Vladimir Sh. YUSUFOV

PARTIALLY ORDERED SYSTEMS OF OPEN SETS. SEMIGROUPS OF OPEN MAPPINGS

Abstract

In this paper the problem, formulated by A.A. Maltsev in the sixties, on characterization of topological spaces by semigroups of open mappings (not necessarily continuous) is solved.

In [1] the following theorem for the class of T_D - spaces (where $T_1 \subset T_D \subset T_0$) is solved by Thron.

Theorem. *(Thron) Let X* and *Y* be *T_D - spaces. The spaces X and Y are homeormorphic if and only if, lattices of all their closed (of all there open) subsets are isomorphic.*

In § 1 theorem 1.3 that is a strengthening of Thron theorem is proved.

In § 2 a subclass A of the class of T_D - spaces is considered. In the semigroup $O(X)$ of all open mappings $X \in A$ the subsemigroups $\widetilde{O}(X)$ are considered and the main theorem of this paper [2] is proved relying on theorem 1.3.

Theorem 2.2. *Let $X, Y \in A$. If semigroups $\widetilde{O}(X)$ and $\widetilde{O}(Y)$ are isomorphic, then spaces X and Y are homeomorphic.*

§ 1. Partially ordered systems of open sets.

1.1. A topological space X is called T_D - space [1] if for the arbitrary point $\xi \in X$ the set $\{\bar{\xi}\} \setminus \{\xi\}$ is closed. We will denote the set $\{\bar{\xi}\} \setminus \{\xi\}$ as in /1/ by $\{\xi\}'$. Obviously, each T_D - space is T_0 - space and each T_1 - space is T_D - space.

In /1/ the following theorem was proved.

Theorem (Thron) Let X, Y be T_D - spaces. The spaces X and Y are homeomorphic if and only if the lattices of all their closed (of all there open) subsets are isomorphic.

1.2. Let X be T_D - space and $\{X_\alpha\}_{\alpha \in A}$ be a system of its open subsets, satisfying the following conditions:

1) $\{X_\alpha\}_{\alpha \in A}$ forms prebase of a topology of the space X.

2) If one point set $\{\xi\}$ of the space X is closed, then $X \setminus \{\xi\} \in \{X_\alpha\}_{\alpha \in A}$.

3) If one point set $\{\xi\}$ of the space X is non-closed, then $X \setminus \{\bar{\xi}\} \in \{X_\alpha\}_{\alpha \in A}$ and $X \setminus \{\xi\}' \in \{X_\alpha\}_{\alpha \in A}$.

Theorem 1.3. *Let X* and *Y* be *T_D - spaces. If their partially ordered systems of open subsets $\{X_\alpha\}_{\alpha \in A}$ and $\{Y_\beta\}_{\beta \in B}$ satisfying the conditions 1), 2), 3) are isomorphic, then the spaces X and Y are homeomorphic.*

Obviously, Thron theorem will follow from theorem 1.3. Denote by Φ an isomorphism of the systems $\{X_\alpha\}_{\alpha \in A}$ *and* $\{Y_\beta\}_{\beta \in B}$.

Lemma 1.4. *Let one-point set $\{\xi\}$ of the space X be closed. Then the set $Y \setminus \Phi(X \setminus \{\xi\})$ consists of a unique point. If one-point set $\{\xi\}$ of the space X is non-closed, then the set $\Phi\left(X \setminus \{\xi\}'\right) \setminus \Phi\left(X \setminus \{\bar{\xi}\}\right)$ consists of a unique point.*

Proof. Note, that if the systems $\{X_\alpha\}_{\alpha \in A}$ and $\{Y_\beta\}_{\beta \in B}$ satisfying the conditions 1), 2), 3) are isomorphic and $X \in \{X_\alpha\}_{\alpha \in A}$, then $Y \in \{Y_\beta\}_{\beta \in B}$ and $\Phi X = Y$. Really, suppose, that $X \in \{X_\alpha\}_{\alpha \in A}$, and $Y \notin \{Y_\beta\}_{\beta \in B}$. Then $\Phi X \neq Y$ and there exists a point $\xi' \in Y$, such that $\xi' \notin \Phi X$. The sistem $\{Y_\beta\}_{\beta\in B}$ is a prebase of Y, and therefore $\xi' \in Y_{\beta_0}$ for some $\beta_0 \in B$. So $\Phi^{-1}Y_{\beta_0} \in \{X_\alpha\}_{\alpha\in A}$. Obviously, $\Phi^{-1}Y_{\beta_0} \subseteq X$ and $Y_{\beta_0} \subseteq \Phi X$. It is impossible, since $\xi' \notin \Phi X$. Let one-point set $\{\xi\}$ of the space X be closed. Suppose, that $|Y\backslash\Phi(X\backslash\{\xi\})| > 1$. Let $\xi' \in Y\backslash\Phi(X\backslash\{\xi\})$ and one-point set $\{\xi'\}$ of the space Y be non-closed. Since $Y\backslash\Phi(X\backslash\{\xi\})$ is a closed set, then $\left\{\bar{\xi}'\right\} \subseteq Y\backslash\Phi(X\backslash\{\xi\})$. Then $\{\xi'\}' \subset Y\backslash\Phi(X\backslash\{\xi\})$, $\Phi(X\backslash\{\xi\}) \subset Y\backslash\{\xi'\}'$.

Hence if follows, that $X\backslash\{\xi\} \subset \Phi^{-1}\left(Y\backslash\{\xi'\}'\right)$. So $\Phi^{-1}\left(Y\backslash\{\xi'\}'\right) = X$ and therefore $Y \in \{Y_\beta\}_{\beta\in B}$. We have

$$\Phi(X\backslash\{\xi\}) \subset Y\backslash\{\xi'\}' \subset Y,\ X\backslash\{\xi\} \subset \Phi^{-1}\left(Y\backslash\{\xi'\}'\right) \subset \Phi^{-1}Y = X.$$

It is impossible, therefore all one-point subsets of the set $Y\backslash\Phi(X\backslash\{\xi\})$ are closed. Let's take any point $\xi'' \in Y\backslash\Phi(X\backslash\{\xi\})$.

Since $|Y\backslash\Phi(X\backslash\{\xi\})| > 1$, then $\Phi(X\backslash\{\xi\}) \subset Y\backslash\{\xi'\}$,
$X\backslash\{\xi\} \subset \Phi^{-1}(Y\backslash\{\xi'\})$.

Thus, $X = \Phi^{-1}(Y\backslash\{\xi'\})$. It is impossible. If one-point set ξ of the space X is non-closed, then $(X\backslash\{\xi\}')\backslash(X\backslash\{\bar{\xi}\}) = \xi$, $(X\backslash\{\bar{\xi}\}) \subset (X\backslash\{\xi\}')$, $\Phi(X\backslash\{\bar{\xi}\}) \subset \Phi(X\backslash\{\xi\}')$. Let $\left|\Phi(X\backslash\{\xi\}')\backslash\Phi(X\backslash\{\bar{\xi}\})\right| > 1$. Let's choose any two points $\xi', \xi'' \in \Phi(X\backslash\{\xi\}')\backslash\Phi(X\backslash\{\bar{\xi}\})$. Since Y is T_0 - space, then there exists a neighboorhood of the point ξ' or ξ'' separating one of them from the other. Suppose for example, there will be found neighboorhood $V_{\xi'}$ of the point ξ' such that $\xi'' \notin V_{\xi'}$. Then $\xi' \notin \left\{\bar{\xi}''\right\}$ and $\{\xi''\} \cap \Phi(X\backslash\{\bar{\xi}\}) = \oslash$. It is clear, that $\Phi(X\backslash\{\bar{\xi}\}) \subseteq \Phi(X\backslash\{\xi\}')\backslash\left\{\bar{\xi}''\right\} \subset Y\backslash\left\{\bar{\xi}''\right\}$.

Hence, it follows, that $(X\backslash\{\bar{\xi}\}) \subset \Phi^{-1}\left(Y\backslash\left\{\bar{\xi}''\right\}\right)$, $X\backslash\Phi^{-1}\left(Y\backslash\left\{\bar{\xi}''\right\}\right) \subset \{\bar{\xi}\}$. The set $X\backslash\Phi^{-1}\left(Y\backslash\left\{\bar{\xi}''\right\}\right)$ is closed and therefore $X\backslash\Phi^{-1}\left(Y\backslash\left\{\bar{\xi}''\right\}\right) \subseteq \{\xi\}'$. So $\xi \notin X\backslash\Phi^{-1}\left(Y\backslash\left\{\bar{\xi}''\right\}\right)$ and $\xi \in \Phi^{-1}\left(Y\backslash\left\{\bar{\xi}''\right\}\right)$. Since $(X\backslash\{\bar{\xi}\}) \subset \Phi^{-1}\left(Y\backslash\left\{\bar{\xi}''\right\}\right)$, then $(X\backslash\{\bar{\xi}\}) \cup \{\xi\} = X\backslash\{\xi\}' \subseteq \Phi^{-1}\left(Y\backslash\left\{\bar{\xi}''\right\}\right)$,

$\Phi(X\backslash\{\xi\}') \subseteq Y\backslash\{\bar{\xi}"\}$ .Hence it would follow, that $\xi'' \notin \Phi(X\backslash\{\xi\}')$. We get contradiction with our supposition. Thus a unique point $\xi' \in Y$ corresponds to $\xi \in X$. Denote this mapping by f i.e. $f\xi \overset{def}{=} \xi'$.

Lemma 1.5. *Let one-point set ξ of the space X be non-closed.*
Then $\Phi(X\backslash\{\bar{\xi}\}) \subseteq Y\backslash\left\{\bar{\xi}'\right\}$.

Proof. By definition of mapping f we have
$\xi' = \Phi(X\backslash\{\xi\}')\backslash\Phi(X\backslash\{\bar{\xi}\})$. If $Y \neq \Phi(X\backslash\{\bar{\xi}\}) \cup \left\{\bar{\xi}'\right\}$ then $\Phi(X\backslash\{\bar{\xi}\}) \neq Y\backslash\left\{\bar{\xi}'\right\}$. Since $\Phi(X\backslash\{\bar{\xi}\}) \cap Y\backslash\left\{\bar{\xi}'\right\} = \oslash$ and $\Phi(X\backslash\{\bar{\xi}\}) \neq Y\backslash\left\{\bar{\xi}'\right\}$, then $\Phi(X\backslash\{\bar{\xi}\}) \subset Y\backslash\left\{\bar{\xi}'\right\}$. In addition $\Phi(X\backslash\{\xi\}') \not\subset Y\backslash\left\{\bar{\xi}'\right\}$, because $\xi' \in \Phi(X\backslash\{\xi\}')$. From $\Phi(X\backslash\{\bar{\xi}\}) \subset Y\backslash\left\{\bar{\xi}'\right\}$ we get, that

$$X\backslash\{\bar{\xi}\} \subset \Phi^{-1}\left(Y\backslash\left\{\bar{\xi}'\right\}\right),\ X\backslash\Phi^{-1}\left(Y\backslash\left\{\bar{\xi}'\right\}\right) \subset \{\bar{\xi}\}.$$

As the set $X\backslash\Phi^{-1}\left(Y\backslash\left\{\bar{\xi}'\right\}\right)$ is closed and the inclusion $X\backslash\Phi^{-1}\left(Y\backslash\left\{\bar{\xi}'\right\}\right)\subset\left\{\bar{\xi}\right\}$ is strict,then $\xi\notin X\backslash\Phi^{-1}\left(Y\backslash\left\{\bar{\xi}'\right\}\right)$ and so $\xi\in\Phi^{-1}\left(Y\backslash\left\{\bar{\xi}'\right\}\right)$. Therefore $\left(X\backslash\left\{\bar{\xi}\right\}\right)\cup\{\xi\}=X\backslash\{\xi\}'\subseteq\Phi^{-1}\left(Y\backslash\left\{\bar{\xi}'\right\}\right)$, $\Phi\left(X\backslash\{\xi\}'\right)\subseteq Y\backslash\left\{\bar{\xi}'\right\}$. Contradiction.

Lemma 1.6. *From lemma 1.5 and* $\Phi\left(X\backslash\{\xi\}'\right)\backslash\xi'=\Phi\left(X\backslash\left\{\bar{\xi}\right\}\right)$ *it follows, that mapping* f *transfers non-closed one-point sets to non-closed ones.*

Lemma 1.7. *Mapping* f *is bijective.*

Proof. Assume, that one-point sets ξ_1 and ξ_2 are non-closed, $\xi_1\neq\xi_2$ and $f\xi_1=f\xi_2=\xi'$. From lemma 1.5 we will get, that $Y\backslash\Phi\left(X\backslash\left\{\bar{\xi}_1\right\}\right)=\left\{\bar{\xi}'\right\}$ and $Y\backslash\Phi\left(X\backslash\left\{\bar{\xi}_2\right\}\right)=\left\{\bar{\xi}'\right\}$. So $\Phi\left(X\backslash\left\{\bar{\xi}_1\right\}\right)=\Phi\left(X\backslash\left\{\bar{\xi}_2\right\}\right)$, $X\backslash\left\{\bar{\xi}_1\right\}=X\backslash\left\{\bar{\xi}_2\right\}$ and $\left\{\bar{\xi}_1\right\}=\left\{\bar{\xi}_2\right\}$. The space X is T_0- space and therefore it follows from $\left\{\bar{\xi}_1\right\}=\left\{\bar{\xi}_2\right\}$, that $\xi_1=\xi_2$. If one-point sets ξ_1 and ξ_2 are closed, then we have, that $Y\backslash\Phi\left(X\backslash\xi_1\right)=\xi'$, $Y\backslash\Phi\left(X\backslash\xi_2\right)=\xi'$, $\Phi\left(X\backslash\xi_1\right)=\Phi\left(X\backslash\xi_2\right)$, $X\backslash\xi_1=X\backslash\xi_2$ and $\xi_1=\xi_2$. Let again one-point set ξ' of the space Y be non-closed. By lemma 1.5 for the isomorphism Φ^{-1} we have, that $\Phi^{-1}\left(Y\backslash\left\{\bar{\xi}'\right\}\right)=X\backslash\left\{\bar{\xi}\right\}$ where ξ is a point of the space X. Again by lemma 1.5 for the isomorphism Φ we will get, that $\Phi\left(X\backslash\left\{\bar{\xi}\right\}\right)=Y\backslash\left\{\overline{f\xi}\right\}$. So$\left\{\bar{\xi}'\right\}=\left\{\overline{f\xi}\right\}$ and $\xi'=f\xi$. If one-point set of the space Y is closed, then we have for the isomorphism Φ^{-1} , that $\Phi^{-1}\left(Y\backslash\xi\right)=X\backslash\xi$, $\Phi\left(X\backslash\xi\right)=Y\backslash\xi'$, $\xi'=Y\backslash\Phi\left(X\backslash\xi\right)$, where ξ is a point of the space X. By definition of mapping f it follows, that $f\xi=Y\backslash\Phi\left(X\backslash\xi\right)$. Thus $f\xi=\xi'$.

Lemma 1.8. *Mappings* f *and* f^{-1} *are open.*

Proof. It suffices to show, that for any element of the class $\{X_\alpha\}_{\alpha\in A}$ is fulfilled $fX_\alpha=\Phi X_\alpha$. Let $\xi\in X_\alpha$ and one-point set ξ be closed. If $f\xi\notin\Phi X_\alpha$, then $\Phi X_\alpha\subseteq Y\backslash f\xi$. Obviously, that $Y\backslash f\xi\in\{Y_\beta\}_{\beta\in B}$. Therefore $X_\alpha\subseteq\Phi^{-1}\left(Y\backslash f\xi\right)=X\backslash\xi$, and it contradicts to $\xi\in X_\alpha$. Now let $\xi\in X_\alpha$ and one-point set ξ be non-closed. If $f\xi\notin\Phi X_\alpha$ then $\left\{\overline{f\xi}\right\}\cap\Phi X_\alpha=\oslash$, as ΦX_α is open. Hence it follows, that $\Phi X_\alpha\subseteq Y\backslash\left\{\overline{f\xi}\right\}$. Taking into account, that $Y\backslash\left\{\overline{f\xi}\right\}\in\{Y_\beta\}_{\beta\in B}$ we would get the inclusion $X_\alpha\subseteq\Phi^{-1}\left(Y\backslash\left\{\overline{f\xi}\right\}\right)=X\backslash\left\{\bar{\xi}\right\}$. It is impossible, as $\xi\in X_\alpha$. Thus, $f\xi\in\Phi X_\alpha$ for any $\xi\in X_\alpha$, i.e. $fX_\alpha\subseteq\Phi X_\alpha$. Analogously one can show that $f^{-1}\left(\Phi X_\alpha\right)\subseteq\Phi^{-1}\left(\Phi X_\alpha\right)=X_\alpha$, $\Phi X_\alpha\subseteq fX_\alpha$. Thus $fX_\alpha=\Phi X_\alpha$ for any element of the family $\{X_\alpha\}_{\alpha\in A}$, and it means, that the mapping f is open. In the same way we can prove that the mapping f^{-1} is open.

§ 2. Semigroups of open mappings.

2.1. Let X be T_D - space satisfying the following conditions:

1) X has an open prebase, each element of which is an image of X under an open mapping.

2) For any $\xi\in X$ there exists open mappings f, g of the space X into itself such that $f\eta=g\eta$, when $\eta\neq\xi$, but $f\xi\neq g\xi$.

3) If one-point set $\{\xi\}$ is closed, then there exists an open mapping $h:X\to X$ such that $hX=X\backslash\{\xi\}$.

4) If one-point set $\{\xi\}$ is non-closed, then there exist open mappings u and ν of the space X into itshelf such that $uX=X\backslash\left\{\bar{\xi}\right\}$, $\nu X=X\backslash\{\xi\}'$.

Let A be a class of all such spaces. In particular, the open subsets of the cube. $I^{\tau}, \tau \geq 1$ and the cube D^{τ}, $\tau \geq \aleph_0$ belong to the class A [3]. Denote by $O(X)$ the semigroup of all open mappings of the space $X \in A$ into itself. $\widetilde{O}(X)$ is a subsemigroup of the semigroup $O(X)$ satisfying the following conditions:

1) The family of the open sets $\{aX\}_{a\in\widetilde{O}(X)}$ is a prebase of the space X,

2) For each $\xi \in X$ there exist $f, g \in \widetilde{O}(X)$ such that $f\eta = g\eta$, when $\eta \neq \xi$, but $f\xi \neq f\xi$,

3) If one-point set $\{\xi\}$ is closed, then there exists $h \in \widetilde{O}(X)$, such that $hX = X \backslash \{\xi\}$,

4) If one-point set ξ is non-closed, then there exist $u, \nu \in \widetilde{O}(X)$ such that $uX = X \backslash \{\bar{\xi}\}$, $\nu X = X \backslash \{\xi\}'$.

Everywhere below by φ we denote an isomorphism of the corresponding semigroups of open mappings.

Theorem 2.2. *Let $X, Y \in A$. If semigroups $\widetilde{O}(X)$ and $\widetilde{O}(Y)$ are isomorphic, then the spaces X and Y are homeomorphic.*

Lemma 2.3. *Let $X \in A$ and $a, b \in \widetilde{O}(X)$.*

$$bX \subseteq aX \longleftrightarrow \forall f, g \in \widetilde{O}(X)\ [fa = ga \rightarrow fb = gb]$$

Proof. If $bX \subseteq aX$, then for any point $\eta \in X$ there will be a point $\xi \in X$, such that $b\eta = a\xi$. Then $gb\eta = ga\xi = fa\xi = fb\eta \rightarrow fb = gb$. Conversely, let $\forall f, g \in \widetilde{O}(X)\ [fa = ga \rightarrow fb = gb]$, where $a, b \in \widetilde{O}(X)$. Assume, that, $bX \backslash aX \neq \oslash$. Then we will take an arbitrary point $\xi = b\eta$ from $bX \backslash aX$. From the condition 2) it follows, that there exist elements $f, g \in \widetilde{O}(X)$, such that $fa = ga$, but $f\xi \neq g\xi$ and so $fb\eta \neq gb\eta \rightarrow fb \neq gb$.

Lemma 2.4. *Let $X, Y \in A$ and $a, b \in \widetilde{O}(X)$. Then from the inclusion $bX \subseteq aX$ it follows the inclusion $(\varphi b)Y \subseteq (\varphi a)Y$, and so if $bX = aX$, then $(\varphi b)Y = (\varphi a)Y$.*

Proof. If for some elements $f', g' \in \widetilde{O}(X)$ it holds $f' \cdot \varphi a = g' \cdot \varphi a$, then there will be found such elements $f, g \in \widetilde{O}(X)$ that $f' = \varphi f, g'^{=} \varphi g$. Obviously $\varphi f \cdot \varphi a = \varphi g \cdot \varphi a, \varphi(fa) = \varphi(ga)$ and $fa = ga$. From lemma 2.3 it follows, that $fb = gb$, and therefore $\varphi f \cdot \varphi b = \varphi g \cdot \varphi b, f' \cdot \varphi b = g' \cdot \varphi b$. Thus, from the equality $f' \cdot \varphi a = g' \cdot \varphi a$ it follows the equality $f' \cdot \varphi b = g' \cdot \varphi b$. Again, using lemma 2.3, we will obtain, that $(\varphi b)Y \subseteq (\varphi a)Y$. It is clear, that if $bX = aX$, then $(\varphi b)Y = (\varphi a)Y$.

Let $a \in \widetilde{O}(X)$. Let us associate to the open set aX of the space X the open set $(\varphi a)Y$ of the space Y. By virtue of lemma 2.4, this mapping will be an isomorphism of the partially-ordered systems of the open sets of the families $\{aX\}_{a\in\widetilde{O}(X)}$ and $\{(\varphi a)Y\}_{\varphi a\in\widetilde{O}(Y)}$ of the spaces X and Y. Now we are under the conditions of theorem 1.3.

2.5. Let X be T_0 space, satisfying the following conditions:

1) Every open subset Ω of the space X is an image of X under an open mapping.

2) For any point $\xi \in X$ there exist open mappings f, g of the space into itself X, such that $f\eta = g\eta$, when $\eta \neq \xi$, but $f\xi \neq g\xi$.

Denote by A_0 the class of all such spaces. Let $\widetilde{O}(X)$ be a subsemigroup of the semigroups of all open mappings of the space $X \in A_0$ into itself such that:

1) For any open subset Ω of the space X there exists an element $h \in \widetilde{O}(X)$, such that $hX = \Omega$.

2) For any point $\xi \in X$ there exist elements $f, g \in \widetilde{O}(X)$, such that $f\eta = g\eta$, when $\eta \neq \xi$, but $f\xi \neq g\xi$.

Theorem 2.6. *Let X, Y be T_1 - spaces of the class A, semigroups $\widetilde{O}(X)$ and $\widetilde{O}(X)$ be isomorphic, $a \in \widetilde{O}(X)$ and a be injective. If φ is an isomorphism of the semigroups $\widetilde{O}(X)$ and $\widetilde{O}(X)$, then $\varphi a = faf^{-1}$ where f is a homomorphism induced by the isomorphism φ.*

Lemma 2.7. *Let X and Y be T_1 - spaces of the class A_0, $a, b \in \widetilde{O}(X)$, $bX \subset aX$ and $|aX \backslash bX| = 1$. Then $|(\varphi a)Y \backslash (\varphi b)Y| = 1$ and, if $\xi \in aX \backslash bX$, then $f\xi \in (\varphi a)Y \backslash (\varphi b)Y$.*

Proof. From lemma 2.4 it follows, that $(\varphi b)Y \subset (\varphi a)Y$. Assume that $|(\varphi a)Y \backslash (\varphi b)Y| > 1$. Let us choose any point $\xi' \in (\varphi a)Y \backslash (\varphi b)Y$. There exists an open mapping $g : Y \to Y$, such that $gY = ((\varphi a)Y) \backslash \{\xi'\}$. We have $(\varphi b)Y \subset gY \subset (\varphi a)Y$, $bX \subset (\varphi^{-1}g)X \subset aX$. As $|aX \backslash bX| = 1$, then we get $(\varphi^{-1}g)X = aX$, $gY = (\varphi a)Y$. It is a contradiction. Thus, $|(\varphi a)Y \backslash (\varphi b)Y| = 1$. There exists an element $c \in \widetilde{O}(X)$, such that $cX = X \backslash \{\xi\}$. Since $\xi \in aX$, then $f\xi \in (\varphi a)Y$. If $f\xi \notin (\varphi a)Y \backslash (\varphi b)Y$, then $f\xi \in (\varphi b)Y$. But $f\xi = Y \backslash (\varphi c)Y$, $f\xi \notin (\varphi c)Y$ and therefore $(\varphi b)Y \nsubseteq (\varphi c)Y$. It is impossible, since $bX \subseteq cX$.

Now, we'll prove the theorem.

Let y be an arbitrary point from Y. There exists an element $b' \in \widetilde{O}(Y)$, such that $Y \backslash b'Y = y$. By the definition of the mapping f, we have, that $X \backslash (\varphi^{-1}b')X = f^{-1}y$. Since a is injective, then $aX \backslash a(\varphi^{-1}b')X = af^{-1}y$.

By lemma 2.7 we have, that

$$faf^{-1}y = (\varphi a)Y \backslash \varphi\left[a\left(\varphi^{-1}b'\right)\right]Y = (\varphi a)Y \backslash (\varphi a)b'Y.$$

But $b'Y = Y \backslash y$, and therefore $(\varphi a)Y \backslash (\varphi a)b'Y = (\varphi a)y$, since the sets $(\varphi a)Y$ and $(\varphi a)b'Y$, differ only by one point. Thus $(\varphi a)y = faf^{-1}y$.

Corollary 2.8. *Let X and Y be T_1- spaces of the class A_0. If semigroups $O(X)$ and $O(Y)$ are isomorphic, then any isomorphism φ between semigroups $O(X)$ and $O(Y)$ maps $OH(X)$ onto $OH(Y)$.*

Theorem 2.9. *Let $X \in A_0$ and Y* be *T_2- spaces of the class A_0. If semigroups $\widetilde{O}(X)$* and *$\widetilde{O}(Y)$ are isomorphic, then the spaces X and Y are homeomorphic.*

Proof. Repeating the proof of lemmas 2.3 and 2.4 we will get, that the lattices of all open subsets of the spaces X and Y are isomorphic. Let $\xi \in X$. In the space X there won't be two different open sets X_1 and X_2 such that $X \backslash \{\bar{\xi}\} \subset X_1$, $X \backslash \{\bar{\xi}\} \subset X_2$, $X_1 \cap X_2 = X \backslash \{\bar{\xi}\}$. If follows from the fact that if $X \backslash \{\bar{\xi}\} \subset X'$, where X' is open, then the set $\{\bar{\xi}\} \backslash X'$ is closed and therefore $\xi \in X'$. Then in the space Y there don't exist two different open sets Y_1 and Y_2, such that $\Phi\left(X \backslash \{\bar{\xi}\}\right) \subset Y_1$, $\Phi\left(X \backslash \{\bar{\xi}\}\right) \subset Y_2$, $Y_1 \cap Y_2 = \Phi\left(X \backslash \{\bar{\xi}\}\right)$. Assume, that $\left|Y \backslash \Phi\left(X \backslash \{\bar{\xi}\}\right)\right| > 1$. There will be found two different points $\xi_1', \xi_2' \in Y \backslash \Phi\left(X \backslash \{\bar{\xi}\}\right)$. The space X is Housdorff, that's why there exist neighborhoods $V_{\xi_1'}$ and $V_{\xi_2'}$ of the points ξ_1' and ξ_2', such that $V_{\xi_1'} \cap V_{\xi_2'} = \Phi$. Then there would be $\left(\Phi\left(X \backslash \{\bar{\xi}\}\right) \cup V_{\xi_1'}\right) \cap \left(\Phi\left(X \backslash \{\bar{\xi}\}\right) \cup V_{\xi_2'}\right) = \Phi\left(X \backslash \{\bar{\xi}\}\right)$. So,$\Phi\left(X \backslash \{\bar{\xi}\}\right) = Y \backslash \{\xi'\}$ where ξ' is a point of the space Y. Suppose, that $\{\bar{\xi}\} \neq \{\xi\}$. There exists a point $\xi_1 \neq \xi$, such that $\xi_1 \in \{\bar{\xi}\}$. Since X is T_0 - space, then $\xi \notin \{\bar{\xi}_1\}$ and that's why $X \backslash \{\bar{\xi}\} \subset X \backslash \{\bar{\xi}_1\} \subset X$, $\Phi\left(X \backslash \{\bar{\xi}\}\right) \subset \Phi\left(X \backslash \{\bar{\xi}_1\}\right) \subset Y$. It contradicts the fact that $\Phi(X \backslash \{\xi\}) = Y \backslash \{\xi'\}$. Thus, for any point $\xi \in X$ we have $\left\{\bar{\xi}\right\} = \{\xi\}$. It means, that X is a T_1 - space. Now we will apply Thron theorem.

I want to express my deep gratitude to A.A.Maltsev for his valuable advices and constant attention to the paper.

References

[1]. Thron W.J. *Lattice equivalence of topological spaces.* Duke Math. J., 1962, 29, pp.671-688

[2]. Yusufov V.Sh. *Open mappings U.M.N.* 1986, v.41, No6, pp.185-186. (Russian)

[3]. Yusufov V.Sh. *Open and open covering mappings of topological spaces.* Proceedings of IMM of NAS of Azerbaijan, v. XXI(XXIX), 2004, pp.187-192.

Vladimir Sh.Yusufov

Institute of Mathematics and Mechanics of NAS of Azerbaijan
9, F. Agayev str., AZ141, Baku, Azerbaijan
Tel.: (99412) 439 47 20 (off.)

Received September 12, 2005; Revised December 15, 2005.
Translated by Nazirova S.H.