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ON THE BASIS PROPERTIES OF ROOT FUNCTIONS OF STURM-LIOUVILLE PROBLEMS WITH AFFINE BOUNDARY CONDITIONS

Abstract

We consider Sturm-Liouville problems with a boundary condition linearly dependent on the eigenparameter. We study basisness of root functions in multiple eigenvalue cases.

Consider the following spectral problem

$$-y'' + q(x)y = \lambda y, \ 0 < x < 1, \tag{0.1}$$

$$y'(0)\sin\beta = y(0)\cos\beta, \ 0 \le \beta < \pi, \tag{0.2}$$

$$y'(1) = (a\lambda + b)y(1), \tag{0.3}$$

where a, b are real constants and $a < 0$, λ is the spectral parameter, $q(x)$ is a real valued and continuous function over the interval $[0, 1]$.

We have discussed in our previous work [1] the basis property in $L_p(0,1)$ $(1 < p < \infty)$ of the system $\{y_n\}$ $(n = 0, \ 1, \ldots; n \neq l)$, where y_n is a root function of (0.1)-(0.3), and l is an arbitrary non-negative integer. Although our study was complete in a sense that we discussed all possible cases of the choice of the root function y_l, there was no analogue of [6 ,Theorem 5]. In this paper we discuss these remaining cases in multiple eigenvalue case and give an example to triple eigenvalue case for which there wasn't any example in known to us references. We heavily rely on the designations of our previous work [1].

Sturm-Liouville problems with eigenparameter-dependent boundary conditions were considered in [2-5].

1. Preliminaries. In this paper we shall study only the multiple eigenvalue case, where all the eigenvalues are real. Following relations were proved in our previous work [1].

Let $y(x, \lambda)$ be a non-zero solution of (0.1), (0.2), and $\varpi(\lambda) = y'(1, \lambda) - (a\lambda + b)y(1, \lambda)$.

Lemma 1.1. *Let y_n, y_m be eigenfunctions corresponding to eigenvalues $\lambda_n \neq \lambda_m$. Then*

$$(y_n, y_m) = -ay_n(1)y_m(1),$$

$$\|y_n\|_2^2 = (y_n, y_n) = -ay_n(1)^2 - y_n(1)\varpi'(\lambda_n).$$

If λ_k is a double eigenvalue $(\lambda_k = \lambda_{k+1})$ then $\varpi'(\lambda_k) = 0$. In this case we define the first associated function y_{k+1} by

$$-y''_{k+1} + q(x)y_{k+1} = \lambda_k y_{k+1} + y_k,$$

$$y'_{k+1}(0)\sin\beta = y_{k+1}(0)\cos\beta,$$

$$y'_{k+1}(1) = (a\lambda_k + b)y_{k+1}(1) + ay_k(1).$$

Following lemma about the existence of the auxiliary associated function y^*_{k+1} was proved in our previous work [1].

Lemma 1.2. *If λ_k is a double eigenvalue then there exists associated function $y^*_{k+1} = y_{k+1} + c_1 y_k$, where c_1 is a constant, for which*

$$(y^*_{k+1}, y_{k+1}) = -a y^*_{k+1}(1) y_{k+1}(1).$$

If in the system of root functions we take two different associated functions $y^{(1)}_{k+1}$, $y^{(2)}_{k+1}$, then we denote their corresponding auxiliary associated functions by $y^{(1)*}_{k+1}$, $y^{(2)*}_{k+1}$. It is easy to show that for these functions following relations are true:

$$(y^{(1)*}_{k+1}, y_n) = -a y^{(1)*}_{k+1}(1) y_n(1),$$

for $n \neq k, k+1$, similar equality for $y^{(2)*}_{k+1}$, and

$$(y^{(1)*}_{k+1}, y^{(2)}_{k+1}) = -a y^{(1)*}_{k+1}(1) y^{(2)}_{k+1}(1) - C y_k(1) \frac{\varpi''(\lambda_k)}{2},$$

$$(y^{(2)*}_{k+1}, y^{(1)}_{k+1}) = -a y^{(2)*}_{k+2}(1) y^{(1)}_{k+1}(1) - D y_k(1) \frac{\varpi''(\lambda_k)}{2},$$

where $C, D \neq 0$.

If λ_k is a triple eigenvalue ($\lambda_k = \lambda_{k+1} = \lambda_{k+2}$) then together with the first order associated function y_{k+1} there exists the second order associated function y_{k+2} for which the following relations hold:

$$-y''_{k+2} + q(x) y_{k+2} = \lambda_k y_{k+2} + y_{k+1},$$

$$y'_{k+2}(0) \sin\beta = y_{k+2}(0)\cos\beta,$$

$$y'_{k+2}(1) = (a\lambda_k + b) y_{k+2}(1) + a y_{k+1}(1).$$

Following well known properties of associated functions play an important role in our investigation. The functions $y_{k+1} + c y_k$ and $y_{k+2} + d y_k$, where c and d are arbitrary constants, are also associated functions of the first and second order respectively. Next we observe that replacing the associated function y_{k+1} by $y_{k+1} + c y_k$, the associated function y_{k+2} changes to $y_{k+2} + c y_{k+1}$.

2. Basisness of root functions.

Theorem 2.1. *If λ_k is a double eigenvalue then the system*

$$\{y_n\} \ (n = 0,\ 1, \ldots; n \neq l, k, k+1) \cup \left\{ y^{(1)}_{k+1},\ y^{(2)}_{k+1} \right\},$$

where $l \neq k,\ k+1$ is a non-negative integer and $y^{(1)}_{k+1} = y_{k+1} + c y_k$ and $y^{(2)}_{k+1} = y_{k+1} + d y_k$ are different associated functions of the first order, is a basis in $L_p(0,1)$ $(1 < p < \infty)$.

Proof. We define the elements of the biorthogonal system

$$\{u_n\} \ (n = 0,\ 1, \ldots; n \neq l, k, k+1) \cup \left\{ u^{(1)}_{k+1},\ u^{(2)}_{k+1} \right\},$$

by

$$u_n(x) = \frac{y_n(x) - \frac{y_n(1)}{y_l(1)} y_l(x)}{B_n}.$$

where $n \neq l, k, k+1$ and $B_n = \|y_n\|_2^2 + a y_n(1)^2$,

$$u_{k+1}^{(1)}(x) = \frac{y_{k+1}^{(2)*}(x) - \frac{y_{k+1}^{(2)*}(1)}{y_l(1)} y_l(x)}{-D y_k(1) \varpi''(\lambda_k)/2},$$

$$u_{k+1}^{(2)}(x) = \frac{y_{k+1}^{(1)*}(x) - \frac{y_{k+1}^{(1)*}(1)}{y_l(1)} y_l(x)}{-C y_k(1) \varpi''(\lambda_k)/2}.$$

The relation $(u_n, y_m) = \delta_{nm}$ can be verified using the mentioned properties of $y_{k+1}^{(1)}$, $y_{k+1}^{(2)}$ and Lemma 1.1.

Following theorems are proved in a similar manner.

Theorem 2.2. *If λ_k is a triple eigenvalue then the system*

$$\{y_n\} \ (n = 0, \ 1, \ldots; n \neq l, k, k+1) \cup \left\{ y_{k+1}^{(1)}, \ y_{k+1}^{(2)} \right\},$$

where $l \neq k, \ k+1, \ k+2$ is a non-negative integer and $y_{k+1}^{(1)} = y_{k+1} + c y_k$ and $y_{k+1}^{(2)} = y_{k+1} + d y_k$ are different associated functions of the first order, is a basis in $L_p(0,1)$ $(1 < p < \infty)$.

Theorem 2.3. *If λ_k is a triple eigenvalue then the system*

$$\{y_n\} \ (n = 0, \ 1, \ldots; n \neq l, k, k+2) \cup \left\{ y_{k+2}^{(1)}, \ y_{k+2}^{(2)} \right\},$$

where $l \neq k, \ k+1, \ k+2$ is a non-negative integer and $y_{k+2}^{(1)} = y_{k+2} + c y_k$ and $y_{k+2}^{(2)} = y_{k+2} + d y_k$ are different associated functions of the second order, is a basis in $L_p(0,1)$ $(1 < p < \infty)$.

Theorem 2.4. *If λ_k is a triple eigenvalue then the system*

$$\{y_n\} \ (n = 0, \ 1, \ldots; n \neq l, k, k+1) \cup \left\{ y_{k+2}^{(1)}, \ y_{k+2}^{(2)} \right\},$$

where $l \neq k, \ k+1, \ k+2$ is a non-negative integer and $y_{k+2}^{(1)} = y_{k+2} + c y_k$ and $y_{k+2}^{(2)} = y_{k+2} + d y_{k+1}$ are associated functions of the second order and $cd \neq 0$ is a basis in $L_p(0,1)$ $(1 < p < \infty)$.

3. Example to triple eigenvalue case. Consider the spectral problem

$$-y'' = \lambda y, \ 0 < x < 1,$$

$$y'(0) = 0, \ y'(1) = a(\lambda - b) y(1).$$

We shall prove that it is possible to find constants a, b such that there exists an eigenvalue λ_k for which $\varpi(\lambda_k) = \varpi'(\lambda_k) = \varpi''(\lambda_k) = 0$, that is λ_k is a triple eigenvalue. Note that for this problem $y(x,\lambda) = \cos\sqrt{\lambda} x$ and $\varpi(\lambda) = -\sqrt{\lambda}\sin\sqrt{\lambda} - a(\lambda - b)\cos\sqrt{\lambda}$. The following lemma will be needed.

Lemma 3.1. Let $f(\lambda) = \sqrt{\lambda}\tan\sqrt{\lambda}$. Then there exists $\xi \in \left(\frac{\pi^2}{4}, \pi^2\right)$ such that $f''(\xi) = 0$.

Proof. Note that

$$f''(\lambda) = \frac{1}{4\lambda\cos^2\sqrt{\lambda}} + \frac{\tan\sqrt{\lambda}}{\sqrt{\lambda}} \left(\frac{1}{2\cos^2\sqrt{\lambda}} - \frac{\tan\sqrt{\lambda}}{4\sqrt{\lambda}} \right).$$

It is easy to check that

$$\lim_{\lambda\to\frac{\pi^2}{4}} f''(\lambda) = -\infty, \quad f''(\pi^2) = \frac{1}{4\pi^2}.$$

Since the function $f''(\lambda)$ is continuous in the interval $(\frac{\pi^2}{4}, \pi^2)$ then there exists $\xi \in \left(\frac{\pi^2}{4}, \pi^2\right)$ such that $f''(\xi) = 0$.

The proof of Lemma 3.1 is complete.

Now we determine the values of a, b for which $\varpi''(\xi) = 0$. Let $a = -f'(\xi)$ and $b = -\frac{f(\xi)}{f'(\xi)} + \xi$. It is easy to verify that for these values of a, b the number $\lambda_k = \xi$ is a triple eigenvalue.

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