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THE ASYMPTOTIC BEHAVIOR OF WEAK SOLUTION OF CAUCHY PROBLEM FOR A CLASS SOBOLEV TYPE SEMILINEAR EQUATION

Abstract

Cauchy problem for a class Sobolev type semi linear equation is considered. The existence and uniqueness of weak solution are proved and the behavior of weak solutions as $t \to +\infty$ are investigated. The considered class of equations in particular includes the semi linear wave equations with dissipation.

In this paper we consider the following Cauchy problem in $[0,\infty) \times R^n$.

$$u_{tt}(t,x) + (-1)^k \Delta^k u_{tt}(t,x) + (-1)^l \Delta^l u(t,x) + (-1)^k \Delta^k u_t + u_t = f(u(t,x)),$$

$$t \in [0,\infty), \ x \in R^n, \tag{1}$$

$$u(0,x) = u_0(x), \ u_t(0,x) = u_1(x), \ x \in R^n, \tag{2}$$

where Δ is a Laplace operator , $u_t = \frac{\partial u}{\partial t}$, $u_{tt} = \frac{\partial^2 u}{\partial t^2}$, but f is differentable function which will be defined below.

In case when $k = 0$ and $f(u) = -|u|^{p-1} u + f_1(u)$, where $|f_1(u)| \leq c\left(1 + |u|^{p-1}\right)$, $u \in R$ the corresponding problem was investigated in different papers. The expound of these results and list of references are given in [1]. The corresponding problem was investigated at paper [2] in case when $k = 0$ and $l = 1$, $n = 1,2,3$ $f(u) = |u|^{p-1} u$ where at $n = 1,2$, $1 + \frac{2}{n} \leq p < +\infty$ and at $n = 3$, $2 < p \leq \frac{n}{n-2}$

In paper [2] proved that for any small data the corresponding Cauchy problem has global weak solutions and also obtained the order of decreasing to zero as $t \to \infty$ of weak solutions.

In this work me investigate the similar problem for equation (1) in the following considerations.

1^0. $k \leq l$.

2^0. The function $f(\cdot)$ defined at some interval $(-a,a) \subset R$ and continuous differentable;

3^0. For any $u \in (-a,a)$

$$|f(u)| \leq c|u|^p, \quad |f'(u)| \leq c|u|^{p-1},$$

where at $n \leq 2(l-k)$, $a > 0$ and $\frac{2(l-k)}{n} + 1 \leq p < \infty$, and at $n > 2(l-k)$, $a = \infty$ and $2 < p \leq \frac{n}{n-2(l-k)}$.

Let $U_\delta \subset \mathcal{H} = \left[W_2^{l-k}(R^n) \cap L_1(R^n)\right] \times \left[L_2(R^n) \cap L_1(R^n)\right]$ be the sehere of radius $\delta > 0$ i.e.

$$U_\delta = \Big\{(u,v) : (u,v) \in H \ \ \|u\|_{W_2^{l-k}(R^n)} + \|u\|_{L_1(R^n)} +$$

$$+\|v\|_{L_2(R^n)}+\|v\|_{L_1(R^n)}<\delta\Big\}.$$

Theorem. *Let conditions* 1^0-3^0 *be satisfied. Then there exists* $\delta_0>0$ *such that for any* $(u_0,u_1)\in U_{\delta_0}$ *the problem has a unique solution* $u\in C\left([0,\infty),\ W_2^{l-k}(R^n)\right)\cap C^1([0,\infty),\ L_2(R^n))$ *and the following estimations are fulfiled:*

$$\sum_{|\alpha|=l-k}\|D^\alpha u(t,\cdot)\|_{L_2(R^n)}\le c_0R_0\,[1+t]^{-\left(\frac{l-k}{2l}+\frac{n}{4l}\right)}$$

$$\|u(t,\cdot)\|_{L_2(R^n)}\le c_0R_0\,(1+t)^{-\frac{n}{4l}},$$

$$\|u_t(t,\cdot)\|_{L_2(R^n)}\le c_0R_0\,(1+t)^{-\gamma},$$

where $c_0>0,\ \gamma=\min\left\{1+\frac{n}{4l},\ \frac{pn}{4l}\right\},\ R_0=\|u_0\|_{W_2^{l-k}(R^n)}+\|u_0\|_{L_1(R^n)}+\|u_1\|_{L_2(R^n)}+\|u_1\|_{L_1(R^n)}$.

Let's introduce the functional space $H=W_2^{l-k}(R^n)\times L_2(R^n)$ with inner product

$$\langle w^1,w^2\rangle=\int_{R^n}\nabla^{l-k}u^1\nabla^{l-k}u^2dx+\int_{R^n}v^1v^2dx,$$

where $\nabla^s=\Delta^{[s/2]}$ if s is even, $\nabla^s=\nabla\Delta^{[s+1/2]}$ if s is odd.

The problem (1)-(2) reduced to the Cauchy problem in Hilbert space H

$$w'=Aw+F(w),\tag{3}$$

$$w(0)=w_0,\tag{4}$$

by substitution $v_1=u$, $v_2=u_t$, where

$$w=\begin{pmatrix}v_1\\ v_2\end{pmatrix},\quad w_0=\begin{pmatrix}u_0\\ u_1\end{pmatrix},\quad G=\left(I+(-1)^k\Delta^k\right)^{-1},$$

$$A=\begin{pmatrix}0 & I\\ (-1)^{l+1}G\Delta^l & G\end{pmatrix},\quad D(A)=W_2^{2(l-k)}(R^n)\times W_2^{l-k}(R^n),$$

$$F(w)=\begin{pmatrix}0\\ Gf(v_1)\end{pmatrix},$$

The operator A generates the strong continuous semi-group in Hilbert space H.

The operator – function $w\to F(w)$ acts from U_δ to H and satisfies local lipshitz shy condition in the sense:

$$\left\|F(w^1)-F(w^2)\right\|\le c\left(\left\|w^1\right\|,\left\|w^2\right\|\right)\cdot\left\|w^1-w^2\right\|,$$

where $\|\cdot\|=\langle\cdot,\cdot\rangle,\ c(\cdot,\cdot)\in C\left(R_+^2,R_1\right),\ R_1=[0,\infty)$.

Hence, for the problem (3)-(4) the conditions of the theorem on local solvability are satisfied (see [3]).

Thus for any $(u_0, u_1) \in \mathcal{H}$ there exists $T' > 0$ such that the problem (1)-(2) has a unique weak solution $u \in C\left([0,T');\ W_2^{l-k}(R)\right) \cap C^1([0,T');\ L_2(R^n))$. It is known that, if

$$E(t) = \|u(t,\cdot)\|_{W_2^{l-k}(R^n)} + \|u(t,\cdot)\|_{L_2(R^n)} \leq c \qquad c = const > 0 \tag{5}$$

then $T' = \infty$.

Now, we prove that apriori estimation (5) is valid for small initial datas.

Let's define the following functional

$$E_1(t) = \frac{1}{2}\|u_t(t,\cdot)\|_{L_2(R^n)} + \frac{1}{2}\sum_{|\alpha|=l-k}\|D^\alpha u(t,\cdot)\|_{L_2(R^n)} + \frac{1}{2}\|u(t,\cdot)\|_{L_2(R^n)}.$$

It follows from the embedding theorem that

$$c^{-1}E_1(t) \leq E(t) \leq cE(t), \tag{6}$$

where $c > 0$ is a constant.

Let $\nu_0(t,x)$ be the solution of the problem

$$\nu_{0tt}(t,x) + (-1)^k \Delta^k \nu_{0tt}(t,x) + (-1)^l \Delta^l \nu_0(t,x) + (-1)^k \Delta^k \nu_{0t}(t,x) +$$

$$+\nu_{0t}(t,x) = 0, \qquad t \in [0,\infty), \quad x \in R^n, \tag{7}$$

$$\nu_0(0,x) = u_0(x), \quad \nu_{0t}(0,x) = u_1(x), \qquad x \in R^n, \tag{8}$$

and $\hat{\nu}_1(t,\xi)$ be a solution of the problem

$$\hat{\nu}_{1tt}(t,\xi) + |\xi|^{2k}\hat{\nu}_{1tt}(t,\xi) + |\xi|^{2l}\hat{\nu}_1(t,\xi) + |\xi|^{2k}\hat{\nu}_{1t}(t,\xi) + \hat{\nu}_{1t}(t,\xi) = 0 \tag{9}$$

$$\hat{\nu}_1(t,\xi) = 0 \quad \hat{\nu}_1(t,\xi) = 1 \tag{10}$$

where $\hat{\nu}_1(t,\xi)$ is the Fourier transformation of of function $\nu_1(t,x)$.

The solution of the problem (1)-(2) can be presented in the following form

$$u(t,x) = \nu_0(t,x) + \int_0^t \nu_1(t-\tau,x) f(u(\tau,x))\,d\tau. \tag{11}$$

As in the paper [4] we can prove that, for the functions $\nu_0(t,x)$ and $\nu_1(t,\xi)$ the following estimations are valid

$$\left\|D_t^i D_x^\alpha \nu_0(t,x)\right\|_{L_2(R^n)} \leq c[1+t]^{-\left(i+\frac{|\alpha|}{2l}+\frac{n}{4l}\right)} \times$$

$$\times\left[\|u_0\|_{W_2^{|\alpha|+(l-k)i}(R^n)} + \|u_0\|_{L_1(R^n)} + \|u_1\|_{W_2^{|\alpha|+(l-k)(i-1)}(R^n)} + \|u_1\|_{L_1(R^n)}\right], \tag{12}$$

$$\left\|D_t^i D_x^\alpha (\nu_1(t,x) * \varphi(x))\right\|_{L_2(R^n)} \leq$$

$$\leq c[1+t]^{-\left(i+\frac{|\alpha|}{2l}+\frac{n}{4l}\right)}\left[\|\varphi\|_{W_2^{|\alpha|+(l-k)(i-1)}(R^n)} + \|\varphi\|_{L_1(R^n)}\right] \tag{13}$$

where

$$u_0 \in W_2^{|\alpha|+(l-k)i}(R^n) \cap L_1(R^n), \quad u_1 \in W_2^{|\alpha|+(l-k)(i-1)}(R^n) \cap L_1(R^n)$$

$$\varphi \in W_2^{|\alpha|+(l-k)(i-1)}(R^n) \cap L_1(R^n), \quad c = const > 0.$$

By substituting $i = 0$ and $|\alpha| = l - k$ into (11)-(13) we get

$$\|D_x^\alpha u(t,\cdot)\|_{L_2(R^n)} \leq c(1+t)^{-d_1}\Big[\|u_0\|_{W_2^{l-k}(R^n)} + \|u_0\|_{L_1(R^n)} +$$

$$+ \|u_1\|_{L_2(R^n)} + \|u_1\|_{L_1(R^n)}\Big] + c\int_0^t (1+t-\tau)^{-d_1} \times$$

$$\times \Big[\|f(u(\tau,\cdot))\|_{L_1(R^n)} + \|f(u(\tau,\cdot))\|_{L_1(R^n)}\Big]d\tau, \tag{14}$$

where $d_1 = \frac{l-k}{2l} + \frac{n}{4l}$.

Further substituting $i = 1$ and $|\alpha| = 0$ into (11)-(13) we get the following inequality

$$\|u_t(t,\cdot)\|_{L_2(R^n)} \leq c(1+t)^{-d_2}\Big[\|u_0\|_{W_2^{l-k}(R^n)} + \|u_0\|_{L_1(R^n)} +$$

$$+ \|u_1\|_{L_2(R^n)} + \|u_1\|_{L_1(R^n)}\Big] + c\int_0^t (1+t-\tau)^{-d_2} \times$$

$$\times \Big[\|f(u(\tau,\cdot))\|_{W_2^{l-k}(R^n)} + \|f(u(\tau,\cdot))\|_{L_1(R^n)}\Big]d\tau, \tag{15}$$

where $d_2 = 1 + \frac{n}{4l}$, $c > 0$.

Similarly, substituting $i = 1$ and $|\alpha| = 0$ into (11)-(13) we get also

$$\|u_t(t,\cdot)\|_{L_2(R^n)} \leq c(1+t)^{-d_3}\Big[\|u_0\|_{L_2(R^n)} + \|u_0\|_{L_1(R^n)} +$$

$$+ \|u_1\|_{L_2(R^n)} + \|u_1\|_{L_1(R^n)}\Big] + c\int_0^t (1+t-\tau)^{-d_3} \times$$

$$\times \Big[\|f(u(\tau,\cdot))\|_{W_2^{-(l-k)}(R^n)} + \|f(u(\tau,\cdot))\|_{L_1(R^n)}\Big]d\tau, \tag{16}$$

where $d_3 = \frac{n}{4l}$

From the conditions 3^0 it follows that

$$\|f(u)\|_{L_2(R^n)} + \|f(u)\|_{L_1(R^n)} \leq c\left(\|u\|^p_{L_{2p}(R^n)} + \|u\|^p_{L_p(R^n)}\right) \tag{17}$$

$$\|\nu\|_{W_2^{-(l-k)}(R^n)} \leq c_1\|\nu\|_{L_2(R^n)} \tag{18}$$

therefore

$$\|f(u)\|_{W_2^{-(l-k)}(R^n)} \leq c_2\|u\|^p_{L_{2p}(R^n)} \tag{19}$$

At the next step we will use the multiplicate inequality

$$\|u\|_{L_q(R^n)} \le c\,\|u\|^{\theta}_{W_2^m(R^n)} \cdot \|u\|^{1-\theta}_{L_2(R^n)}, \tag{20}$$

where $q > 2$, $0 < \theta \le 1$ and at $n > 2m$, $2 < q < \frac{2n}{n-2m}$ (see [5]).

In the other side

$$\|u\|_{W_2^m(R^n)} \le c\left(\sum_{|\alpha|=m} \|D_x^{\alpha} u\|_{L_2(R^n)} + \|u\|_{L_2(R^n)}\right) \tag{21}$$

$c > 0$ (see [5])

From (20)-(21) we get

$$\|u\|_{L_q(R^n)} \le c \sum_{|\alpha|=m} \|D^{\alpha} u\|^{\theta}_{L_2(R^n)} \cdot \|u\|^{1-\theta}_{L_2(R^n)} + c\,\|u\|_{L_2(R^n)} \tag{22}$$

Taking into account 2^0 from (22) we get

$$\|f(u)\|_{W_2^{-(l-k)}(R^n)} + \|f(u)\|_{L_2(R^n)} \le$$

$$\le c_1 \left[\sum_{|\alpha|=m} \|D^{\alpha} u\|^{p\theta_1}_{L_2(R^n)} \cdot \|u\|^{p(1-\theta_1)}_{L_2(R^n)} + \|u\|^{p}_{L_2(R^n)}\right], \tag{23}$$

where $\theta_1 = \frac{n}{2(l-k)}\left(1 - \frac{1}{p}\right)$.

Similarly,

$$\|f(u)\|_{L_1(R^n)} \le c_2 \left[\sum_{|\alpha|=m} \|D^{\alpha} u\|^{p\theta_2}_{L_2(R^n)} \cdot \|u\|^{p(1-\theta_2)}_{L_2(R^n)} + \|u\|^{p}_{L_2(R^n)}\right], \tag{24}$$

where $\theta_2 = \frac{n}{l-k}\left(\frac{1}{2} - \frac{1}{p}\right)$.

Taking into account (19), (23) and (24) in the (14)-(16) we have the following estimation

$$\|D_x^{\alpha} u(t,\cdot)\|_{L_2(R^n)} \le c\,(1+t)^{-d_1} R_0 + c_4 \int_0^t (1+t-\tau)^{d_1} \phi(\tau)\,d\tau, \quad |\alpha| = l-k \tag{25}$$

$$\|u(t,\cdot)\|_{L_2(R^n)} \le c\,(1+t)^{-d_2} R_0 + c_5 \int_0^t (1+t-\tau)^{d_2} \phi(\tau)\,d\tau, \tag{26}$$

$$\|u(t,\cdot)\|_{L_2(R^n)} \le c\,(1+t)^{-d_3} R_1 + c_6 \int_0^t (1+t-\tau)^{d_3} \phi(\tau)\,d\tau, \tag{27}$$

where

$$R_0 = \|u_0\|_{W_2^{l-k}(R^n)} + \|u_0\|_{L_1(R^n)} + \|u_1\|_{L_2(R^n)} + \|u_1\|_{L_1(R^n)},$$

$$R_1 = \|u_0\|_{L_2(R^n)} + \|u_0\|_{L_1(R^n)} + \|u_1\|_{W_2^{-(l-k)}(R^n)} + \|u_1\|_{L_1(R^n)},$$

$$\phi(t) = \sum_{|\alpha|=l-k} \|D^\alpha u(t,\cdot)\|^{p\theta_1}_{L_2(R^n)} \cdot \|u(t,\cdot)\|^{p(1-\theta_1)}_{L_2(R^n)} +$$

$$+ \sum_{|\alpha|=l-k} \|D^\alpha u(t,\cdot)\|^{p\theta_2}_{L_2(R^n)} \cdot \|u(t,\cdot)\|^{p(1-\theta_2)}_{L_2(R^n)} + \|u(t,\cdot)\|^{p}_{L_2(R^n)}.$$

Denote

$$R_1(t) = (1+t)^{d_1} \sum_{|\alpha|=l-k} \|D^\alpha u(t,\cdot)\|_{L_2(R^n)}, \quad R_2(t) = (1+t)^{d_2} \|u_t(t,\cdot)\|_{L_2(R^n)},$$

$$R_3(t) = (1+t)^{d_3} \|u(t,\cdot)\|_{L_2(R^n)}.$$

From (25)-(27) we get

$$R_1(t) \le c_7 R_0 + c_8 (1+t)^{d_1} \int_0^t (1+t-\tau)^{-d_1} \psi(\tau) L(\tau)\, d\tau, \tag{28}$$

$$R_2(t) \le c_9 R_0 + c_{10} (1+t)^{d_2} \int_0^t (1+t-\tau)^{-d_2} \psi(\tau) L(\tau)\, d\tau, \tag{29}$$

$$R_3(t) \le c_{11} R_1 + c_{12} (1+t)^{d_3} \int_0^t (1+t-\tau)^{-d_3} \psi(\tau) L(\tau)\, d\tau, \tag{30}$$

where

$$\psi(\tau) = (1+\tau)^{(\theta_1 p d_1 + (1-\theta_1) p d_3)} + (1+\tau)^{(\theta_2 p d_1 + (1-\theta_2) p d_3)} + (1+\tau)^{-p d_3},$$

$$L(\tau) = R_1^{\theta_1 p}(\tau) R_3^{(1-\theta_1)p}(\tau) + R_1^{\theta_2 p}(\tau) R_3^{(1-\theta_2)p}(\tau) + R_3^{p}(\tau).$$

At the next step we will use the following statement.

Lemma (see [6]). *Let* $0 < d \le \eta$. *Then*

$$(1+t)^{-d} \int_0^t (1+t-\tau)^{-d} (1+\tau)^{-\eta}\, d\tau \le c, \tag{31}$$

where $c > 0$ *doesn't depend on* $t > 0$.

If $p > \max\left\{\frac{l-k}{n} + \frac{3}{2}, 2\right\}$, then using (31) the (28), (30) we get

$$R_1(t) \le c_7 R_0 + c_{13} \sup_{0 \le \tau \le t} L(\tau), \tag{32}$$

$$R_3(t) \le c_9 R_1 + c_{13} \sup_{0 \le \tau \le t} L(\tau), \tag{33}$$

Denote by $R(t) = \sup\limits_{0\le\tau\le t} R_1(t) + \sup\limits_{0\le\tau\le t} R_3(t)$ and taking into account that $R_1 \le c_{14} R_0$ we get from (32)-(33) that

$$R(t) \le c R_0 + c R^p(t), \quad c > 0. \tag{34}$$

For a sufficiently small R_0 the function

$$f(x) = cx^p - x + cR_0$$

has a positive root.

Let M be the smallest positive root of function $f(x)$. Then for sufficiently small R_0 from (34) it follows, that

$$R(t) \le M, \quad t > 0. \tag{35}$$

Hence it follows from (35) that the following asymptotic estimation takes place

$$\sum_{|\alpha|=l-k} \|D^\alpha u(t,\cdot)\|_{L_2(R^n)} \le c_0 R_0 (1+t)^{-d_1}, \ \|u(t,\cdot)\|_{L_2(R^n)} \le c_0 R_0 (1+t)^{-d_3}. \tag{36}$$

Taking into account the last estimations in (29) we get

$$R_2(t) \le M_1 (1+t)^{d_2-\gamma},$$

where $M_1 > 0$ doesn't depend on $t > 0$.

From this we get the following asymptotic estimation

$$\|u_t(t,\cdot)\|_{L_2(R^n)} \le c(1+t)^{-\gamma}. \tag{37}$$

Thus, taking into account (6), (36) and (37), for the solution of the problem (1)-(2) the apriori estimation (5) is fulfilled.

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