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ON SOLVABILITY OF A CLASS OF FOURTH ORDER OPERATOR-DIFFERENTIAL EQUATIONS

Abstract

In a separable Hilbert space H it was researched a regular solvability of a class of operator-differential equations of fourth order, main part of which contains operator-differential equations with a multiple characteristic. Moreover, a basic operator is normal with spectrum in some sector. Conditions of solvability of the given operator-differential equation on the axis were obtained. These conditions are expressed only by the properties of operator coeffcients of the given operator differential equation.

Let H be a separable Hilbert space, A be a normal inverse operator in H, and operators A_j $(j = 0, ..., 4)$ be linear, in general, nonbounded operators. We suppose fulfillment of the following conditions:

a) spectrum of the operator A is contained in sector

$$S_\varepsilon = \{\lambda : |\arg \lambda| \le \varepsilon\}, \ 0 \le \varepsilon < \pi/2;$$

b) operators $B_j = A_j A^j$ $(j = 0, ..., 4)$ are bounded in H.

When conditions a) and b) hold, operator A has expansion $A = UC$, where U is unitary, and C is a positive definite self-adjoined operator. Obviously, $D(A) = D(A^*) = D(C)$ and $\|Ax\| = \|A^*x\| \, \|Cx\|$ at $x \in D(A)$.

Denote by $L_2(R : H)$ $(R = (-\infty, \infty))$ a Hilbert space of the vector-functions $f(t)$ with the value in H, measurable and square integrable by Bochner and suppose

$$\|f\|_{L_2(R;H)} = \left(\int_{-\infty}^{\infty} \|f(t)\|^2 \, dt \right)^{1/2}.$$

Further introduce a Hilbert space [1]

$$W_2^4(R : H) = \left\{ u | u^{(4)} \in L_2(R : H), \ A^4 \in L_2(R : H) \right\}$$

with the norm

$$\|u\|_{W_2^4(R:H)} = \left(\left\| u^{(4)} \right\|^2_{L_2(R;H)} + \left\| A^4 u \right\|^2_{L_2(R:H)} \right)^{1/2}.$$

Now consider in space H the operator-differential equation

$$\left(-\frac{d^2}{dt^2} + A^2 \right)^2 u(t) + \sum_{j=0}^{4} A_{4-j} u^{(j)}(t) = f(t), \quad t \in R, \tag{1}$$

where

$$f(t) \in L_2(R : H), \quad u(t) \in W_2^4(R : H).$$

Definition 1. *If vector-function* $u(t) \in W_2^4(R:H)$ *satisfies equation (1) almost everywhere in* R, *so this vector-function is said to be a regular solution of equation (1).*

Definition 2. *If at any* $f(t) \in L_2(R:H)$ *there exists a regular solution of equation (1)* $u(t)$, *which satisfies inequality*

$$\|u\|_{W_2^4(R:H)} \leq const \, \|f\|_{L_2(R:H)},$$

so, we'll say, that equation (1) is regularly solvable.

In the given paper we find conditions on operator coefficients of equation (1), providing a regular solvability of equation (1).

Note, that equation (1) was researched in the paper [2, 3] for self-adjoined positive operator A. Existence and uniqueness of some holomorphic solutions for second order operator-differential equations are considered by the author in the papers [4, 5].

Note, that for characteristic equation in the main part of equation (1) has simple roots, i.e. when the main part has the form $P_0 u = u^{(4)} + A^4 u$, analogies of equation (1) for operator-differential equations were considered in the papers [6, 7].

Denote by

$$P_0 u = \left(-\frac{d^2}{dt^2} + A^2\right)^2 u(t), \quad u(t) \in W_2^4(R:H),$$

$$P_1 u = \sum_{j=0}^{4} A_{4-j} u^{(j)}(t), \quad u(t) \in W_2^4(R:H)$$

and

$$Pu = P_0 u + P_1 u, \quad u \in W_2^4(R:H).$$

Further denote by $H_4 = D\left(A^4\right)$ and $D(R:H_4)$ a set of the infinitely differentiable vector-functions with the values in H_4. As is known, $D(R:H_4)$ is dense everywhere in $W_2^4(R:H)$.

The following theorem is true:

Theorem 1. *Let a normal inverse operator* A *satisfy condition a). then operator* P_0 *isomorfically maps space* $W_2^4(R:H)$ *onto* $L_2(R:H)$.

Proof. It is obvious, that at $u \in D(R:H_4)$ the following inequalities hold

$$\|P_0 u\|_{L_2(R:H)} = \left\|\left(-\frac{d^2}{dt^2} + A^2\right) u\right\|_{L_2(R:H)} =$$

$$= \left\|\frac{d^4}{dt^4} + A^4 u - 2A^2 \frac{d^2 u}{dt^2}\right\|_{L_2(R:H)} \leq$$

$$\leq \left(\left\|\frac{d^4 u}{dt^4}\right\|^2_{L_2(R:H)} + \left\|A^4 u\right\|^2_{L_2(R:H)} + 4\left\|A^2 \frac{d^2 u}{dt^2}\right\|^2_{L_2(R:H)}\right)$$

On the other hand from the theorem on intermediate derivative it follows, that [1]

$$\left\|A^2 \frac{d^2 u}{dt^2}\right\|_{L_2(R:H)} = \left\|C^2 \frac{d^2 u}{dt^2}\right\|_{L_2(R:H)} \leq$$

$$\leq const\left(\left\|C^4u\right\|^2_{L_2(R:H)}+\left\|u^{(4)}\right\|^2_{L_2(R:H)}\right)=$$

$$=const\left(\left\|A^2u\right\|^2_{L_2(R:H)}+\left\|u^{(4)}\right\|^2_{L_2(R:H)}\right).$$

Thus

$$\left\|P_0u\right\|_{L_2(R:H)}\leq const\left\|u\right\|_{W_2^2(R:H)}. \tag{2}$$

It is obvious that inequality (2) is true also for vector-functions from $W_2^4\left(R:H\right)$.

Prove, that for any $f\left(t\right)\in L_2\left(R:H\right)$ there exists a regular solution of the equation (1). Indeed, let $\hat{f}\left(\xi\right)$ be a Fourier transformation of vector-function $f\left(t\right)$, then denote by

$$u\left(t\right)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}P_0^{-1}\left(i\xi\right)\hat{f}\left(\xi\right)e^{i\xi t}d\xi\equiv$$

$$\equiv\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left(\xi^2+A^2\right)^{-2}\hat{f}\left(\xi\right)e^{i\xi t}d\xi,\quad t\in R.$$

It is obvious, that vector-function $u\left(t\right)$ satisfies equation (1) almost everywhere in $R\left(-\infty,\infty\right)$. Let's show, that $u\left(t\right)\in W_2^4\left(R:H\right)$.

Let $\hat{u}\left(\xi\right)$ be a Fourier transformation of vector-function $u\left(t\right)$. Then, from Plansherel theorem it follows that

$$\left\|u\right\|^2_{W_2^4(R:H)}=\left\|u^{(4)}\right\|^2_{L_2(R:H)}+\left\|A^4u\right\|^2_{L_2(R:H)}=$$

$$=\left\|\xi^4\hat{u}\left(\xi\right)\right\|^2_{L_2(R:H)}+\left\|A^4\hat{u}\left(\xi\right)\right\|^2_{L_2(R:H)}=$$

$$=\left\|\xi^4P_0^{-1}\left(i\xi\right)\hat{f}\left(\xi\right)\right\|^2_{L_2(R:H)}+\left\|A^4P_0^{-1}\left(i\xi\right)\hat{f}\left(\xi\right)\right\|^2_{L_2(R:H)}\leq$$

$$\leq\sup_{\xi\in R}\left\|\xi^4P_0^{-1}\left(i\xi\right)\right\|^2\left\|\hat{f}\left(\xi\right)\right\|^2_{L_2(R:H)}+$$

$$+\sup_{\xi\in R}\left\|A^4P_0^{-1}\left(i\xi\right)\right\|^2\left\|\hat{f}\left(\xi\right)\right\|^2_{L_2(R:H)}=$$

$$=\left(\sup_{\xi\in R}\left\|\xi^4P_0^{-1}\left(i\xi\right)\right\|^2_{L_2(R_+:H)}+\right.$$

$$\left.+\sup_{\xi\in R}\left\|A^4P_0^{-1}\left(i\xi\right)\right\|^2_{L_2(R:H)}\right)\left\|f\right\|^2_{L_2(R:H)} \tag{3}$$

It is obvious, that at $\xi\in R$ the following inequalities hold ($\mu=re^{i\varphi}\in\sigma\left(A\right)$):

$$\left\|\xi^4P_0^{-1}\left(i\xi\right)\right\|=\left\|\xi^4\left(\xi^2E+A^2\right)^{-2}\right\|=\sup_{\mu\in\sigma(A)}\left|\xi^4\left(\xi^2+\mu^2\right)^2\right|=$$

$$=\sup_{\substack{r>0\\|\psi|\leq\varepsilon}}\left|\xi^4\left(\xi^2+r^2e^{2i\varphi}\right)^{-2}\right|=\sup_{\substack{r>0\\|\psi|\leq\varepsilon}}\left|\xi^4\left(\xi^2+r^4+2\xi^2r^2\cos2\varphi\right)^{-1}\right|\leq$$

$$\leq \sup_{r>0}\left|\xi^4\left(\xi^4+r^4+2\xi^2r^2\cos 2\varepsilon\right)^{-1}\right|. \tag{4}$$

At $0\leq\varepsilon\leq\pi/4$ it follows from inequality (4), that $(\cos 2\varepsilon\geq 0)$

$$\left|\xi^4P_0^{-1}(i\xi)\right|\leq \sup_{r>0}\left|\xi^4\left(\xi^4+r^4\right)^{-1}\right|\leq 1, \tag{5}$$

and at $\pi/4\leq\varepsilon<\pi/2$ $(\cos 2\varepsilon\leq 0)$, using Cauchy inequality, from (4) we get:

$$\left\|\xi^4P_0^{-1}(i\xi)\right\|\leq \sup_{r>0}\left|\xi^4\left(\xi^4+r^4+\left(\xi^4+r^4\right)\cos 2\varepsilon\right)^{-1}\right|\leq$$

$$\sup_{r>0}\left(\xi^4\left(\xi^4+r^4\right)^{-1}\right)(1+\cos 2\varepsilon)^{-1}\leq\frac{1}{2\cos^2\varepsilon}. \tag{6}$$

Consequently, from inequality (5), (6) it follows, that:

$$\sup_{\xi\in R}\left\|\xi^4P_0^{-1}(i\xi)\right\|\leq C_4(\varepsilon), \tag{7}$$

where

$$C_4(\varepsilon)=\begin{cases}1, & at \quad 0\leq\varepsilon\leq\pi/4\\ \dfrac{1}{2\cos^2\varepsilon}, & at \quad \pi/4\leq\varepsilon\leq\pi/2\end{cases} \tag{8}$$

Now estimate the norm $\left\|A^4P_0^{-1}(i\xi)\right\|$. For any $\xi\in R$

$$\left\|A^4P_0^{-1}(i\xi)\right\|=\left\|A^4\left(\xi^2E+A^2\right)^{-2}\right\|\leq\sup_{\mu\in\sigma(A)}\left|\mu^4\left(\xi^2+\mu^{2-2}\right)\right|=$$

$$=\sup_{\substack{r>0\\|\varphi|\leq\varepsilon}}\left|r^4\left(\xi^4+r^4+2\xi^2r^2\cos 2\varphi\right)^{-1}\right|\leq\sup_{r>0}\left|r^4\left(\xi^4+r^4+2\xi^2r^2\cos 2\varepsilon\right)\right|.$$

Analogously, we get

$$\sup_{r>0}\left|r^4\left(\xi^4+r^4+2\xi^2r^2\cos 2\varepsilon\right)\right|\leq C_0(\varepsilon),$$

where $C_0(\varepsilon)=C_4(\varepsilon)$, i.e.

$$C_0(\varepsilon)=\begin{cases}1, & at \quad 0\leq\varepsilon\leq\pi/4\\ \dfrac{1}{2\cos^2\varepsilon}, & at \quad \pi/4\leq\varepsilon<\pi/2\end{cases} \tag{9}$$

Consequently, for any $\xi\in R$

$$\left\|A^4P_0^{-1}(i\xi)\right\|\leq C_0(\varepsilon) \tag{10}$$

From inequalities (8) and (9), taking into account inequalities (3), we get

$$\|u\|_{W_2^4(R:H)}\leq\sqrt{2}C_0(\varepsilon)\|f\|_{L_2(R:H)},$$

i.e. $u\in W_2^4(R:H)$.

Show, that homogeneous equation $P_0u = 0$ has only zero solution from $W_2^4(R:H)$. It is obvious, that a general solution of equation $P_0u = 0$ from space $W_2^4(R:H)$ has the form

$$u_0(t) = \begin{cases} e^{-tA}\varphi_1 + Ate^{-tA}\varphi_2, & t>0 \\ e^{tA}\varphi_3 + Ate^{-tA}\varphi_4, & t<0 \end{cases},$$

where $\varphi_1, \varphi_2, \varphi_3$ and $\varphi_4 \in D\left(C^{7/2}\right)$. Since at $t = 0$, at $j = 0, 1, 2, 3$, $u_0^{(j)}(+0) = u_0^{(j)}(-0) = 0$, then we get, that $\varphi_1 = \varphi_2 = \varphi_3 = \varphi_4 = 0$, consequently, $u_0(t) = 0$. Then, using Banach theorem on inverse operator, we get, that P_0 isomorphically maps space $W_2^4(R:H)$ onto $L_2(R:H)$. The theorem is proved.

Lemma 1. *Let conditions a) and b) hold. So, operator* $P : W_2^4(R:H) \to L_2(R:H)$ *is bounded.*

Proof. From theorem 1 follows, that it suffices to prove boundedness of operator $P_1 : W_2^4(R:H) \to L_2(R:H)$.

Since

$$\|P_1u\|_{L_2(R:H)} \le \sum_{j=0}^{4} \left\|A_{4-j}u^{(j)}\right\|_{L_2(R:H)} \le \sum_{j=0}^{4} \|B_{4-j}\| \left\|A^{4-j}u\right\|_{L_2(R:H)} =$$

$$= \sum_{j=0}^{4} \|B_{4-j}\| \left\|C^{4-j}u^{(j)}\right\|_{L_2(R:H)},$$

using theorem on intermediate derivatives in the last inequality, we get

$$\|P_1u\|_{L_2(R:H)} \le \sum_{j=0}^{4} \|B_{4-j}\| \, C_j \, \|u\|_{W_2^4(R:H)} = const.$$

The lemma is proved.

The following theorem on estimations of the intermediate derivatives hold.

Theorem 2. *Let a normal inverse operator A satisfy condition a). Then, the following estimations hold*

$$\left\|A^4u\right\|_{L_2(R:H)} \le C_0(\varepsilon) \|P_0u\|_{L_2(R:H)} \tag{11}$$

$$\left\|A^3\frac{du}{dt}\right\|_{L_2(R:H)} \le C_1(\varepsilon) \|P_0u\|_{L_2(R:H)} \tag{12}$$

$$\left\|A^2\frac{d^2u}{dt^2}\right\|_{L_2(R:H)} \le C_2(\varepsilon) \|P_0u\|_{L_2(R:H)} \tag{13}$$

$$\left\|A^3\frac{d^3u}{dt^3}\right\|_{L_2(R:H)} \le C_3(\varepsilon) \|P_0u\|_{L_2(R:H)} \tag{14}$$

$$\left\|\frac{d^4u}{dt^4}\right\|_{L_2(R:H)} \le C_4(\varepsilon) \|P_0u\|_{L_2(R:H)} \tag{15}$$

where

$$C_0(\varepsilon) = C_4(\varepsilon) = \begin{cases} 1, & at \quad 0 \le \varepsilon \le \pi/4 \\ \dfrac{1}{2\cos^2\varepsilon}, & at \quad \pi/4 \le \pi/2 \end{cases}$$

$$C_1(\varepsilon)=C_3(\varepsilon)=\begin{cases}\dfrac{3^{3/4}}{4}, & at \quad 0\le\varepsilon\le\pi/4\\ \dfrac{3^{3/4}}{8\cos^2\varepsilon}, & at \quad 0\le\varepsilon<\pi/2\end{cases}$$

$$C_2(\varepsilon)=\frac{1}{4\cos^2\varepsilon},\ 0\le\varepsilon<\pi/2.$$

Proof. Inequalities (11) and (15) follow from inequalities (!0) and (7), respectively.

Further, using theorem 1 and Plansharel theorem, we have

$$\left\|A^3\frac{du}{dt}\right\|_{L_2(R:H)}=\left\|A^3\xi\hat{u}(\xi)\right\|_{L_2(R:H)}=$$

$$=\left\|A^3\xi P_0^{-1}(i\xi)\hat{f}(\xi)\right\|_{L_2(R:H)}\le\sup_{\xi}\left\|A^3\xi P_0^{-1}(i\xi)\right\|\left\|\hat{f}\right\|_{L_2(R:H)} \tag{16}$$

Since at any $\xi\in R$ the following inequalities hold

$$\left\|A^3\xi P_0^{-1}(i\xi)\right\|=\sup_{\mu\in\sigma(A)}\left|\mu^3\xi\left(\xi^3+\mu^2\right)^{-2}\right|=$$

$$=\sup_{\substack{r>0\\|\psi|\le\varepsilon}}\left|r^3\xi\left(\xi^4+r^4+2\xi^2r^2\cos2\varphi\right)^{-1}\right|\le$$

$$\le\sup_{r>0}\left|r^4\left(\xi^4+r^4+2\xi^2r^2\cos2\varepsilon\right)^{-1}\right| \tag{17}$$

At $0\le\varepsilon\le\pi/4$ we get

$$\sup_{r>0}\left|r^3\xi\left(\xi^4+r^4+2\xi^2r^2\cos2\varepsilon\right)^{-1}\right|\le\left|r^3\xi\left(\xi^4+r^4\right)^{-1}\right|\le\frac{3^{3/4}}{4} \tag{18}$$

and at $\pi/4\le\varepsilon<\pi/2$ we have

$$\sup_{r>0}\left|r^3\xi\left(\xi^4+r^4+2\xi^2r^2\cos2\varepsilon\right)^{-1}\right|\le\sup_{\tau>0}\left|\tau\left(\tau^4+1+2\tau^2\cos2\varepsilon\right)^{-1}\right|\le$$

$$\le\sup_{\tau>0}\left|\tau\left(\tau^4+1\right)\right|(1+\cos2\varepsilon)^{-1}=\frac{3^{3/4}}{8\cos^2\varepsilon}=C_1(\varepsilon). \tag{19}$$

From inequality (17), taking inequalities (18) and (19) into account, we get, that at any $\xi\in R$ the following inequality is true

$$\left\|A^3\xi P_0^{-1}(i\xi)\right\|\le C_1(\varepsilon),$$

i.e.

$$\left\|A^3\frac{du}{dt}\right\|_{L_2(R:H)}\le C_1(\varepsilon)\|f\|_{L_2(R:H)}=C_1(\varepsilon)\|P_0u\|_{L_2(R;H)},$$

Analogously, we prove that

$$\left\|A\frac{d^3u}{dt^3}\right\|_{L_2(R:H)}\le C_3(\varepsilon)\|P_0u\|_{L_2(R;H)}.$$

Now, we prove inequality (13)

Since

$$\left\| A^2 \frac{d^2 u}{dt^2} \right\|_{L_2(R:H)} = \left\| A^2 \xi^2 \hat{u}(\xi) \right\|_{L_2(R:H)} =$$

$$\left\| A^2 \xi^2 \left(\xi^2 E + A^2 \right)^{-2} \hat{f}(\xi) \right\|_{L_2(R:H) \leq} \tag{20}$$

Analogously, we get, that at any $\xi \in R$

$$\left\| A^2 \xi^2 \left(\xi^2 E + A^2 \right)^{-2} \right\| = \sup_{\mu \in \sigma(A)} \left| \mu^2 \xi^2 \left(\xi^2 + \mu^2 \right)^{-2} \right| \leq$$

$$\leq \sup_{r > 0} \left| r^2 \xi^2 \left(\xi^4 + r^4 + 2\xi^2 r^2 \cos 2\varepsilon \right)^{-1} \right| \leq$$

$$\leq \sup_{\tau > 0} \left| \tau^2 \left(\tau^4 + 1 + 2\tau^2 \cos 2\varepsilon \right)^{-1} \right| \leq \frac{1}{4 \cos^2 \varepsilon} = C_2(\varepsilon) \tag{21}$$

Taking into account inequalities (21) and (20), we get that

$$\left\| A^2 \frac{d^2 u}{dt^2} \right\|_{L_2(R_+:H)} \leq C_2(\varepsilon) \left\| P_0 u \right\|_{L_2}.$$

The theorem is proved.

Theorem 3. *Let operators A and A_j $(j = 0, ..., 4)$ satisfy conditions a) and b), moreover*

$$\alpha(\varepsilon) = \sum_{j=0}^{4} C_{4-j}(\varepsilon) \left\| B_j \right\| < 1. \tag{22}$$

Then, equation (1) is regularly solvable.

Proof. Write equation (1) in the form of the operator equation:

$$Pu = P_0 u + P_1 u = f, \tag{23}$$

where $f \in L_2(R:H)$, $u \in W_2^4(R:H)$. By theorem 1 operator P_0^{-1} exists and bounded. Then, at any $\omega \in L_2(R:H)$ there exists $u \in W_2^4(R:H)$ such that $P_0^{-1}\omega = u$.

Then, the norm of the operator $P_1 P_0^{-1} : L_2(R_+ : H) \to L_2(R:H)$ is less than $\alpha(\varepsilon) < 1$.

Indeed, taking theorem 3 into account, we get

$$\left\| P_1 P_0^{-1} \omega \right\| = \left\| P_1 u \right\| \leq \sum_{j=0}^{4} \left\| B_{4-j} \right\| \left\| A^{4-j} u^{(j)} \right\|_{L_2} \leq$$

$$\leq \left(\sum_{j=0}^{4} C_j(\varepsilon) \left\| B_{4-j} \right\| \right) \left\| P_0 u \right\|_{L_2(R:H)} = \alpha(\varepsilon) \left\| \omega \right\|_{L_2(R:H)}. \tag{24}$$

We can write equation (23) in the form

$$\omega P_1 P_0^{-1} \omega = f.$$

Then, from (24) it follows, that

$$\omega = \left(E + P_1 P_0^{-1}\right)^{-1} f,$$

and

$$u = P_0^{-1}\left(E + P_1 P_0^{-1}\right)^{-1} f.$$

Hence, we easily get, that

$$\|u\|_{W_2^4(R:H)} \le const\, \|f\|_{L_2(R:H)}\,.$$

The theorem is proved.

References

[1]. Lions J.-L., Majenes E. *Inhomogeneous boundary-value problems and their applications.* Moscow: "Mir", 1971, 371 p. (Russian)

[2]. Mirzoev S.S. *Problems of theory of solvability of the boundary-value problems for operator-differential equations in a Hilbert space and related spectral problems.* Abstract of Ph.D. dissertation, Baku, BSU, 1994, 32 p. (Russian)

[3]. Gumbataliyev R.Z. *On the solvability of initial boundary-value problems for a class of operator-differential equations of fourth order.* Izvestiya AN Azerb., phys.-math. sciences series, v.XVII, No1-3, 1996, pp.68-76. (Russian)

[4]. Mirzoev S.S., Amirov L.I. *On the existence of the holomorphic solutions of second order elliptic type operator-differential equations.* Izvestiya AN Azerb., phys.-math. tech. sciences series. Baku, 1998, v.XVIII, No2, pp.61-66. (Russian)

[5]. Amirova L.L. *On the solvability of boundary-value problems for a class of the second order operator-differential equations.* Proceedings Of IMM Of NASA, Baku, 1999, v.XI, pp.16-20.

[6]. Mirzoev S.S. *On the multiple completeness of a part of eigen and adjoint vectors of the polynomial operator bundles responding to boundary-value problems on semi-axis.* Functional analysis i ego prilozh. Moscow1983, v.17, issue 2, pp.84-85. (Russian)

[7]. Mirzoev S.S. *Multiple completeness of a part of root vectors of polynomial operator bundles of fourth order with a normal leading part.* Spectral theory of operators. Baku, "Elm", 1982, pp.148-162. (Russian)

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