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INITIAL BOUNDARY-VALUE PROBLEM FOR A CLASS OF QUASILINEAR SOBOLEV TYPE EQUATIONS

Abstract

The mixed problem is considered for a class of quasilinear Sobolev type equations. The existence and uniqueness of local solutions is proved. For some class of quasilinear equations is also considered for which the corresponding mixed problem is solvable "in large".

Let $\Omega \in R^n$ be boundary domain with smooth boundary Γ. Let's consider in the cylinder $Q = [0,T] \times \Omega$ the following mixed problem

$$u_{tt} + a_k(t)\,\Delta^k u_{tt} + b_l(t)\,\Delta^l u = f\left(t, x, \delta_{l-k+s}u, \delta_s u_t\right), \tag{1}$$

with the boundary conditions

$$\Delta^i u(t,x) = 0,\ (t,x) \in [0,T] \times \Gamma,\ i = 0,1,...,l-1, \tag{2}$$

and initial conditions

$$u(0,x) = u_0(x),\ u_t(0,x) = u_1(x),\ x \in \Omega, \tag{3}$$

where $\delta_i \vartheta = \left(\vartheta, \dfrac{\partial\vartheta}{\partial x_1}, ..., \dfrac{\partial\vartheta}{\partial x_n}, \dfrac{\partial^2\vartheta}{\partial x_1^2}, ..., \dfrac{\partial^i\vartheta}{\partial x_n^i}\right),\ s \le 2k.$

When f doesn't depend on $\delta_s u_t$ and $s = 0$ problem (1)-(3) is considered in the paper [1], where the theorem on local solvability is established.

In the present paper a theorem on local solvability at $s \le 2k$ is proved and in some cases theorems on solvability "in large" are proved.

Let's determine the number $r = r(n,k,s)$ in the following way:

$$r = \begin{cases} \dfrac{2n}{n + 2(2k+s)}, & n > 2(2k+s), \\ 1 < r \le 2,\ n = 2(2k+s), \\ 1,\ n < 2(2k+s), \end{cases}$$

Assume that the following conditions are fulfilled:

1^0.$s \le 2k,\ k \le l.$

2^0.$a_k(t) = (-1)^k a(t),\ b_k(t) = (-1)^k b(t)$, where $a(\cdot),\ b(\cdot) \in C^1[0,T]$

3^0. The function $f(t,x,\xi,\eta)$ is determined for all $t \in [0,T]$, $x \in \overline{\Omega}$, $\xi = \{\xi_\alpha\}_{\substack{\alpha=(\alpha_1,...,\alpha_n)\in R^{\chi_1} \\ |\alpha| \le l-k+s}}$, $\eta = \{\eta_\beta\}_{\substack{\beta=(\beta_1,...,\beta_n)\in R^{\chi_2} \\ |\beta| \le s}}$ is continuously differentiable with respect to $(t,\xi,\eta) \in [0,T] \times R^{\chi_1+\chi_2}$, where $\chi_1 = \frac{(n+l-k+s)!}{n!(l-k+s)!}$.

4^0. Let for all $(t,x,\xi,\eta)\in[0,T]\times\overline{\Omega}\times R^{\chi_1+\chi_2}$ the following estimations be fulfilled:

$$\left|f\left(t,x,\xi,\eta\right)\right|,\ \left|f_t\left(t,x,\xi,\eta\right)\right|\le c\left(\sum_{|\alpha|<l-k+s-n/2}\left|\xi_\alpha\right|,\sum_{|\beta|<s-n/2}\left|\eta_\beta\right|\right)\times$$

$$\times\left[g\left(x\right)+\sum_{l-k+s-n/2\le|\alpha|<l-k+s}\left|\xi_\alpha\right|^{p(\alpha)/r}+\sum_{s-n/2\le|\beta|<s}\left|\eta_\beta\right|^{q(\beta)/r}\right],$$

where $c\left(\cdot,\cdot\right)\in C\left(R_+^2,R_+\right),\ R_+=[0,\infty),\ g\left(\cdot\right)\in L_r\left(\Omega\right)$ and $p\left(\alpha\right),\ q\left(\beta\right)$ satisfy the following conditions:

at $|\alpha|=l-k+s-\dfrac{n}{2}$ $p\left(\alpha\right)\in\left[\dfrac{1}{r},\infty\right)$;

at $|\alpha|<l-k+s-\dfrac{n}{2}$ $p\left(\alpha\right)\le\dfrac{2n}{n-2\left(l-k+s-|\alpha|\right)}$;

at $|\beta|=s-\dfrac{n}{2}$ $q\left(\beta\right)\in\left[\dfrac{1}{r},\infty\right)$;

at $|\beta|>s-\dfrac{n}{2}$ $q\left(\beta\right)\le\dfrac{2n}{n-2\left(s-|\beta|\right)}$.

5^0. At all $(t,x,\xi,\eta)\in[0,T]\times\overline{\Omega}\times R^{\chi_1+\chi_2}$ the following estimations are fulfilled:

$$\left|f_{\xi_\lambda}\left(t,x,\xi,\eta\right)\right|\le c_\lambda\left(\sum_{|\beta|<l-k+s-n/2}\left|\xi_\beta\right|,\sum_{|\gamma|<s-n/2}\left|\eta_\gamma\right|\right)\times$$

$$\times\left[g_\lambda\left(x\right)+\sum_{l-k+s-n/2\le|\beta|<l-k+s}\left|\xi_\beta\right|^{p_\lambda(\beta)}+\sum_{s-n/2\le|\gamma|<s}\left|\eta_\gamma\right|^{q_\lambda(\gamma)}\right],$$

where $c_\lambda\left(\cdot,\cdot\right)\in C\left(R_+^2,R_+\right)$ and $p_\lambda\left(\beta\right),\ q_\lambda\left(\gamma\right)$ satisfy the following conditions:

a) if $|\lambda|<l-k+s-\dfrac{n}{2}$, then

1) $g_r\left(x\right)\in L_2\left(\Omega\right)$;

2) at $|\beta|=l-k+s-\dfrac{n}{2}$ $p_\lambda\left(\beta\right)\in\left[\frac{1}{r},\infty\right)$ and

at $|\beta|>l-k+s-\dfrac{n}{2}$ $p_\lambda\left(\beta\right)\le\dfrac{2n}{r\left[n-2\left(l-k+s-|\beta|\right)\right]}$;

3) at $|\gamma|=s-\dfrac{n}{2}$ and $q_\lambda\left(\gamma\right)\in\left[\dfrac{1}{r},\infty\right)$ and

at $|\gamma|>s-\dfrac{n}{2}$ $q_\lambda\left(\gamma\right)\le\dfrac{2n}{r\left[n-2\left(s-|\gamma|\right)\right]}$;

b) if $|\lambda|=l-k+s-\dfrac{n}{2}$, then

1) $g_\lambda\left(x\right)\in L_q\left(\Omega\right)$ $q\ge1$,

2) at $|\beta|=l-k+s-\dfrac{n}{2}$ $p_\lambda\left(\beta\right)\in\left[\dfrac{1}{r},\infty\right)$ and

at $|\beta|>l-k+s-\dfrac{n}{2}$ $p_\lambda\left(\beta\right)\le\dfrac{2n}{r\left[n-2\left(l-k+s-|\beta|\right)\right]}$;

3) at $|\gamma|=s-\dfrac{n}{2}$ $q_\lambda\left(\gamma\right)\in\left[\dfrac{1}{r},\infty\right)$ and

at $|\gamma| > s-\dfrac{n}{2}$ $\quad q_{\lambda}(\gamma) \leq \dfrac{2n(1-r)}{r[n-2(s-|\gamma|)]}$;

c) if $|\lambda| > l-k+s-\dfrac{n}{2}$

1) $g_{\lambda}(x) \in L_{\frac{n}{l+k-|\lambda|}}(\Omega)$;

2) at $|\beta| = l-k+s-\dfrac{n}{2}$ $\quad p_{\lambda}(\beta) \in \left[\dfrac{1}{r}, \infty\right)$ and

at $|\beta| > l-k+s-\dfrac{n}{2}$ $\quad p_{\lambda}(\beta) \leq \dfrac{2n-r[n-2(l-k+s-|\lambda|)]}{r[n-2(l-k+s-|\beta|)]}$;

3) at $|\gamma| = s-\dfrac{n}{2}$ $\quad q_{\gamma} \in \left[\dfrac{1}{r}, \infty\right)$ and

at $|\gamma| > s-\dfrac{n}{2}$ $\quad q_{\lambda}(\gamma) \leq \dfrac{2n-r[n-2(l-k+s-|\lambda|)]}{r[n-2(s-|\gamma|)]}$

6^0. For all $(t,x,\xi,\eta) \in [0,T] \times \overline{\Omega} \times R^{\chi_1+\chi_2}$ the following estimations are fulfilled

$$\left|f_{\eta_\theta}(t,x,\xi,\eta)\right| \leq \tilde{c}_\theta\left(\sum_{|\beta|<l-k+s-n/2}|\xi_\beta|, \sum_{|\gamma|<s-n/2}|\eta_\gamma|\right) \times$$

$$\times\left[g_\theta(x) + \sum_{l-k+s-n/2\leq\beta<l-k+s}|\xi_\beta|^{\tilde{p}_\theta(\beta)} + \sum_{s-n/2\leq\gamma<s}|\eta_\gamma|^{\tilde{q}_\theta(\gamma)}\right],$$

where $\tilde{c}_\theta(\cdot,\cdot) \in C\left(R_+^2, R_+\right)$ and $\tilde{p}_\theta(\beta)$, $\tilde{q}_\theta(\gamma)$ satisfy the following conditions:

a) if $|\theta| < s-\dfrac{n}{2}$, then

1) $\tilde{g}_\theta(\cdot) \in L_r(\Omega)$;

2) at $|\beta| = l-k+s-\dfrac{n}{2}$ $\quad \tilde{p}_\theta(\beta) \in \left[\dfrac{1}{r}, \infty\right)$ and

at $|\beta| > l-k+s-\dfrac{n}{2}$ $\quad \tilde{p}_\theta(\beta) \leq \dfrac{2n}{r[n-2(l-k+s-|\beta|)]}$;

3) at $|\gamma| = s-\dfrac{n}{2}$ $\quad q_\theta(\gamma) \in \left[\dfrac{1}{r}, \infty\right)$ and

at $|\gamma| > s-\dfrac{n}{2}$ $\quad \tilde{q}_\theta(\gamma) \leq \dfrac{2n}{r[n-2(s-|\gamma|)]}$;

b) if $|\theta| = s-\dfrac{n}{2}$, then

1) $\tilde{g}_\theta(\cdot) \in L_p(\Omega)$ $\quad p \in [r, \infty)$

2) at $|\beta| = l-k+s-\dfrac{n}{2}$ $\quad \tilde{p}_\theta(\beta) \in \left[\dfrac{1}{r}, \infty\right)$ and

at $|\beta| > l-k+s-\dfrac{n}{2}$ $\quad \tilde{p}_\theta(\beta) \leq \dfrac{2n(1-r)}{r[n-2(l-k+s-|\beta|)]}$;

3) at $|\gamma| = s-\dfrac{n}{2}$ $\quad \tilde{q}_\theta(\gamma) \in \left[\dfrac{1}{r}, \infty\right)$,

at $|\gamma| > s-\dfrac{n}{2}$ $\quad \tilde{q}_\theta(\gamma) \leq \dfrac{2n(1-r)}{r[n-2(s-|\gamma|)]}$;

c) if $|\theta| > s-\dfrac{n}{2}$, then

1) $\tilde{g}_\theta(\cdot) \in L_{\frac{n}{2k-|\theta|}}(\Omega)$;

2) at $|\beta| = l-k+s-\dfrac{n}{2}$ $\quad \tilde{p}_\theta(\beta) \in \left[\dfrac{1}{r}, \infty\right)$ and

at $|\beta| > l-k+s-\dfrac{n}{2}$ $\quad\tilde{p}_\theta(\beta) \le \dfrac{2n - r[n-2(s-|\theta|)]}{r[n-2(l-k+s-|\beta|)]}$;

3) at $|\gamma| = s-\dfrac{n}{2}$ $\quad\tilde{q}_\theta(\gamma) \in \left[\dfrac{1}{r}, \infty\right)$,

at $|\gamma| > s-\dfrac{n}{2}$ $\quad\tilde{q}_\theta(\gamma) \le \dfrac{2n - r[n-2(s-|\theta|)]}{r[n-2(s-|\gamma|)]}$;

Let's determine the Hilbert space $\mathcal{H}_s = \hat{H}^{l-k+s} \times \hat{H}^s$ with the scalar product

$$\left\langle w^1, w^2 \right\rangle = \int\limits_\Omega \nabla^{l-k+s} u_1 \cdot \nabla^{l-k+s} u_2 dx + \int\limits_\Omega \nabla^s \vartheta_1 \cdot \nabla^s \vartheta_2 dx,$$

where $w^i = \begin{pmatrix} u_i \\ v_i \end{pmatrix}$, $u_i \in \hat{H}^{l-k+s}$, $v_i \in \hat{H}^s$, $i = 1, 2$;

$\hat{H}^m = \left\{ u : u \in W_2^m(\Omega),\ \Delta^i u|_\Gamma = 0,\ i = 0, 1, ..., \left(\dfrac{m}{2}\right)\right\}$, $\left(\dfrac{m}{2}\right) = \dfrac{m}{2} - 1$ if m is even $\left(\dfrac{m}{2}\right) = \left[\dfrac{m}{2}\right]$, if m is odd.

Let's determine also the space $\mathcal{H}_0 = \hat{H}^{2(l-k)+s} \times \hat{H}^{l-k+s}$.

Theorem 1. *Let conditions 1^0-6^0 be satisfied. Then at any $(u_0, u_1) \in \mathcal{H}_0$ there exists such $t_0 = \varphi(\|(u_0, u_1)\|_{\mathcal{H}})$ that problem (1)-(3) has a unique solution*

$$u(\cdot) \in C\left([0, T']; \hat{H}^{2(l-k)+s}\right) \cap C^1\left([0, T']; \hat{H}^{l-k+s}\right) \cap C^2\left([0, T']; \hat{H}^s\right)$$

where $T' \in (0, t_0]$, $\varphi(\cdot) \in C(R_+; R_+)$.

If for solution of problem (1)-(3) a priori estimation

$$E(t) \equiv \|u(t, \cdot)\|_{\hat{H}^{l-k+s}} + \left\|u'(t, \cdot)\right\|_{\hat{H}^s} \le c(E(0)), \tag{4}$$

$c(\cdot) \in C(R_+; R_+)$ is fulfilled then $t_0 = T$. In the other case there exists such $T_1 \in (0, T)$, that

$$\lim_{t \to T_1 - 0} E(t) = \infty.$$

Introduce the basic moments of the proof of theorem 1.

Substituting $v_1 = u$, $v_2 = u_t$ we can lead to problem (1)-(3) the Cauchy problem

$$w_t = L(t) w + F(t, w) \tag{5}$$

$$w(0) = w_0 \tag{6}$$

in the Hilbert space $\mathcal{H}$, where $D(L(t)) = \mathcal{H}_0$ and at each $t \in (0, T)$ $L(t) - \omega I$ generates the strong continuous construction semi-group in $\mathcal{H}$ $(\omega > 0)$. The operator function $L(t)$ is strongly continuously differentiable.

Further, it is proved that $F(t, w)$ satisfies all conditions of the theorem on the existence and uniqueness of solutions of Cauchy problem in Banach space (see [2]).

We introduce only proofs of fulfillment of Lipschitz local condition:

$$\left\|F\left(t, w^1\right) - F\left(t, w^2\right)\right\|_{\mathcal{H}} \le c\left(\left\|w^1\right\|_{\mathcal{H}}, \left\|w^2\right\|_{\mathcal{H}}\right) \cdot \left\|w^1 - w^2\right\|_{\mathcal{H}},$$

at $n > 2(2k+s)$, where $c(\cdot) \in C\left(R_+^2; R_+\right)$.

Let $w^i = \left(u^i, \vartheta^i\right) \in \mathcal{H}_0,\ i = 1, 2,$, then

$$\left\|F\left(t, w^1\right) - F\left(t, w^2\right)\right\|_{\mathcal{H}} \leq \left\|f\left(t, \delta_{l-k+s}u^1, \delta_s\vartheta^1\right) - f\left(t, \delta_{l-k+s}u^2, \delta_s\vartheta^2\right)\right\|_{H^{s-2k}}$$

Using imbedding theorem

$$\hat{H}^{2k-s} \subset L_{r'}(\Omega) \subset L_2(\Omega) \subset L_r(\Omega) \subset \hat{H}^{s-2k},$$

where $r = \dfrac{n+2(2k+s)}{2n}$, $r' = \dfrac{1-r}{r}$, from the last inequality we'll get

$$\left\|F\left(t, w^1\right) - F\left(t, w^2\right)\right\|_{\mathcal{H}} \leq$$

$$\leq C\left\|f\left(t, \delta_{l-k+s}u^1, \delta_s\vartheta^1\right) - f\left(t, \delta_{l-k+s}u^2, \delta_s\vartheta^2\right)\right\|_{L_r(\Omega)}. \tag{7}$$

From (7) it follows that

$$\left\|F\left(t, w^1\right) - F\left(t, w^2\right)\right\|_{\mathcal{H}}^r \leq$$

$$\leq \int_\Omega \left|\int_0^1 \left[\sum_{|\alpha| \leq l-k+s} f_{\xi_\alpha}\left(t, u^\tau, \vartheta^\tau\right) D^a\left(u_1 - u_2\right) + \right.\right.$$

$$\left.\left. + \sum_{|\theta| \leq s} f_{\eta\theta}\left(t, u^\tau, \vartheta^\tau\right) D^\theta\left(\vartheta_1 - \vartheta_2\right)\right] d\tau\right|^r dx,$$

where $u^\tau = (1-\tau)u_1 + \tau u_2$, $\vartheta^\tau = (1-\tau)\vartheta_1 + \tau\vartheta_2$.

Hence using conditions 3^0-4^0 we'll get

$$\left\|F\left(t, w^1\right) - F\left(t, w^2\right)\right\|_{\mathcal{H}}^r \leq$$

$$\leq \int_\Omega\int_0^1 \left|\sum_{|\lambda| \leq l-k+s} c_\lambda\left(\sum_{|h| \leq l-k+s-n/2}\left|D^h u^\tau\right|, \sum_{|\lambda| s-n/2}\left|D^\lambda\vartheta^\tau\right|\right)\left[g_\lambda(x) + \right.\right.$$

$$+ \sum_{l-k+s-n/2 \leq |\beta| < l-k+s}\left(\left|D^\beta u_1\right|^{p_\lambda(\beta)} + \left|D^\beta u_2\right|^{p_\lambda(\beta)}\right) +$$

$$\left. + \sum_{s-n/2 \leq |\gamma| < s}\left(\left|D^\gamma\vartheta_1\right|^{q_\lambda(\gamma)} + \left|D^\gamma\vartheta_2\right|^{q_\lambda(\gamma)}\right)\right] \times$$

$$\times\left|D^\lambda\left(u_1 - u_2\right)\right| + \sum_{|\theta| \leq s} c_\theta\left(\sum_{|h| \leq l-k+s-n/2}\left|D^h u^\tau\right|, \sum_{|\lambda| \leq s-n/2}\left|D^\lambda\vartheta^\tau\right|\right) \times$$

$$\times\left[\tilde{g}_\theta(x) + \sum_{l-k+s-n/2 \leq |\beta| < l-k+s}\left(\left|D^\beta u_1\right|^{\tilde{p}_\theta(\beta)} + \left|D^\beta u_2\right|^{\tilde{p}_\theta(\beta)}\right) + \right.$$

$$+\sum_{s-n/2\leq|\gamma|<s}\left(|D^{\gamma}\vartheta_1|^{\tilde{q}_{\theta}(\gamma)}+|D^{\gamma}\vartheta_2|^{\tilde{q}_{\theta}(\gamma)}\right)\Bigg]\,D^{\theta}\left(\vartheta_1-\vartheta_2\right)\Bigg|\,d\tau dx. \qquad (8)$$

Further, we'll show only how addends are estimated from above

$$J_{\lambda}=\int\limits_{\Omega}\int\limits_{0}^{1}c_{\lambda}^{r}\left(\sum_{|h|\leq l-k+s-n/2}\left|D^{h}u^{\tau}\right|,\sum_{|\lambda|\leq s-n/2}\left|D^{\lambda}\vartheta^{\tau}\right|\right)\times$$

$$\times\sum_{l-k+s-n/2\leq|\beta|<l-k+s}\left|D^{h}u_1\right|^{p_{\lambda}(\beta)r}\left|D^{\lambda}\left(u_1-u_2\right)\right|^{r}dxd\tau.$$

Other addends are estimated analogously.

Allowing for $m>\frac{n}{2}$ $\hat{H}^{m}\subset C\left(\overline{\Omega}\right)$ from the last we'll get:

$$J_{\lambda}\leq\varphi_{\lambda}^{\tau}\left(\|u^{\tau}\|_{\hat{H}^{\ l-k+s}},\|\vartheta^{\tau}\|_{\hat{H}^{\ s}}\right)\times$$

$$\times\sum_{l-k+s-n/2\leq|\beta|<l-k+s}\int\limits_{\Omega}\left|D^{\beta}u_1\right|^{p_{\lambda}(\beta)r}\left|D^{\lambda}\left(u_1-u_2\right)\right|^{r}dx,$$

where $\varphi_{\lambda}(\cdot,\cdot)\in C^{2}\left(R_{+}^{2};R_{+}\right)$.

In case $|\lambda|<l-k+s-\dfrac{n}{2}$ we have the following estimation:

$$J_{\lambda}\leq\varphi_{\lambda}^{\tau}\left(\|u^{\tau}\|_{\hat{H}^{l-k+s}},\|\vartheta^{\tau}\|_{\hat{H}^{s}}\right)\times$$

$$\times\|u_1-u_2\|_{\hat{H}^{l-k+s}}\sum_{l-k+s-n/2\leq|\beta|<l-k+s}\int\limits_{\Omega}\left|D^{\beta}u_1\right|^{p_{\lambda}(\beta)r}dx.$$

In view of condition 5^{0} at $|\beta|=l-k+s-\dfrac{n}{2}$ $\quad p_{\lambda}(\beta)\geq\dfrac{1}{r}$ and at $|\beta|=l-k+s-\dfrac{n}{2}$ $\ p_{\lambda}(\beta)\leq\dfrac{2n}{r\left[n-2\left(l-k+s-|\beta|\right)\right]}$.

Therefore using imbedding theorem

$$H^{l-k+s-n/2}\subset L_{p_{\lambda}(\beta)r}\qquad\text{(see [3])}$$

from (8)-(9) we'll get that

$$J_{\lambda}\leq\psi^{r}\left(\left\|w^{1}\right\|_{H_s},\ \left\|w^{2}\right\|_{H_s}\right)\|w_1-w_2\|_{\mathcal{H}},$$

where $\psi(\cdot,\cdot)\in C\left(R_{+}^{2};R_{+}\right)$.

Other addends are estimated analogously.

Thus, for problem (5)-(6) all conditions of theorems on the existence and uniqueness of Cauchy problem are fulfilled for quasilinear hyperbolic equations in Hilbert space $\mathcal{H}$ (see [2]).

Now let's consider some class of nonlinear functions f for which for solutions of corresponding mixed problem a priori estimation (4) is fulfilled.

7^0. Let $s \geq k$.

8^0. Assume that G_1 and G_2 are some Banach spaces, moreover $\hat{H}^{l-k+s} \subset G_1$, $\hat{H}^s \subset G_2$. Let at any $u \in C\left([0,T];\ \hat{H}^{2(l-k)+s}\right) \cap C^1\left([0,T];\hat{H}^{l-k+s}\right) \cap C^2\left([0,T];\hat{H}^s\right)$ the following one-sided and two-sided estimations are fulfilled:

$$\int_0^t \int_\Omega f\left(\tau, x, \delta_{l-k+s} u\left(\tau, x\right), \delta_s u_\tau\left(\tau, x\right)\right) u_\tau\left(\tau, x\right) dx d\tau +$$

$$+c_1 \left\|u\left(\tau, \cdot\right)\right\|_{G_1}^{p_1} + c_2 \int_0^t \left\|u\left(\tau, \cdot\right)\right\|_{G_2}^{p_2} d\tau \leq c_3 +$$

$$+c_4 \int_0^t \left[\int_\Omega \left(\left|\nabla^l u\left(\tau, x\right)\right|^2 dx + \left|\nabla^k u_\tau\left(\tau, x\right)\right|^2\right) dx + c_1 \left\|u\left(\tau, \cdot\right)\right\|_{G_1}^{p_1}\right] d\tau,$$

where $c_1, c_2, c_3, c_4 \geq 0$;

$$\left\|f\left(t, x, \delta_{l-k+s} u\left(t, x\right), \delta_s u_t\left(t, x\right)\right)\right\|_{\hat{H}^j} \leq$$

$$\leq c\left(\|u\|_{G_1}, \|u_\tau\|_{G_2}, \|u\|_{\hat{H}^{k+j-1}}, \|u_t\|_{\hat{H}^{k+j-1}}\right)\left[1 + \|u\|_{\hat{H}^{l-k+j}} + \|u_t\|_{\hat{H}^{k+j}}\right]$$

$$j = 1, 2, ..., j_0,\ j_0 \geq s - k$$

Theorem 2. *Let conditions 1^0-8^0 be satisfied. Then at any $(u_0, u_1) \in \mathcal{H}_0$ problem (1)-(3) has a unique solution*

$$u \in C\left([0,T];\ \hat{H}^{2(l-k)+s}\right) \cap C^1\left([0,T];\hat{H}^{l-k+s}\right) \cap C^2\left([0,T];\hat{H}^s\right).$$

Consider nonlinear equation in the domain $[0,T] \times \Omega$

$$u_{tt} + (-1)^k \Delta^k u_{tt} + (-1)^l \Delta^l u = \varphi\left(u\right) + \psi\left(u_t\right) + h\left(t, x, u, u_t\right) \tag{9}$$

with boundary conditions (2) and with initial conditions (3).

Assume that the following conditions be satisfied

1. $k \leq s \leq 2k,\ k \leq l.$
2. φ and ψ are continuously differentiable functions.
9. Let the following one-sided estimations be satisfied:

$$\int_0^u \varphi\left(\tau\right) d\tau \leq -\eta_1 \left|u\right|^{\rho_1+1} + \eta_2 \left|u\right|^2 + \eta_3,$$

$$\int_0^u \varphi\left(\tau\right) d\tau \leq -\eta_1 \left|u\right|^{\rho_1+1} + \eta_2 \left|u\right|^2 + \eta_3,$$

and for all $(t,x,u,\vartheta)\in[0,T]\times\overline{\Omega}\times R^2$ the following estimations be satisfied

$$|h(t,x,u,\vartheta)|\le c\left(1+\eta_1\xi_1|u|^{\frac{(\rho_1+1)(\rho_2+1)}{\rho_2}}+\eta_1|u|^{\frac{\rho_1+1}{2}}+\xi_1|\vartheta|^{\frac{\rho_2+1}{2}}\right),$$

where $\eta_i\ge 0,\ \xi_i\ge 0,\ i=1,2,3,\ c>0,\ \rho_1,\rho_2\ge 0.$

3. At $n\ge 2(l-k+s)$ the following two-sided estimations be satisfied:

$$|\varphi(u)|\le c\left(1+|u|^{p_n}\right),$$

$$|\varphi'(u)|\le c\left(1+|u|^{p'_n}\right),$$

where $c>0,\ p_{2(l-k+s)}\in[1;\infty),\ p'_{2(l-k+s)}\in[1;\infty)$ and at $n>2\ (l-k+s)$, $p_n\le\dfrac{2n}{[n-2(l-k+s)]\,r},\ p'_n\le\dfrac{2n-r[n-2(l-k+s)]}{[n-2(l-k+s)]\,r}.$

4. At $n\ge 2s$ the following two-sided estimations be satisfied:

$$|\psi(\vartheta)|\le c\left(1+|\vartheta|^{q_n}\right),$$

$$|\psi'(\vartheta)|\le c\left(1+|\vartheta|^{q'_n}\right),$$

where $c>0,\ q_{2s}\in[1;\infty)$ and at $n>2s,\ q_n\le\dfrac{2n}{(n-2s)\,r},\ q'_n\le\dfrac{2n-r(n-2s)}{(n-2s)\,r}.$

5. $h(t,x,u,\vartheta)$ is continuously differentiable function in the domain $[0,T]\times\overline{\Omega}\times R^2$.

6. At $l-k+s-\dfrac{n}{2}>0,\ s-\dfrac{n}{2}\le 0$ the following estimation is fulfilled:

$$|h(t,x,u,\vartheta)|,\ |h_t(t,x,u,\vartheta)|\le c(u)\left(1+|\vartheta|^q\right),$$

where $c(\cdot)\in C^1(R)$ at $s-\dfrac{n}{2}=0\ \ q\in\left(\dfrac{1}{r},\infty\right)$ and at $s-\dfrac{n}{2}<0\ q\le\dfrac{2n}{(n-2s)\,r}.$

At $l-k+s-\dfrac{n}{2}=0$

$$|h(t,x,u,\vartheta)|,\ |h_t(t,x,u,\vartheta)|\le c\left(1+|u|^p+|\vartheta|^q\right),$$

where $c>0,\ p\in\left[\dfrac{1}{r},\infty\right)$ and $q\le\dfrac{2n}{(n-2s)\,r}.$

At $l-k+s-\dfrac{n}{2}<0$

$$|h(t,x,u,\vartheta)|,\ |h_t(t,x,u,\vartheta)|\le c\left(1+|u|^p+|\vartheta|^q\right),$$

where $c>0,\ p\le\dfrac{2n}{[n-2(l-k+s)]\,r},\ q\le\dfrac{2n}{(n-2s)\,r}.$

7. At $l-k+s-\dfrac{n}{2}>0,\ s-\dfrac{n}{2}\le 0$ the following estimation is fulfilled:

$$|h_t(t,x,u,\vartheta)|\le c(u)\left(1+|\vartheta|^q\right),$$

where $c(\cdot)\in C(R)$ at $s-\dfrac{n}{2}=0\ \ q\in[1,\infty)$, and at $s-\dfrac{n}{2}<0\ \ q\le\dfrac{2n}{(n-2s)\,r}.$

At $l-k+s-\frac{n}{2}\leq 0$ the following estimation is fulfilled

$$\left|h'_u(t,x,u,\vartheta)\right|\leq c\left(1+|u|^p+|\vartheta|^q\right),$$

where at $l-k+s-\frac{n}{2}=0,\ s-\frac{n}{2}=0\ \ p\in[1,\infty)$ and $q\in[1,\infty)$,

at $l-k+s-\frac{n}{2}=0,\ s-\frac{n}{2}<0\ \ p\in[1,\infty)$ and $q\leq\frac{2n}{(n-2s)\,r}$,

at $l-k+s-\frac{n}{2}<0$

$$p\leq\frac{2n-r\left[n-2\left(l-k+s\right)\right]}{\left[n-2\left(l-k+s\right)\right]r},\ \ q\leq\frac{2n-r\left[n-2\left(l-k+s\right)\right]}{(n-2s)\,r}.$$

8. At $l-k+s-\frac{n}{2}\geq 0,\ s-\frac{n}{2}\leq 0$ the following estimation is fulfilled:

$$\left|h'_\vartheta(t,x,u,\vartheta)\right|\leq c(u)\left(1+|\vartheta|^q\right)$$

where $c(\cdot)\in C(R)$ at $s-\frac{n}{2}=0,\ q\in[1,\infty)$, and at $s-\frac{n}{2}<0\quad q\leq\frac{2n}{(n-2s)\,r}$.

At $l-k+s-\frac{n}{2}\leq 0$ and $s-\frac{n}{2}\leq 0$ the following estimation is fulfilled

$$\left|h'_\vartheta(t,x,u,\vartheta)\right|\leq c\left(1+|u|^p+|\vartheta|^q\right),$$

where $c>0$ at $l-k+s-\frac{n}{2}=0,\ s-\frac{n}{2}=0\quad p\in[1,\infty)$ and $q\in[1,\infty)$,

at $l-k+s-\frac{n}{2}<0\ \ p\in[1,\infty)$ and $q\leq\frac{2n}{(n-2s)\,r}$,

at $l-k+s-\frac{n}{2}<0\quad p\leq\frac{2n-r\,(n-2s)}{\left[n-2\left(l-k+s\right)\right]r},\ \ q\leq\frac{2n-r\,(n-2s)}{(n-2s)\,r}$.

From foregoing conditions follows the following theorem on solvability "in large".

Theorem 3. *Let conditions 1^0-8^0 be satisfied. Then at any $(u_0,u_1)\in\hat{H}^{2(l-k)+s}\times\hat{H}^{l-k+s}$ problem (9), (2), (3) has a unique solution*

$$u\in C\left([0,T];\ \hat{H}^{2(l-k)+s}\right)\cap C^1\left([0,T];\hat{H}^{l-k+s}\right)\cap C^2\left([0,T];\hat{H}^{s}\right).$$

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