APPLIED PROBLEMS OF MATHEMATICS AND MECHANICS

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TORSION OF RADILLY LAMINATED SPHERE OF SMALL THICKNESS

Abstract

In the paper we study mode of deformation of torsion problem of radially spherical shell with arbitrary number of alternating rigid and soft layers. It is shown that for laminated spherical shell there exist weakly damping boundary layer solutions. They may penetrate sufficiently deep and change the pattern of mode of deformation far from the ends.

In [1-3] it is shown that spectrum of homogeneous solutions for laminated bodies with alternating rigid and soft layers are divided into the “lowest” and “highest” parts, and meanwhile some applied theory that takes into account weakly damping boundary layer solutions correspond to the lowest part of a spectrum. The method of the indicated papers was generalized in [4] in problems of stationary torsional oscillations of radially-laminated cylinder with alternating rigid and soft layers. Propagation of axially symmetric waves in radially laminated cylindrical wave guide was studied in [5].

In the present paper we investigate mode of deformation of torsion problem of radially laminated spherical shell with arbitrary number of alternating rigid and soft layers. It is shown that for a laminated spherical shell there exist weakly damping boundary layer solutions. In spite of the fact that a stressed state balanced in section responds to these solutions, they may penetrate sufficiently deep and essentially change the stressed state pattern far from the ends.

1. Let’s consider the torsion problem for radially laminated spherical shell of small thickness with alternating rigid and soft layers. Let the shell consist of $n = 2l-1$ layers and assume that external layers are rigid. Provide each rigid layer with odd number $j = 1, 3, .., n$ and soft one with even number $s = 2, 4, ..., n-1$. Assume that elastic properties of all rigid and soft layers are the same: shear modulus $G_j = G_g,\ G_s = G_m$. We denote internal radius of the ”k”-th layer by r_{ok}, external one by r_{1k} $\left(k=\overline{1,n}\right)$. Assume that in spherical system of coordinates the shell occupies the volume $\Gamma = \{r \in [r_{01}; r_{1n}],\ \theta \in [\theta_0; \theta_1],\ \varphi \in [0; 2\pi]\}$.

Equilibrium equations of the ”k”-th layer in permutations are of the form:

$$\Delta u_{\varphi k} - \frac{1}{r^2 \sin\theta} u_{\varphi k} = 0 \tag{1.1}$$

where $\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial \theta^2} + ctg\theta \cdot \frac{\partial}{\partial \theta}\right)$, $u_{\varphi k} = u_{\varphi k}(r,\theta)$ are the components of a shift vector of the ”k”-th layer.

Assume that spherical part of the boundary is stressless:

$$\sigma_{r\varphi}^{(1)}(r_{01},\theta) = 0, \quad \sigma_{r\varphi}^{(n)}(r_{1n},\theta) = 0 \tag{1.2}$$

Here $\sigma_{r\varphi}^{(k)} = G_k\left(\frac{\partial u_{\varphi k}}{\partial r} - \frac{u_{\varphi k}}{r}\right)$ are the stress tensor components.

We assume the combination of layers rigid that means that the following conjugation conditions are fulfilled:

$$\sigma_{r\varphi}^{(1)}(r_{1t},\theta)=\sigma_{r\varphi}^{(t+1)}(r_{0,t+1},\theta) \tag{1.3}$$

$$u_{\varphi t}(r_{1t},\theta)=u_{\varphi,t+1}(r_{0,t+1},\theta) \tag{1.4}$$

$$(t=1,2,...,n-1)$$

Assume that arbitrary boundary conditions keeping a spherical shell in equilibrium state are given at the ends, i.e. in conical sectors $\theta=\theta_0,\ \theta=\theta_1\ (\theta_0<\theta_1)$.

We'll search for the solution of problem (1.1)-(1.4) in the following form:

$$u_{\varphi k}(r,\theta)=u_k(r)\,m'(\theta) \tag{1.5}$$

where the function $m(\theta)$ satisfies the Legendre equation

$$m''(\theta)+m'(\theta)\,ctg\theta+\left(z^2-\frac{1}{4}\right)m(\theta)=0 \tag{1.6}$$

After substitution of (1.5) into (1.1)-(1.4) allowing for (1.6) we get the spectral problem:

$$u_k''(r)+\frac{2}{r}u_k'(r)-\left(z^2-\frac{1}{4}\right)\frac{u_k(r)}{r^2}=0 \tag{1.7}$$

$$G_1\left(u_1'(r)-\frac{1}{r}u_1(r)\right)\Big|_{r_{01}}=0,\quad G_n\left(u_n'(r)-\frac{1}{r}u_n(r)\right)\Big|_{r_{1n}}=0 \tag{1.8}$$

$$G_t\left(u_t'(r)-\frac{1}{r}u_t(r)\right)\Big|_{r_{1t}}=G_{t+1}\left(u_{t+1}'(r)-\frac{1}{r}u_{t+1}(r)\right)\Big|_{r_{0,t+1}} \tag{1.9}$$

$$\begin{gathered}u_t(r)|_{r_{1t}}=u_{t+1}(r)|_{r_{0,t+1}}\\ \left(k=\overline{1,n};\ t=\overline{1,n-1}\right)\end{gathered} \tag{1.10}$$

2.We take small parameter $p=\frac{G_m}{G_g}$ as relative character of rigidity and study spectral problem (1.7)-(1.10) as $p\to 0$. We'll distinguish two cases: condition $p\to 0$ is equivalent to a) $G_m\to 0,\ G_g$ is fixed; b) $G_g-\infty,\ G_m$ is fixed.

Take the layer of number "k" and for it we consider the auxiliary problem:

$$\begin{cases} u_k''(r)+\frac{2}{r}u_k'(r)-\left(z^2-\frac{1}{4}\right)\frac{1}{r^2}u_k(r)=0\\ u_k(r)|_{r_{0k}}=V_k^-,\quad u_k(r)|_{r_{1k}}=V_k^+\end{cases} \tag{2.1}$$

Solution of problem (2.1) is of the from

$$u_k(r)=\frac{1}{r^{1/2}sh\left(z\ln\left(\frac{r_{1k}}{r_{0k}}\right)\right)}\left[V_k^-r_{0k}^{1/2}sh\left(z\ln\left(\frac{r_{1k}}{r}\right)\right)+\right.$$

$$\left.+V_k^+r_{1k}^{1/2}sh\left(z\ln\left(\frac{r}{r_{0k}}\right)\right)\right] \tag{2.2}$$

We return to initial problem (1.7)-(1.10) and satisfy boundary conditions (1.8) and conjugation conditions (1.9) and (1.10) by means of (2.2), we get the following homogeneous algebraic system

$$A(z,p)\,\bar{u} = A_0(z)\,\bar{u} + pA_1(z)\,\bar{u} = \bar{0} \tag{2.3}$$

Here we take into account $V_k^+ = V_{k+1}^-$ and introduce $2l$ -dimensional vector of the form

$$\bar{u} = \left(V_1^-, V_1^+, V_3^-, V_3^+, ..., V_n^-, V_n^+,\right)^T,$$

$A_0(z)$, $A_1(z)$ are block matrices of the form

$$A_0 = \left\| \begin{matrix} A_1^{(0)} & 0.......0 \\ 0 & A_3^{(0)}.......0 \\ & .... \\ 0 & 0.......A_n^{(0)} \end{matrix} \right\|$$

$$A_1 = \left\| \begin{matrix} A_2^{(1)} & A_{(2)}^{(2)} & 0......0 & 0 & 0 \\ A_2^{(3)} & A_4^{(1)} + A_2^{(4)} & A_4^{(2)}...0 & 0 & 0 \\ ..... & .... & .... & .... & .... \\ 0 & 0 & 0...A_{n-3}^{(3)} & A_{n-1}^{(1)} + A_{n-3}^{(4)} & A_{n-1}^{(2)} \\ 0 & 0 & 0......0 & A_{n-1}^{(3)} & A_{n-1}^{(4)} \end{matrix} \right\|$$

$$A_s^{(1)} = \frac{1}{\Delta_s^{(1)}} \left\| \begin{matrix} 0 & 0 \\ 0 & -a_{11}^{(s)} \end{matrix} \right\|, \quad A_s^{(2)} = \frac{1}{\Delta_s^{(1)}} \left\| \begin{matrix} 0 & 0 \\ -a_{12}^{(s)} & 0 \end{matrix} \right\|$$

$$A_s^{(3)} = \frac{1}{\Delta_s^{(1)}} \left\| \begin{matrix} 0 & -a_{21}^{(s)} \\ 0 & 0 \end{matrix} \right\|, \quad A_s^{(4)} = \frac{1}{\Delta_s^{(1)}} \left\| \begin{matrix} -a_{22}^{(s)} & 0 \\ 0 & 0 \end{matrix} \right\|$$

$$A_j^{(0)} = \frac{1}{\Delta_j^{(1)}} \left\| \begin{matrix} a_{11}^{(j)} & a_{12}^{(j)} \\ a_{21}^{(j)} & a_{22}^{(j)} \end{matrix} \right\|$$

$$\Delta_k^{(1)} = -2r_{0k}^{-1/2} r_{1k}^{-1/2} sh\left(z\ln\left(\frac{r_{1k}}{r_{0k}}\right)\right);$$

$$a_{11}^{(k)} = r_{0k}^{-3/2} r_{1k}^{-1/2} \left(2zch\left(z\ln\left(\frac{r_{1k}}{r_{0k}}\right)\right) + 3sh\left(z\ln\left(\frac{r_{1k}}{r_{0k}}\right)\right)\right);$$

$$a_{12}^{(k)} = -2zr_{0k}^{-2}; \quad a_{21}^{(k)} = 2zr_{1k}^{-2};$$

$$a_{22}^{(k)} = r_{0k}^{-1/2} r_{1k}^{-3/2} \left(3sh\left(z\ln\left(\frac{r_{1k}}{r_{0k}}\right)\right) - 2zch\left(z\ln\left(\frac{r_{1k}}{r_{0k}}\right)\right)\right);$$

$$j = 1,3,...,n; \quad s = 2,4,...,n-1; \quad k = 1,2,3,...,n$$

Let's study the structure of the spectrum of the matrix $A(z,p)$ for small p. For this we research the spectrum of limit matrix $A(z,0) = A_0(z)$. The spectrum $A_0(z)$ is the combination of spectra of matrix blocks of its constituents, i.e.

$$\wedge_g(0) = \bigcup_{j=1,3,..}^{n} \wedge_j(0)$$

Here $\wedge_g(0)$ is a spectrum of the operator $A_0(z)$, and $\wedge_j(0)$ are the spectra of the operators $A_j^{(0)}(z)$ that are two-dimensional matrices.

The considered limit case $A_0(z)\bar{u}=\bar{0}$ $\left(A_j^{(0)}\bar{u}_j=\bar{0},\ \bar{u}_j=\left(V_j^-;V_j^+\right)^T\right)$ corresponds to the case of absence of stresses on lateral surfaces of rigid layers, i.e. it corresponds to the case a) that is equivalent to the consideration of l separately unconnected between themselves rigid layers with free lateral surfaces.

Equating determinants of $A_j^{(0)}$to zero, we get

$$\left(z^2-\frac{9}{4}\right)sh\left(z\ln\left(\frac{r_{1j}}{r_{0j}}\right)\right)=0 \tag{2.4}$$

Hence it follows that each branch $\wedge_j(0)$ consists of eigen-value $z_0=\pm\frac{3}{2}$ and denumerable set of eigen-values of the form

$$z_{0jq}=\pm\frac{i\pi q}{\ln\left(\frac{r_{1j}}{r_{0j}}\right)}\quad(q=1,2,3,...;\quad j=1,3,...,n) \tag{2.5}$$

It is seen that passage to limit as $p\to 0$ is impossible if z is the root of one of the equations

$$sh\left(z\ln\left(\frac{r_{1s}}{r_{0s}}\right)\right)=0\quad(s=2,4,...,n-1)$$

This indicates that limit spectrum may have additional branches.

To determine that limit spectrum has additional branches corresponding to case b) $G_g\to\infty,\ G_m$ is fixed, we consider the following auxiliary problem:

$$u_k''(r)+\frac{2}{r}u_k'(r)-\left(z^2-\frac{1}{4}\right)\frac{1}{r^2}u_k(r)=0 \tag{2.6}$$

$$G_k\left(u_k'(r)-\frac{1}{r}u_k(r)\right)\Big|_{r_{0k}}=\tau_k^-,\qquad G_k\left(u_k'(r)-\frac{1}{r}u_k(r)\right)\Big|_{r_{1k}}=\tau_k^+ \tag{2.7}$$

Solution of problem (2.6) (2.7) is of the form:

$$u_k(r)=-\frac{1}{G_k r^{1/2}\left(z^2-\frac{9}{4}\right)sh\left(z\ln\left(\frac{r_{1k}}{r_{0k}}\right)\right)}\times$$

$$\times\left[\tau_k^- r_{0k}^{3/2}\left(zch\left(z\ln\left(\frac{r_{1k}}{r}\right)\right)-\frac{3}{2}sh\left(z\ln\left(\frac{r_{1k}}{r}\right)\right)\right)-\right.$$

$$\left.-\tau_k^+ r_{1k}^{3/2}\left(zch\left(z\ln\left(\frac{r}{r_{0k}}\right)\right)+\frac{3}{2}sh\left(z\ln\left(\frac{r}{r_{0k}}\right)\right)\right)\right] \tag{2.8}$$

We satisfy conditions (1.8)-(1.10) by means of (2.8) and get homogeneous algebraic system of the form

$$B(z,p)\overline{\tau}=B_0(z)\overline{\tau}+pB_1(z)\overline{\tau}=\overline{0} \tag{2.9}$$

Here allowing for $\tau_1^-=\tau_n^+=0$ and $\tau_k^+=\tau_{k+1}^-$ we introduce the vector

$\overline{\tau}=\left(\tau_2^-,\tau_2^+,\tau_4^-,\tau_4^+,...,\tau_{n-1}^-,\tau_{n-1}^+\right)^T$, where $B_0\left(z\right)$ and $B_1\left(z\right)$ are block matrices of the form

$$B_0=\left\|\begin{array}{c} B_2^{(0)}\ 0.......0 \\ 0\ \ B_4^{(0)}.......0 \\ .... \\ 0\ \ \ 0......B_{n-1}^{(0)} \end{array}\right\|$$

$$B_1=\left\|\begin{array}{cccc} B_1^{(1)}+\ B_3^{(4)} & B_3^{(2)} & 0......0 & 0 \\ B_3^{(3)} & B_3^{(1)}+B_5^{(4)} & B_5^{(2)}....0 & 0 \\ .... & .... & .... & .... \\ 0 & 0 & 0....B_{n-2}^{(3)} & B_{n-2}^{(1)}+B_n^{(4)} \end{array}\right\|$$

$$B_s^{(0)}=\frac{1}{\Delta_s^{(2)}}\left\|\begin{array}{cc} b_{11}^{(s)} & b_{12}^{(s)} \\ b_{21}^{(s)} & b_{22}^{(s)} \end{array}\right\|$$

$$B_j^{(1)}=\frac{1}{\Delta_j^{(2)}}\left\|\begin{array}{cc} -b_{22}^{(j)} & 0 \\ 0 & 0 \end{array}\right\|,\quad B_j^{(2)}=\frac{1}{\Delta_j^{(2)}}\left\|\begin{array}{cc} 0 & 0 \\ -b_{12}^{(j)} & 0 \end{array}\right\|$$

$$B_j^{(3)}=\frac{1}{\Delta_j^{(2)}}\left\|\begin{array}{cc} 0 & -\ b_{21}^{(j)} \\ 0 & 0 \end{array}\right\|,\quad B_j^{(4)}=\frac{1}{\Delta_j^{(2)}}\left\|\begin{array}{cc} 0 & 0 \\ 0 & -\ b_{11}^{(j)} \end{array}\right\|$$

$$\Delta_k^{(2)}=2\left(z^2-\frac{9}{4}\right)r_{0k}^{-3/2}r_{1k}^{-3/2}sh\left(z\ln\left(\frac{r_{1k}}{r_{0k}}\right)\right)$$

$$b_{11}^{(k)}=r_{0k}^{-1/2}r_{1k}^{-3/2}\left(3sh\left(z\ln\left(\frac{r_{1k}}{r_{0k}}\right)\right)-2zch\left(z\ln\left(\frac{r_{1k}}{r_{0k}}\right)\right)\right);$$

$$b_{12}^{(k)}=2zr_{0k}^{-2};\quad b_{21}^{(k)}=-2zr_{1k}^{-2};$$

$$b_{22}^{(k)}=r_{0k}^{-3/2}r_{1k}^{-1/2}\left(2zch\left(z\ln\left(\frac{r_{1k}}{r_{0k}}\right)\right)+3sh\left(z\ln\left(\frac{r_{1k}}{r_{0k}}\right)\right)\right)$$

Denote spectrum of limit operator $B\left(z,0\right)=B_0\left(z\right)$ by $\wedge_m\left(0\right)$ where

$$\wedge_m\left(0\right)=\bigcup_{s=2,4,..}^{n-1}\wedge_s\left(0\right).$$

Each branch of $\wedge_s\left(0\right)$ corresponds to the condition of absence of permutations on lateral surfaces of soft layers and is determined by the roots of the equation

$$sh\left(z\ln\left(\frac{r_{1s}}{r_{0s}}\right)\right)=0\quad\left(s=2,4,...,n-1\right)\tag{2.10}$$

Hence it follows that each branch of $\wedge_s\left(0\right)$consists of denumerable set of eigen-values of the form

$$z_{0sq}=\pm\frac{i\pi q}{\ln\left(\frac{r_{1s}}{r_{0s}}\right)}\quad\left(q=1,2,...;\ s=2,4,...,n-1\right)$$

Applying the theory of perturbations of linear operators [6] to algebraic system (2.3), (2.9) we get:

Theorem: *Spectrum* $\wedge(p)$ *of problem* (1.8)-(1.10) *is represented in the form:*

$$\wedge(p) = \wedge_0(p) \cup \wedge_-^{(1)}(p) \cup \wedge_-^{(2)}(p) \cup \wedge_+(p)$$

moreover

1) $\wedge_0(p)$ *consists of eigen-value* $z_0 = \pm\frac{3}{2}$.

2) $\wedge_-^{(1)}(p)$ *consists of* $(l-1)$ *real eigen-values of the form*

$$z_t = \frac{3}{2} + p\gamma_t + O\left(p^2\right) \qquad (2.11)$$

where γ_t are non-zero eigen-values of homogeneous Jacobian algebraic system

$$C\overline{X} - \gamma B\overline{X} = \overline{0} \qquad (2.12)$$

here $\overline{X} = (X_1, X_3, ..., X_n)^T$, $B = diag\,\|b_{jj}\|$

$$C = \left\| \begin{array}{cccc} -c_{11} & c_{11} & 0........0 & 0 \\ c_{11} & -(c_{11}+c_{33}) & c_{33}.......0 & 0 \\ .. & .... & .... & ..... \\ 0 & 0 & 0...c_{n-2,n-2} & -c_{n-2,n-2} \end{array} \right\|$$

$$b_{jj} = \frac{r_{1j}^3 - r_{0j}^3}{3} \quad (j = 1, 3, ..., n)$$

$$c_{jj} = \frac{r_{1j}^3 \cdot r_{0,j+2}^3}{r_{0,j+2}^3 - r_{1j}^3} \quad (j = 1, 3, ..., n-2)$$

3) $\wedge_-^{(2)}(p)$ consists of $(l-1)$ real eigen-values of the form

$$z_t = -\frac{3}{2} + p\gamma_t + O\left(p^2\right) \qquad (2.13)$$

where γ_t are non-zero eigen-values of homogeneous Jacobian algebraic system

$$C\overline{X} + \gamma B\overline{X} = \overline{0} \qquad (2.14)$$

4) $\wedge_+(p)$ consists of n sets of eigen-values of the form

$$z_{kq} = \pm\frac{i\pi q}{2\varepsilon\omega_k} + O\left(p^\delta\right) \qquad (2.15)$$

where $\varepsilon = \frac{1}{2}\ln\left(\frac{r_{1n}}{r_{01}}\right)$ is a small parameter characterizing thin-shell property of a spherical shell, $\omega_k = \frac{1}{2\varepsilon}\ln\left(\frac{r_{1k}}{r_{0k}}\right)$; $k = \overline{1,n}$, $q = 1, 2, 3, ...,$; $\delta = 1$, if $z_{k+1,q} \neq z_{kq}$, $z_{kq} \neq z_{k-1,q}$ and $\delta = \frac{1}{2}$ if $z_{k+1,q} = z_{kq}$ or $z_{kq} = z_{k-1,q}$

3. Construct eigen-functions for small p.

Penetrating solution

$$u_{\varphi k}^{(0)}(r,\theta) = -r\left[A_0 \sin\theta + B_0\left(\frac{1}{2}\sin\theta \cdot ln\left(ctg^2\frac{\theta}{2}\right) + ctg\theta\right)\right] \qquad (3.1)$$

corresponds to eigen-values $z_0 = \pm\frac{3}{2}$ here A_0 and B_0 are arbitrary constants.

Eigen functions $u_k(r)$ corresponding to $\wedge_-^{(1)}(p)$, $\wedge_-^{(2)}(p)$ are expressed by the solutions $\overline{X}_t = (X_{t1}, X_{t3}, ..., X_{tn})^T$ of systems (2.12), (2.14) in the following way.

$$u_k^{(t)} = u_{kt0}(r) + O(p) \tag{3.2}$$

$$u_{jt0} = rX_{tj}$$

$$u_{st0} = \frac{1}{r^2\left(r_{1s}^3 - r_{0s}^3\right)}\left[r_{1,s-1}r_{0s}^2\left(r_{1s}^3 - r^3\right)X_{t,s-1} + r_{1s}^2 r_{0,s+1}\left(r^3 - r_{0s}^3\right)X_{t,s+1}\right] \tag{3.3}$$

Assume $z_{kt} \in \wedge_+(p)$ and cite expressions for eigen-functions:

1) $z_{qt}(0) = z_{0qt} = \frac{i\pi t}{2\varepsilon\omega_q} \in \wedge_q(0)$; ($q$ is an odd number)

In this case

$$u_k^{(t)}(r) = u_{k0}^{(t)}(r) + O(p) \tag{3.4}$$

$$u_{q0}^{(t)} = (-1)^e X_{qt}\left(\frac{r_{0q}}{r}\right)^{1/2}\cos\left(\frac{\pi e}{2\varepsilon\omega_q}\ln\left(\frac{r_{1q}r_{0q}}{r^2}\right)\right) \text{ for } t = 2e$$

$$u_{q0}^{(t)} = (-1)^e X_{qt}\left(\frac{r_{0q}}{r}\right)^{1/2}\sin\left(\frac{\pi(1+2e)}{4\varepsilon\omega_q}\ln\left(\frac{r_{0q}r_{1q}}{r^2}\right)\right) \text{ for } t = 2e+1$$

$$u_{q-1,0}^{(t)} = X_{qt}\left(\frac{r_{1q-1}}{r}\right)^{1/2}\frac{sh\left(z_{0qt}\ln\left(\frac{r}{r_{0q-1}}\right)\right)}{sh\left(2\varepsilon\omega_{q-1}z_{0qt}\right)}$$

$$u_{q+1,0}^{(t)} = X_{qt}\left(\frac{r_{0q}}{r}\right)^{1/2}\frac{ch\left(2\varepsilon\omega_q z_{0qt}\right)}{sh\left(2\varepsilon\omega_{q+1}z_{0qt}\right)}sh\left(z_{0qt}\ln\left(\frac{r_{1q+1}}{r}\right)\right)$$

2) $z_{lt}(0) = z_{0lt} \in \wedge_l(0)$; ($l$ is an even number)

$$u_k^{(t)}(r) = u_{k0}^{(t)}(r) + O(p) \tag{3.5}$$

$$u_{l0}^{(t)} = \frac{(-1)^d r^{-\frac{1}{2}} r_{0l}^{3/2} p^{-\frac{1}{2}} Y_{lt}}{2\left(z_{0lt}^2 - \frac{9}{4}\right)}\times$$

$$\times\left[2z_{0lt}sh\left(\frac{z_{0lt}\ln\left(\frac{r^2}{r_{0l}r_{1l}}\right)}{2}\right) + 3ch\left(\frac{z_{0lt}\ln\left(\frac{r_{0l}r_{1l}}{r^2}\right)}{2}\right)\right] \text{ for } t = 2d$$

$$u_{l0}^{(t)} = \frac{(-1)^{d+1} ir^{-\frac{1}{2}} r_{0l}^{3/2} p^{-\frac{1}{2}} Y_{lt}}{2\left(z_{0lt}^2 - \frac{9}{4}\right)}\times$$

$$\times\left[3sh\left(\frac{z_{0lt}\ln\left(\frac{r_{0l}r_{1l}}{r^2}\right)}{2}\right) - 2z_{0lt}ch\left(\frac{z_{0lt}\ln\left(\frac{r_{0l}r_{1l}}{r^2}\right)}{2}\right)\right] \text{ for } t = 2d+1$$

$$u_{l-1,0}^{(t)} = \frac{r^{-\frac{1}{2}} r_{1l-1}^{3/2} p^{\frac{1}{2}} Y_{lt}}{\left(z_{0lt}^2 - \frac{9}{4}\right) sh\left(2\varepsilon\omega_{l-1}z_{0lt}\right)}\times$$

$$\times\left[z_{0lt}ch\left(z_{0lt}\ln\left(\frac{r}{r_{0l-1}}\right)\right) + \frac{3}{2}sh\left(z_{0lt}\ln\left(\frac{r}{r_{0l-1}}\right)\right)\right]$$

$$u_{l+1,0}^{(t)} = -\frac{r^{-\frac{1}{2}} r_{0l}^{3/2} p^{\frac{1}{2}} Y_{lt}}{\left(z_{0lt}^2 - \frac{9}{4}\right) sh\left(2\varepsilon\omega_{l+1}z_{0lt}\right)}\times$$

$$\times ch\left(2\varepsilon\omega_l z_{0lt}\right)\left[z_{0lt} ch\left(z_{0lt}\ln\left(\frac{r_{1l+1}}{r}\right)\right)-\frac{3}{2}sh\left(z_{0lt}\ln\left(\frac{r_{1l+1}}{r}\right)\right)\right]$$

3) $z_{kt}(0)=z_{0kt}\in\bigcap\limits_{\alpha=l}^{l+2q}\wedge_\alpha(0)$; ($l$ is an odd number)

$$u_k^{(t)}(r)=u_{k0}^{(t)}(r)+O(p) \tag{3.6}$$

$$u_{l-1.0}^{(t)}(r)=\left(\frac{r_{1l-1}}{r}\right)^{1/2}X_{lt}\frac{sh\left(z_{0kt}\ln\left(\frac{r}{r_{0l-1}}\right)\right)}{sh\left(2\varepsilon\omega_{l-1}z_{0kt}\right)}$$

$$u_{\alpha,0}^{(t)}=(-1)^h X_{\alpha t}\left(\frac{r_{0\alpha}}{r}\right)^{\frac{1}{2}}\cos\left(\frac{\pi h}{2\varepsilon\omega_\alpha}\ln\left(\frac{r_{1\alpha}r_{0\alpha}}{r^2}\right)\right)\text{ for } t=2h$$

$$u_{\alpha,0}^{(t)}=(-1)^h X_{\alpha t}\left(\frac{r_{0\alpha}}{r}\right)^{\frac{1}{2}}\sin\left(\frac{\pi(1+2h)}{4\varepsilon\omega_\alpha}\ln\left(\frac{r_{1\alpha}r_{0\alpha}}{r^2}\right)\right)\text{ for } t=2h+1$$

(where $\alpha=l;\ l+2;...;l+2q$)

$$u_{\alpha+1,0}^{(t)}=\frac{1}{sh\left(2\varepsilon\omega_{\alpha+1}z_{0kt}\right)}\times$$

$$\times\left[\left(\frac{r_{0\alpha}}{r}\right)^{1/2}sh\left(z_{0kt}\ln\left(\frac{r_{1\alpha+1}}{r}\right)\right)ch\left(2\varepsilon\omega_\alpha z_{0kt}\right)X_{\alpha t}+\right.$$

$$\left.+\left(\frac{r_{1\alpha+1}}{r}\right)^{1/2}sh\left(z_{0kt}\ln\left(\frac{r}{r_{0\alpha+1}}\right)\right)X_{\alpha+2t}\right)\right]$$

$$\alpha=l,l+2,...,l+2q-2$$

$$u_{l+2q+1,0}^{(t)}=\left(\frac{r_{0l+2q}}{r}\right)^{\frac{1}{2}}X_{l+2q,t}\times$$

$$\times\frac{ch\left(2\varepsilon\omega_{l+2q}z_{0kt}\right)sh\left(z_{0kt}\ln\left(\frac{r_{1l+2q+1}}{r}\right)\right)}{sh\left(2\varepsilon\omega_{l+2q+1}z_{0kt}\right)}$$

4) $z_{kt}(0)=z_{0kt}\in\wedge_{q-2}(0)\bigcap\wedge_{q+2}(0)$ (q is an odd number)

$$u_k^{(t)}(r)=u_{k0}^{(t)}(r)+O(p) \tag{3.7}$$

$$u_{q-3.0}^{(t)}=\left(\frac{r_{1q-3}}{r}\right)^{\frac{1}{2}}\frac{sh\left(z_{0kt}\ln\left(\frac{r}{r_{0q-3}}\right)\right)}{sh\left(2\varepsilon\omega_{q-3}z_{0kt}\right)}X_{q-2,t}$$

$$u_{q-2,0}^{(t)}=(-1)^e X_{q-2,t}\left(\frac{r_{0q-2}}{r}\right)^{\frac{1}{2}}\cos\left(\frac{\pi e}{2\varepsilon\omega_{q-2}}\ln\left(\frac{r_{1q-2}r_{0q-2}}{r^2}\right)\right)\text{ for } t=2e$$

$$u_{q-2,0}^{(t)}=(-1)^e X_{q-2,t}\left(\frac{r_{0q-2}}{r}\right)^{\frac{1}{2}}\sin\left(\frac{\pi(1+2e)}{4\varepsilon\omega_{q-2}}\ln\left(\frac{r_{0q-2}r_{1q-2}}{r^2}\right)\right)\text{ for } t=2e+1$$

$$u_{q-1,0}^{(t)}=X_{q-2,t}\left(\frac{r_{0q-2}}{r}\right)^{\frac{1}{2}}\frac{sh\left(z_{0kt}\ln\left(\frac{r_{1q-1}}{r}\right)\right)}{sh\left(2\varepsilon\omega_{q-1}z_{0kt}\right)}ch\left(2\varepsilon\omega_{q-2}z_{0kt}\right)$$

$$u_{q,0}^{(t)}=0$$

$$u_{q+2,0}^{(t)} = (-1)^e X_{q+2,t} \left(\frac{r_{0q+2}}{r} \right)^{\frac{1}{2}} \cos \left(\frac{\pi e}{2\varepsilon \omega_{q+2}} \ln \left(\frac{r_{1q+2} r_{0q+2}}{r^2} \right) \right) \text{ for } t = 2e$$

$$u_{q+2,0}^{(t)} = (-1)^e X_{q+2,t} \left(\frac{r_{0q+2}}{r} \right)^{\frac{1}{2}} \sin \left(\frac{\pi (1+2e)}{4\varepsilon \omega_{q+2}} \ln \left(\frac{r_{0q+2} r_{1q+2}}{r^2} \right) \right) \text{ for } t = 2e+1$$

$$u_{q+1,0}^{(t)} = X_{q+2,t} \left(\frac{r_{1q+1}}{r} \right)^{\frac{1}{2}} \frac{sh \left(z_{0kt} \ln \left(\frac{r}{r_{0q+1}} \right) \right)}{sh \left(2\varepsilon \omega_{q+1} z_{0kt} \right)}$$

$$u_{q+3,0}^{(t)} = \left(\frac{r_{0q+2}}{r} \right)^{\frac{1}{2}} X_{q+2,t} \frac{sh \left(z_{0kt} \ln \left(\frac{r_{1q+3}}{r} \right) \right)}{sh \left(2\varepsilon \omega_{q+3} z_{0kt} \right)}$$

5) $z_{kt}(0) = z_{0kt} \in \bigcap\limits_{d=l,l+2}^{l+2q} \wedge_d (0)$ (l is an odd number)

$$u_k^{(t)}(r) = u_{k0}^{(t)}(r) + O(p) \tag{3.8}$$

$$u_{h0}^{(t)} = \frac{(-1)^{\beta} r^{-\frac{1}{2}} r_{0h}^{3/2} p^{-\frac{1}{2}}}{2 \left(z_{0kt}^2 - \frac{9}{4} \right)} Y_{ht} \times$$

$$\times \left[2 z_{0kt} sh \left(\frac{z_{0kt} \ln \left(\frac{r^2}{r_{1h} r_{0h}} \right)}{2} \right) + 3ch \left(\frac{z_{0kt} \ln \left(\frac{r_{0h} r_{1h}}{r^2} \right)}{2} \right) \right] \text{ for } t = 2\beta$$

$$u_{h0}^{(t)} = \frac{(-1)^{\beta+1} i r^{-\frac{1}{2}} r_{0h}^{3/2} p^{-\frac{1}{2}}}{2 \left(z_{0kt}^2 - \frac{9}{4} \right)} Y_{ht} \times$$

$$\times \left[3sh \left(\frac{z_{0kt} \ln \left(\frac{r_{0h} r_{1h}}{r^2} \right)}{2} \right) - 2 z_{0kt} ch \left(\frac{z_{0kt} \ln \left(\frac{r_{0h} r_{1h}}{r^2} \right)}{2} \right) \right] \text{ for } t = 2\beta + 1$$

$(h = l;\ l+2, ..., l+2q)$

$$u_{l-1,0}^{(t)} = \frac{r^{-\frac{1}{2}} r_{1l-1}^{3/2} p^{\frac{1}{2}} Y_{lt}}{\left(z_{0kt}^2 - \frac{9}{4} \right) sh \left(2\varepsilon \omega_{l-1} z_{0kt} \right)} \times$$

$$\times \left[z_{0kt} ch \left(z_{0kt} \ln \left(\frac{r}{r_{0l-1}} \right) \right) + \frac{3}{2} sh \left(z_{0kt} \ln \left(\frac{r}{r_{0l-1}} \right) \right) \right]$$

$$u_{h+1,0}^{(t)} = \frac{r^{-\frac{1}{2}} p^{\frac{1}{2}}}{\left(z_{0kt}^2 - \frac{9}{4} \right) sh \left(2\varepsilon \omega_{h+1} z_{0kt} \right)} \times$$

$$\times \left\{ Y_{h+2,t} r_{1h+1}^{3/2} \left[z_{0kt} ch \left(z_{0kt} \ln \left(\frac{r}{r_{0h+1}} \right) \right) + \frac{3}{2} sh \left(z_{0kt} \ln \left(\frac{r}{r_{0h+1}} \right) \right) \right] - \right.$$

$$\left. - Y_{ht} r_{0h}^{3/2} \left[z_{0kt} ch \left(z_{0kt} \ln \left(\frac{r_{1h+1}}{r} \right) \right) - \frac{3}{2} sh \left(z_{0kt} \ln \left(\frac{r_{1h+1}}{r} \right) \right) \right] ch \left(2\varepsilon \omega_h z_{0kt} \right) \right\};$$

$(h = l;\ l+2, ..., l+2q-2)$

$$u_{l+2q+1,0}^{(t)} = - \frac{r^{-\frac{1}{2}} p^{\frac{1}{2}} r_{0,l+2q}^{3/2} Y_{l+2q,t}}{\left(z_{0kt}^2 - \frac{9}{4} \right) sh \left(2\varepsilon \omega_{l+2q+1} z_{0kt} \right)} \times$$

$$\times\left[z_{0kt}ch\left(z_{0kt}\ln\left(\frac{r_{1l+2q+1}}{r}\right)\right)-\frac{3}{2}sh\left(z_{0kt}\ln\left(\frac{r_{1l+2q+1}}{r}\right)\right)\right]ch\left(2\varepsilon\omega_{l+2q}z_{0kt}\right)$$

6) $z_{kt}(0)=z_{0kt}\in\wedge_{\alpha}(0)\bigcap\wedge_{\alpha+1}(0)\bigcap\wedge_{\alpha+2}(0)$ (α is an odd number)

$$u_k^{(t)}(r)=u_{k0}^{(t)}(r)+O\left(p^{1/2}\right)\tag{3.9}$$

$$u_{\alpha,0}^{(t)}=(-1)^{q_1}X_{\alpha t}\left(\frac{r_{0\alpha}}{r}\right)^{1/2}\cos\left(\frac{\pi q_1}{2\varepsilon\omega_\alpha}\ln\left(\frac{r_{0\alpha}r_{1\alpha}}{r^2}\right)\right)\text{ for }t=2q_1$$

$$u_{\alpha,0}^{(t)}=(-1)^{q_1}X_{\alpha t}\left(\frac{r_{0\alpha}}{r}\right)^{1/2}\sin\left(\frac{\pi(1+2q_1)}{4\varepsilon\omega_\alpha}\ln\left(\frac{r_{0\alpha}r_{1\alpha}}{r^2}\right)\right)\text{ for }t=2q_1+1$$

$$u_{\alpha+2,0}^{(t)}=(-1)^{q_2}X_{\alpha+2,t}\left(\frac{r_{0\alpha+2}}{r}\right)^{1/2}\cos\left(\frac{\pi q_2}{2\varepsilon\omega_{\alpha+2}}\ln\left(\frac{r_{0\alpha+2}r_{1\alpha+2}}{r^2}\right)\right)\text{ for }t=2q_2$$

$$u_{\alpha+2,0}^{(t)}=(-1)^{q_2}X_{\alpha+2,t}\left(\frac{r_{0\alpha+2}}{r}\right)^{1/2}\sin\left(\frac{\pi(1+2q_2)}{4\varepsilon\omega_\alpha}\ln\left(\frac{r_{0\alpha+2}r_{1\alpha+2}}{r^2}\right)\right)$$

for $t=2q_2+1$

$$u_{\alpha+3,0}^{(t)}=\left(\frac{r_{0\alpha+2}}{r}\right)^{1/2}X_{\alpha+2t}\frac{sh\left(z_{0kt}\ln\left(\frac{r_{1\alpha+3}}{r}\right)\right)}{sh\left(2\varepsilon\omega_{\alpha+3}z_{0kt}\right)}ch\left(2\varepsilon\omega_{\alpha+2}z_{0kt}\right)$$

$$u_{\alpha-1,0}^{(t)}=\left(\frac{r_{1\alpha-1}}{r}\right)^{1/2}X_{\alpha t}\frac{sh\left(z_{0kt}\ln\left(\frac{r}{r_{0\alpha-1}}\right)\right)}{sh\left(2\varepsilon\omega_{\alpha-1}z_{0kt}\right)}$$

$$u_{\alpha+1,0}^{(t)}=(-1)^{q_3}\left(\frac{r_{0\alpha}}{r}\right)^{1/2}X_{\alpha t}ch\left(2\varepsilon\omega_\alpha z_{0kt}\right)ch\left(z_{0kt}\ln\left(\frac{r_{0\alpha+1}r_{1\alpha+1}}{r^2}\right)\right)$$

for $t=2q_3$

$$u_{\alpha+1,0}^{(t)}=(-1)^{q_3+1}iX_{\alpha t}\left(\frac{r_{0\alpha}}{r}\right)^{1/2}ch\left(2\varepsilon\omega_\alpha z_{0kt}\right)sh\left(\frac{z_{0kt}\ln\left(\frac{r_{0\alpha+1}r_{1\alpha+1}}{r^2}\right)}{2}\right)$$

for $t=2q_3+1$

7) $z_{kt}(0)=z_{0kt}\in\wedge_{\alpha-1}(0)\bigcap\wedge_{\alpha}(0)$; ($\alpha$ is an odd number)

$$u_k^{(t)}(r)=u_{k0}^{(t)}(r)+O\left(p^{1/2}\right)\tag{3.10}$$

$$u_{\alpha0}^{(t)}=(-1)^{q}X_{\alpha t}\left(\frac{r_{0\alpha}}{r}\right)^{1/2}\cos\left(\frac{\pi q}{2\varepsilon\omega_\alpha}\ln\left(\frac{r_{0\alpha}r_{1\alpha}}{r^2}\right)\right)\text{ for }t=2q$$

$$u_{\alpha0}^{(t)}=(-1)^{q}X_{\alpha t}\left(\frac{r_{0\alpha}}{r}\right)^{1/2}\sin\left(\frac{\pi(1+2q)}{4\varepsilon\omega_\alpha}\ln\left(\frac{r_{0\alpha}r_{1\alpha}}{r^2}\right)\right)\text{ for }t=2q+1$$

$$u_{\alpha+1,0}^{(t)}=X_{\alpha t}\left(\frac{r_{0\alpha}}{r}\right)^{1/2}\frac{ch\left(2\varepsilon\omega_\alpha z_{0kt}\right)}{sh\left(2\varepsilon\omega_{\alpha+1}z_{0kt}\right)}sh\left(z_{0kt}\ln\left(\frac{r_{1\alpha+1}}{r}\right)\right)$$

$$u_{\alpha-1,0}^{(t)}=X_{\alpha t}\frac{(-1)^{t_{\alpha-1}}p^{-1/2}}{2\varepsilon\omega_{\alpha-1}\delta_1}\left(\frac{r_{1\alpha-1}}{r}\right)^{1/2}sh\left(z_{0kt}\ln\left(\frac{r}{r_{0\alpha-1}}\right)\right),$$

where $\delta_1^2=\frac{z_{0kt}^2}{\left(9-4z_{0kt}^2\right)\varepsilon^2\omega_{\alpha-1}\omega_\alpha}$.

4. We represent permutations in the following form:

$$u_{\varphi k}(r,\theta)=u_{\varphi k}^{(0)}(r,\theta)+\sum_{t=1}^{\infty}u_{kt}(r)\,m_t'(\theta) \tag{4.1}$$

The second addend contains permutations corresponding to eigen-values (2.11), (2.13), (2.15).

For stress we get

$$\sigma_{\theta\varphi}^{(k)}(r,\theta)=\frac{2G_kB_0}{\sin^2\theta}+\sum_{t=1}^{\infty}G_k\frac{1}{r}u_{kt}(r)\left[\left(\frac{1}{4}-z_t^2\right)m_t(\theta)-2ctg\theta\cdot m_t'(\theta)\right] \tag{4.2}$$

For torques of M stresses acting in the section $\theta=const$ we get

$$M=2\pi\sin^2\theta\sum_{k=1}^{n}\int_{r_{0k}}^{r_{1k}}\sigma_{\theta\varphi}^{(k)}r^2dr \tag{4.3}$$

we substitute (4.2) into (4.3)

$$M=4\pi B_0\sum_{k=1}^{n}G_kb_{kk}+2\pi\sin^2\theta\times$$

$$\times\sum_{t=1}^{\infty}\left[\left(\left(\frac{1}{4}-z_t^2\right)m_t(\theta)-2ctg\theta\cdot m_t'(\theta)\right)\sum_{k=1}^{n}\left(G_k\int_{r_{0k}}^{r_{1k}}u_{kt}(r)\,rdr\right)\right] \tag{4.4}$$

Multiplying the both sides of (1.7) by G_kr^3, integrating the obtained one by $[r_{0k};\ r_{1k}]$ and summing with respect to k from 1 to n we get:

$$\left(z_t^2-\frac{1}{4}\right)\sum_{k=1}^{n}\int_{r_{0k}}^{r_{1k}}G_ku_{kt}(r)\,rdr=\sum_{k=1}^{n}\int_{r_{0k}}^{r_{1k}}G_k\left(u_{kt}''(r)+\frac{2}{r}u_{kt}'(r)\right)r^3dr \tag{4.5}$$

By integration by parts using conditions (1.8) – (1.10) we finally get from (4.5)

$$\sum_{k=1}^{n}G_k\int_{r_{0k}}^{r_{1k}}u_{kt}(r)\,rdr=0. \tag{4.6}$$

Substituting (4.6) into (4.4) for M we have

$$M=4\pi B_0\sum_{k=1}^{n}G_kb_{kk} \tag{4.7}$$

The constant B_0 at the absence of external efforts on lateral surfaces, is proportional to torques of M stresses acting on the segment $\theta=const$.

Stress state corresponding to eigen-values (2.11), (2.13) and (2.15) is self-balanced at each section of $\theta=const$.

It follows from the theorem that elementary solutions corresponding to $\wedge_+(p)$ for $p\to 0,\ \varepsilon\to 0$ quickly damp by receding from conical sections (their aggregate will be called "strong boundary layer")

It follows from the theorem that boundary solutions corresponding to $\wedge_-^{(1)}(p)$, $\wedge_-^{(2)}(p)$ for small $p \to 0,\ \varepsilon \to 0$ weakly damp by receding from conical sections of spherical shell and may essentially correct the penetrating solution (the aggregate of these solutions will be called "weak boundary layer").

The above mentioned affirms that Saint-Venants principle may be violated and generally accepted formulation of Saint Venant's principle is of conditional character. In a series of cases it requires essential correction [7].

5. As an example we consider the three-layer spherical shell. In this case $z_t \in \wedge_-^{(1)}(p)$, $z_t \in \wedge_-^{(2)}(p)$

$$\gamma = \pm \frac{c_{11}(b_{11}+b_{33})}{b_{11}b_{33}}; \quad \overline{X} = (X_1;\ X_3)^T$$

$$X_1 = \left(\frac{b_{33}}{b_{11}(b_{11}+b_{33})}\right)^{1/2}; \quad X_3 = -\left(\frac{b_{11}}{b_{33}(b_{11}+b_{33})}\right)^{1/2}$$

i.e.

$$z = \pm\left(\frac{3}{2} - \frac{c_{11}(b_{11}+b_{33})}{b_{11}b_{33}}p + O\left(p^2\right)\right)$$

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