T)$ is equivalent to one introduced earlier. Analogous concept is true for the operator $f_{l^-}^- (T)$, defined by formula (11) too.

Denote by $\mu_n^\pm : l^\pm \leftrightarrow l_n^\pm$ the function inverse to $v_n^\pm$ on $l^\pm$. We say, that the curve l belongs to class $\Gamma_{\lambda-\mu}^\pm$, if besides conditions 1)-3) it is fulfilled also the following condition:

4) expression

$$\frac{[\mu_n^\pm]' (\lambda) - 1}{dist \left(\mu_n^\pm (\lambda), l \right)}$$

uniformly converges to zero as $n \to \infty$.

Let $\Gamma_{\lambda-\mu} = \Gamma_{\lambda-\mu}^+ \cap \Gamma_{\lambda-\mu}^-$. We'll assume, that operator $T \in E_p (\Gamma_{\lambda-\mu})$, if

$$\int_l \left\| R_{\mu_n^+(\lambda)} (T) - R_{\mu_n^-(\lambda)} (T) \right\|^p |d\lambda| \leq M < +\infty,$$

uniformly on n. Assuming, that $T \in E_q (\Gamma_{\lambda-\mu})$, $f \in \mathcal{A}$, for operator $f(T)$:

$$f(T) \overset{def}{\equiv} f_{l^+}^+ (T) + f_{l^-}^- (T)$$

we have:

$$f(T) = -\frac{1}{2\pi i} \lim_n \left[\int_l f(\lambda) \left(\mu_n^+ \right)' (\lambda) R_{\mu_n^+(\lambda)} (T) - \right.$$

$$\left. - \int_l f(\lambda) \left(\mu_n^- \right)' (\lambda) R_{\mu_n^-(\lambda)} (T) \right] d\lambda.$$

Hence:

$$\|f(T)\| = \frac{1}{2\pi} \lim_n || \int_l f(\lambda) \times$$

$$\times \left\{ \left[\left(\mu_n^+ \right)' - 1 \right] R_{\mu_n^+} (T) + R_{\mu_n^+} (T) - \left[\left(\mu_n^- \right)' - 1 \right] R_{\mu_n^-} (T) + R_{\mu_n^-} (T) \right\} d\lambda \right\| \leq$$

$$\leq \frac{\|f\|_{C(l)}}{2\pi} \lim_n \left[\int_l \left| \left(\mu_n^+ \right)' - 1 \right| \cdot \left\| R_{\mu_n^+} (T) \right\| \cdot |d\lambda| + \right.$$

$$\left. + \int_l \left| \left(\mu_n^- \right)' - 1 \right| \cdot \left\| R_{\mu_n^-} (T) \right\| \cdot |d\lambda| + \int_l \left\| R_{\mu_n^+(\lambda)} - R_{\mu_n^-(\lambda)} \right\| \cdot |d\lambda| \right].$$

If there hold $l \in \Gamma_{\lambda-\mu}$, $T \in E_1(\Gamma_{\lambda-\mu})$, from this inequality we have:

$$\|f(T)\| \leq M_1 \|f\|_{C_l}.$$

Repeating reasoning of item 2, we arrive at the following theorem.

Theorem 5. *Let Jordan curve l belong to class $\Gamma_{\lambda-\mu}$ and $T \in E_1(\Gamma_{\lambda-\mu})$, $B = B^*$. Then there exists a unique family of projects $\{E_\lambda\}_{\lambda\in l} \subset \mathcal{L}(B)$, such that for $\forall f \in \mathcal{A}$, the operator $f(T)$ has representation*

$$f(T) = \int_l f(\lambda)\, dE_\lambda,$$

and for $\sigma(T) \subset l$ it holds

$$I = \int_l 1dE_\lambda, \quad T = \int_l \lambda dE_\lambda.$$

Remark. Denote by $BE_p^+(l^+)$ a class of analytical in some domain D^+ operator-functions T_λ, which for arbitrary admissible sequence of Jordan curves $\{l_n^+\} \subset D^+$ satisfy

$$\sup_n \int_{l_n^+} |x^* T_\lambda x|^p\, |d\lambda| \leq M_{x^*x}^{\pm} < +\infty$$

for $\forall x^* \in B^*$, $\forall x \in B$, where $M_{x^*x}^+$ is a constant that depends only on x^* and x.

We can relax requirements on resolvents of the considered operators in previous statements, requiring theirs belonging to corresponding spaces $BE_p^{\pm}$.

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