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INITIAL BOUNDARY VALUE PROBLEMS FOR A CLASS OF THIRD ORDER OPERATOR-DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

Abstract

In the paper on a positive axis we consider initial boundary value problems for a class of operator-differential equations whose principal part undergoes discontinuity. Here obtaining upper estimation of the norm of the intermediate derivatives operators by the principal part of the considered equations we find sufficient conditions of regular soliability of the considered initial boundary value problems.

Let H be a separable Hilbert space, A be a selfadjoint positive-definite operator in H $(A = A^* > cE,\ c > 0,\ E$ is a uniqueoperator).

We denote by $L_2(R_+; H)$ a Hilbert space of all vector-functions determined in $R_+ = [0; +\infty)$ with values from H that have the finite norm

$$\|f\|_{L_2(R_+;H)} = \left(\int_0^{+\infty} \|f(t)\|_H^2 \, dt\right)^{1/2}.$$

Then we determine the following Hilbert spaces:

$$W_2^3(R_+; H) = \left\{ u(t) \,/ \frac{d^3u(t)}{dt^3} \in L_2(R_+; H),\ A^3u(t) \in L_2(R_+; H) \right\},$$

$$\overset{0}{W}{}_2^3(R_+; H) = \left\{ u(t) \,/ u(t) \in W_2^3(R_+; H),\ \frac{d^s u(0)}{dt^s} = 0,\ s = 0;1;2 \right\},$$

$$\overset{0}{W}{}_2^3(R_+; H; i) = \left\{ u(t) \,/ u(t) \in W_2^3(R_+; H),\ \frac{d^i u(0)}{dt^i} = 0 \right\}$$

with norm

$$\|u\|_{W_2^3(R_+;H)} = \left(\left\| \frac{d^3u}{dt^3} \right\|_{L_2(R_+;H)}^2 + \left\|A^3u\right\|_{L_2(R_+;H)}^2 \right)^{1/2},$$

where i is a fixed integer, and i may take only one of the values: $i = 0;1;2$ (on spaces $L_2(R_+; H)$ and $W_2^3(R_+; H)$ in detail see [1; ch. 1]).

Here and in sequel the derivatives are understood in the sense of theory of distributions.

Under $L(X,Y)$ we will understand a set of linear bounded operators acting from the Hilbert space X to another Hilbert space Y, and $L_\infty(R_+; B)$ is a set of B - valued essentially bounded operator functions in R_+, where B is a Banach space.

Now let's consider the following third order operator differential equation:

$$Q\left(\frac{d}{dt}\right) u(t) \equiv \frac{d^3u(t)}{dt^3} + \rho(t) A^3u(t) + A_1(t) \frac{d^2u(t)}{dt^2} +$$

$$+A_2(t)\frac{du(t)}{dt}=f(t),t\in R_+ \qquad (1)$$

at fulfillment the initial boundary condition

$$\frac{d^i u(0)}{dt^i}=0, \qquad (2)$$

where i is a fixed integer that can take only one of the values: $i=0;1;2;$ $f(t)\in L_2(R_+;H)$, $u(t)\in W_2^3(R_+;H)$, $A_k(t)$ $(k=1;2)$ are linear, generally speaking, unbounded operators determined almost for all $t\in R_+$, but $\rho(t)=\alpha$, if $0\le t\le T$ and $\rho(t)=\beta$ if $T<t<+\infty$, moreover, α and β are positive, generally speaking, unequal numbers and for definiteness we will assume $\alpha\le\beta$.

Definition 1. *If the vector function $u(t)\in W_2^3(R_+;H)$ satisfies equation (1) almost everywhere in R_+, it will be said to be a regular solution of equation (1).*

Definition 2. *If for any $f(t)\in L_2(R_+;H)$ there exists a regular solution of equation (1), that satisfies initial boundary condition (2) in the sense*

$$\lim_{t\to 0}\left\|A^{3-i-1/2}\frac{d^i u(t)}{dt^i}\right\|=0$$

and there holds the inequality

$$\|u\|_{W_2^3(R_+;H)}\le cons\,\|f\|_{L_2(R_+;H)},$$

we will say that problem (1), (2) is regularly solvable.

Note that problems on solvability for initial boundary value problems (1), (2) were studied in the case $\rho(t)\equiv 1,\ t\in R_+$ and $A_k(t)=A_k\,(k=1;2),t\in R_+$ in the papers [2], [3] and for $i=0$ in the case when only $A_k(t)=A_k\,(k=1;2),\ t\in R_+$ in the paper [4]. The case $i=0$ with variable operator coefficients in the perturbed part of the equation was considered in the paper [5]. Here alongside with the case $i=0$ we study the remaining two cases of initial boundary value problems on an semi-axis R_+ for the equation(1): $i=1$ and $i=2$. Besides, we state a new approach, to the estimation of the norms of intermediate derivatives operators, participating in the perturbed part through the main part of equation (1) in subspaces $\overset{0}{W}{}_2^3(R_+;H;i)$ $(i=0;1;2)$ that leads to improvement of the estimations known before and thereby admits to obtain new theorems on regular solvability of initial boundary value problems of the form (1), (2) in the cases $i=1$ and $i=2$ but in the case $i=0$ to improve the results of [4], [5].

First of all let's consider the main part of equation (1)

$$Q_0\left(\frac{d}{dt};A\right)u(t)\equiv\frac{d^3u(t)}{dt^3}+\rho(t)A^3u(t)=f(t),$$

where $f(t)\in L_2(R_+;H)$. Let's denote by $Q_0^{(i)}$ an operator acting from the space $\overset{0}{W}{}_2^3(R_+;H;i)$ to $L_2(R_+;H)$ in the following way:

$$Q_0^{(i)}u(t)\equiv Q_0\left(\frac{d}{dt};A\right)u(t),u(t)\in\overset{0}{W}{}_2^3(R_+;H;i).$$

Then the following theorem is true.

Theorem 1. *The operator $Q_0^{(i)}$ realizes an isomorphism from the space $\overset{0}{W}{}_2^3 (R_+; H; i)$ to $L_2 (R_+; H)$.*

Notice that the case $i = 0$ was proved in the paper [4] and found its generalization in [5]. The cases $i = 1$ and $i = 2$ are proved similarly.

Now we denote by $Q_1^{(i)}$ an operator acting from the space $\overset{0}{W}{}_2^3 (R_+; H; i)$ to $L_2 (R_+; H)$ in the following way:

$$Q_1^{(i)} u (t) \equiv A_1 (t) \frac{d^2 u (t)}{dt^2} + A_2 (t) \frac{du (t)}{dt}, u (t) \in \overset{0}{W}{}_2^3 (R_+; H; i) .$$

It holds the following theorem.

Theorem 2. *Let $A = A^* > cE,\ c > 0$ and $A_k (t) A^{-k} \in L_\infty (R_+; L (H, H))$ $(k = 1; 2)$.*

Then the operator $Q_1^{(i)}$ is a bounded operator from the space $\overset{0}{W}{}_2^3 (R_+; H; i)$ to the space $L_2 (R_+; H)$.

Proof. Since for any vector function $u (t) \in \overset{0}{W}{}_2^3 (R_+; H; i)$

$$\left\| Q_1^{(i)} u \right\|_{L_2(R_+;H)} \le \sum_{k=1}^{2} \sup_t \left\| A_{3-k} (t) A^{-3+k} \right\|_{H\to H} \left\| A^{3-k} \frac{d^k u}{dt^k} \right\|_{L_2(R_+;H)},$$

we apply the known theorem on intermediate derivatives [1; ch. 1] and from the inequality obtain

$$\left\| Q_1^{(i)} u \right\|_{L_2(R_+;H)} \le const \left\| u \right\|_{W_2^3(R_+;H)} .$$

The theorem is proved.

Theorem 1 implies that $\left\| Q_0^{(i)} u \right\|_{L_2(R_+;H)}$ is a norm in the space $\overset{0}{W}{}_2^3 (R_+; H; i)$, equivalent to the initial norm $\|u\|_{W_2^3(R_+;H)}$. Since it is known that the operators of intermediate derivatives

$$A^{3-k} \frac{d^k}{dt^k} : \overset{0}{W}{}_2^3 (R_+; H; i) \to L_2 (R_+; H)\ \ (k = 1, 2)$$

are continuous [1; ch. 1], the norms of these operators may be estimated by $\left\| Q_0^{(i)} u \right\|_{L_2(R_+;H)}$. But we first study some properties of polynomial operator bundles that we shall need in our further investigations.

Let $q_k = \frac{2^{2/3} \beta^{k/3}}{3\alpha^{\frac{3-k}{3}}}, k = 1; 2.$

We consider the following operator bundles that depend on the parameter $\gamma \in \left[0; q_k^{-1}\right),\ k = 1; 2:$

$$Q_k (\lambda; \gamma; A) = \frac{1}{\beta} (i\lambda)^6 E + \alpha A^6 - \gamma (i\lambda)^{2k} A^{6-2k},\ k = 1; 2.$$

The typical polynomials responding to them will have the following forms:

$$Q_k (\lambda; \gamma; \sigma) = \frac{1}{\beta} (i\lambda)^6 + \alpha\sigma^6 - \gamma (i\lambda)^{2k} \sigma^{6-2k},\ k = 1; 2,$$

where $\sigma \in \sigma(A)$ is a spectrum of the operator A.

The following lemma is true.

Lemma 1. *Let* $\gamma \in \left[0; q_k^{-1}\right)$, $k = 1;2$. *The polynomial operator bundles* $Q_k(\lambda;\gamma;A)$, $k = 1;2$, *are invertible on an imaginary axis and they admit the following representations:*

$$Q_k(\lambda;\gamma;A) = F_k(\lambda;\gamma;A)\,F_k(-\lambda;\gamma;A), \; k = 1;2,$$

moreover

$$F_k(\lambda;\gamma;A) = \prod_{n=1}^{3}\left(\frac{1}{\sqrt[6]{\beta}}\lambda E - \sqrt[6]{\alpha}\,\omega_{k,n}(\gamma)\,A\right) \equiv$$

$$\equiv \frac{1}{\sqrt{\beta}}\lambda^3 E + d_{1,k}(\gamma)\,\lambda^2 A + d_{2,k}(\gamma)\,\lambda A^2 + \sqrt{\alpha}A^3,$$

where $\operatorname{Re}\omega_{k,n}(\gamma) < 0$, $n = 1;2;3$, *and the numbers* $d_{1,k}(\gamma)$, $d_{2,k}(\gamma)$ *satisfy the following systems of equations:*

1) for $k = 1$ *2) for* $k = 2$

$$\left\{\begin{array}{l} d_{1,1}^2(\gamma) - \frac{2d_{2,1}(\gamma)}{\sqrt{\beta}} = 0, \\ 2\sqrt{\alpha}d_{1,1}(\gamma) - d_{2,1}^2(\gamma) = \gamma; \end{array}\right. \qquad \left\{\begin{array}{l} \frac{2d_{2,2}(\gamma)}{\sqrt{\beta}} - d_{1,2}^2(\gamma) = \gamma, \\ 2\sqrt{\alpha}d_{1,2}(\gamma) - d_{2,2}^2(\gamma) = 0. \end{array}\right. \tag{3}$$

Proof. Let $\lambda = i\xi, \xi \in R = (-\infty;+\infty)$. Then t is clear that for the typical polynomials we have be following relations:

$$Q_k(\lambda;\gamma;\sigma) = Q_k(i\xi;\gamma;\sigma) = \sigma^6\left(\frac{1}{\beta}\left(\frac{\xi^2}{\sigma^2}\right)^3 + \alpha\right)\left[1 - \gamma\frac{\left(\frac{\xi^2}{\sigma^2}\right)^k}{\frac{1}{\beta}\left(\frac{\xi^2}{\sigma^2}\right)^3 + \alpha}\right] \geq$$

$$\geq \sigma^6\left(\frac{1}{\beta}\left(\frac{\xi^2}{\sigma^2}\right)^3 + \alpha\right)\left[1 - \gamma\sup_{\frac{\xi^2}{\sigma^2}\geq 0}\frac{\left(\frac{\xi^2}{\sigma^2}\right)^k}{\frac{1}{\beta}\left(\frac{\xi^2}{\sigma^2}\right)^3 + \alpha}\right], \; k = 1;2.$$

Because of

$$\sup_{\frac{\xi^2}{\sigma^2}\geq 0}\frac{\left(\frac{\xi^2}{\sigma^2}\right)^k}{\frac{1}{\beta}\left(\frac{\xi^2}{\sigma^2}\right)^3 + \alpha} = q_k, \; k = 1;2,$$

we get

$$Q_k(i\xi;\gamma;\sigma) > 0 \tag{4}$$

for $\gamma \in \left[0; q_k^{-1}\right)$, $k = 1;2$. It follows from (4) that the polynomials $Q_k(\lambda;\gamma;\sigma)$ have no roots on an imaginary axis for $\gamma \in \left[0; q_k^{-1}\right)$, $k = 1;2$. Obviously each of typical polynomials $Q_k(\lambda;\gamma;\sigma)$ for $\sigma \in \sigma(A)$ has exactly three roots from the left half plane. Since these polynomials are homogeneous with respect to the arguments λ and σ we can represent them in the following forms:

$$Q_k(\lambda;\gamma;\sigma) = F_k(\lambda;\gamma;\sigma)\,F_k(-\lambda;\gamma;\sigma), \; k = 1;2, \tag{5}$$

where $F_k(\lambda;\gamma;\sigma)\prod_{n=1}^{3}\left(\frac{1}{\sqrt[6]{\beta}}\lambda - \omega_{k,n}(\gamma)\sqrt[6]{\alpha}\sigma\right) \equiv \frac{1}{\sqrt{\beta}}\lambda^3 + d_{1,k}(\gamma)\,\lambda^2\sigma +$

$d_{2,k}(\gamma)\lambda\sigma^2+\sqrt{\alpha}\sigma^3$, moreover, $\operatorname{Re}\omega_{k,n}(\gamma)<0, k=1;2;3$, and the numbers $d_{1,k}(\gamma)$, $d_{2,k}(\gamma)$ satisfy the following systems of equations obtained from relations (5) by comparing coefficients at the equal powers

1) for $k=1$
$$\begin{cases} d_{1,1}^2(\gamma)-\frac{2d_{2,1}(\gamma)}{\sqrt{\beta}}=0, \\ 2\sqrt{\alpha}d_{1,1}(\gamma)-d_{2,1}^2(\gamma)=\gamma; \end{cases}$$

2) for $k=2$
$$\begin{cases} \frac{2d_{2,2}(\gamma)}{\sqrt{\beta}}-d_{1,2}^2(\gamma)=\gamma, \\ 2\sqrt{\alpha}d_{1,2}(\gamma)-d_{2,2}^2(\gamma)=0. \end{cases}$$

In sequel, using spectral expansion of the operator A we get the proof of the lemma from equality (5). The lemma is proved.

Now, let's formulate the theorem that plays an essential part in further investigations an indicates importance of study of spectral properties of polynomial bundles $Q_k(\lambda;\gamma;A)$ and $F_k(\lambda;\gamma;A)$, $k=1;2$ introduced earlier.

Theorem 3. *Let $\gamma\in\left[0;q_k^{-1}\right)$. Then for any $u(t)\in W_2^3(R_+;H)$ it is valid the equality*

$$\left\|\frac{1}{\beta}\frac{d^3u}{dt^3}+\sqrt{\alpha}A^3u\right\|^2_{L_2(R_+;H)}-\gamma\left\|A^{3-k}\frac{d^ku}{dt^k}\right\|^2_{L_2(R_+;H)}=$$
$$=\left\|F_k\left(\frac{d}{dt};\gamma;A\right)u\right\|^2_{L_2(R_+;H)}+\left(G_k(\gamma)\psi,\psi\right)_{H^3}, \tag{6}$$

where $H^3=\overset{3}{\underset{m=1}{\oplus}}H$,

$$G_k(\gamma)=\begin{bmatrix} \sqrt{\alpha}d_{2,k}(\gamma) & \sqrt{\alpha}d_{1,k}(\gamma) & 0 \\ \sqrt{\alpha}d_{1,k}(\gamma) & d_{1,k}(\gamma)d_{2,k}(\gamma) & \frac{d_{2,k}(\gamma)}{\sqrt{\beta}} \\ 0 & \frac{d_{2,k}(\gamma)}{\sqrt{\beta}} & \frac{d_{1,k}(\gamma)}{\sqrt{\beta}} \end{bmatrix},$$

$$\psi=\left(\psi_p=A^{3-p-1/2}\frac{d^pu(0)}{dt^p}\right)^2_{p=0}.$$

Proof. Now let's define the space $D^3(R_+;H)$ a set of infinitely differentiable functions with values in $D\left(A^3\right)$ having a compact support in R_+. Since the space $D^3(R_+;H)$ is dense in $W_2^3(R_+;H)$ (see [1; ch. 1]), it suffices to prove the theorem for vector-functions $u(t)\in D^3(R_+;H)$. Then

$$\left\|F_k\left(\frac{d}{dt};\gamma;A\right)u\right\|^2_{L_2(R_+;H)}=\frac{1}{\beta}\left\|\frac{d^3u}{dt^3}\right\|^2_{L_2(R_+;H)}+d_{1,k}^2(\gamma)\left\|A\frac{d^2u}{dt^2}\right\|^2_{L_2(R_+;H)}+$$
$$+d_{2,k}^2(\gamma)\left\|A^2\frac{du}{dt}\right\|^2_{L_2(R_+;H)}+\alpha\left\|A^3u\right\|^2_{L_2(R_+;H)}+$$
$$+2\frac{d_{1,k}(\gamma)}{\sqrt{\beta}}\operatorname{Re}\left(\frac{d^3u}{dt^3},A\frac{d^2u}{dt}\right)_{L_2(R_+;H)}+$$
$$+2\frac{d_{2,k}(\gamma)}{\sqrt{\beta}}\operatorname{Re}\left(\frac{d^3u}{dt^3},A^2\frac{du}{dt}\right)_{L_2(R_+;H)}+2\frac{\sqrt{\alpha}}{\sqrt{\beta}}\operatorname{Re}\left(\frac{d^3u}{dt^3},A^3u\right)_{L_2(R_+;H)}+$$
$$+2d_{1,k}(\gamma)d_{2,k}(\gamma)\operatorname{Re}\left(A\frac{d^2u}{dt^2},A^2\frac{du}{dt}\right)_{L_2(R_+;H)}+$$

$$+2\sqrt{\alpha}d_{1,k}(\gamma)\operatorname{Re}\left(A\frac{d^2u}{dt^2},A^3u\right)_{L_2(R_+;H)}+2\sqrt{\alpha}d_{2,k}(\gamma)\operatorname{Re}\left(A^2\frac{du}{dt},A^3u\right)_{L_2(R_+;H)}$$

and here applying integration by parts we have

$$\left\|F_k\left(\frac{d}{dt};\gamma;A\right)u\right\|^2_{L_2(R_+;H)}=\frac{1}{\beta}\left\|\frac{d^3u}{dt^3}\right\|^2_{L_2(R_+;H)}+\alpha\left\|A^3u\right\|^2_{L_2(R_+;H)}+$$
$$+\left(d^2_{1,k}(\gamma)-2\frac{d_{2,k}(\gamma)}{\sqrt{\beta}}\right)\left\|A\frac{d^2u}{dt^2}\right\|^2_{L_2(R_+;H)}+$$
$$+\left(d^2_{2,k}(\gamma)-2\sqrt{\alpha}d_{1,k}(\gamma)\right)\left\|A^2\frac{du}{dt}\right\|^2_{L_2(R_+;H)}-\frac{d_{1,k}(\gamma)}{\sqrt{\beta}}\left\|\psi_2\right\|^2- \tag{7}$$
$$-2\frac{d_{2,k}(\gamma)}{\sqrt{\beta}}\operatorname{Re}(\psi_2,\psi_1)-2\frac{\sqrt{\alpha}}{\sqrt{\beta}}\operatorname{Re}(\psi_2,\psi_0)+$$
$$+\left(\frac{\sqrt{\alpha}}{\sqrt{\beta}}-d_{1,k}(\gamma)d_{2,k}(\gamma)\right)\left\|\psi_1\right\|^2-2\sqrt{\alpha}d_{1,k}(\gamma)\operatorname{Re}(\psi_1,\psi_0)-\sqrt{\alpha}d_{2,k}(\gamma)\left\|\psi_0\right\|^2.$$

If we calculate $\left\|\frac{1}{\sqrt{\beta}}\frac{d^3u}{dt^3}+\sqrt{\alpha}A^3u\right\|^2_{L_2(R_+;H)}$ similar to $\left\|F_k\left(\frac{d}{dt};\gamma;A\right)u\right\|^2_{L_2(R_+;H)}$, we find

$$\left\|\frac{1}{\sqrt{\beta}}\frac{d^3u}{dt^3}+\sqrt{\alpha}A^3u\right\|^2_{L_2(R_+;H)}=$$
$$=\frac{1}{\beta}\left\|\frac{d^3u}{dt^3}\right\|^2_{L_2(R_+;H)}+\alpha\left\|A^3u\right\|^2_{L_2(R_+;H)}-2\frac{\sqrt{\alpha}}{\sqrt{\beta}}\operatorname{Re}(\psi_2,\psi_0)+\frac{\sqrt{\alpha}}{\sqrt{\beta}}\left\|\psi_1\right\|^2. \tag{8}$$

Taking into account (8) in (7) by lemma 1 we get the truth of equality (6). The theorem is proved.

The following one follows from theorem 3.

Corollary 1. *If* $u(t)\in \overset{0}{W}{}^3_2(R_+;H)$ *and* $\gamma\in\left[0;q_k^{-1}\right)$, *then*

$$\left\|\frac{1}{\sqrt{\beta}}\frac{d^3u}{dt^3}+\sqrt{\alpha}A^3u\right\|^2_{L_2(R_+;H)}-\gamma\left\|A^{3-k}\frac{d^ku}{dt^k}\right\|^2_{L_2(R_+;H)}=\left\|F_k\left(\frac{d}{dt};\gamma;A\right)u\right\|^2_{L_2(R_+;H)}. \tag{9}$$

Obviously, the norms $\|u\|_{W^3_2(R_+;H)}$ and $\left\|\frac{1}{\sqrt{\beta}}\frac{d^3u}{dt^3}+\sqrt{\alpha}A^3u\right\|_{L_2(R_+;H)}$ are equivalent in the spaces $\overset{0}{W}{}^3_2(R_+;H)$ and $\overset{0}{W}{}^3_2(R_+;H;i)$.

Let's calculate the numbers:

$$a_{0,k}=\sup_{0\neq u\in \overset{0}{W}{}^3_2(R_+;H)}\left\|A^{3-k}\frac{d^ku}{dt^k}\right\|_{L_2(R_+;H)}\left\|\frac{1}{\sqrt{\beta}}\frac{d^3u}{dt^3}+\sqrt{\alpha}A^3u\right\|^{-1}_{L_2(R_+;H)},$$

$$a_{i,k}=\sup_{0\neq u\in \overset{0}{W}{}^3_2(R_+;H;i)}\left\|A^{3-k}\frac{d^ku}{dt^k}\right\|_{L_2(R_+;H)}\left\|\frac{1}{\sqrt{\beta}}\frac{d^3u}{dt^3}+\sqrt{\alpha}A^3u\right\|^{-1}_{L_2(R_+;H)}.$$

First we calculate $a_{0,k}$.

Lemma 2. *The numbers* $a_{0,k} = q_k^{1/2}$, $k = 1;2$.

Proof. In equality (9), passing to limit as $\gamma \to q_k^{-1}$, we have that for any vector-function $u(t) \in \overset{0}{W}{}_2^3(R_+;H)$ the following inequality is true

$$\left\| \frac{1}{\sqrt{\beta}} \frac{d^3u}{dt^3} + \sqrt{\alpha} A^3 u \right\|_{L_2(R_+;H)}^2 \geq q_k^{-1} \left\| A^{3-k} \frac{d^k u}{dt^k} \right\|_{L_2(R_+;H)}^2 ,$$

and thereby we get

$a_{0,k} \leq q_k^{1/2}$, $k = 1;2$.

Now we are to show that here it holds an equality. For this purpose for any $\eta > 0$ it suffices to construct a vector function $u_\eta(t) \in \overset{0}{W}{}_2^3(R_+;H)$, such that the functional

$$\varepsilon(u_\eta) \equiv \left\| \frac{1}{\sqrt{\beta}} \frac{d^3u_\eta}{dt^3} + \sqrt{\alpha} A^3 u_\eta \right\|_{L_2(R_+;H)}^2 - \left(q_k^{-1} + \eta\right) \left\| A^{3-k} \frac{d^k u_\eta}{dt^k} \right\|_{L_2(R_+;H)}^2 < 0.$$

The construction method of the function $u_\eta(t)$ is similar to the method of the papers [6], [7].

The lemma is proved.

By $\overset{0}{W}{}_2^3(R_+;H) \subset \overset{0}{W}{}_2^3(R_+;H;i)$, then $a_{i,k} \geq a_{0,k} = q_k^{1/2}$, $k = 1;2$. Notice that for any vector-function $u(t) \in \overset{0}{W}{}_2^3(R_+;H;i)$ and $\gamma \in \left[0; q_k^{-1}\right)$ it is fulfilled the equality

$$\left\| \frac{1}{\sqrt{\beta}} \frac{d^3u}{dt^3} + \sqrt{\alpha} A^3 u \right\|_{L_2(R_+;H)}^2 - \gamma \left\| A^{3-k} \frac{d^k u}{dt^k} \right\|_{L_2(R_+;H)}^2 =$$

$$= \left\| F_k\left(\frac{d}{dt};\gamma;A\right) u \right\|_{L_2(R_+;H)}^2 + \left(G_k(\gamma;i)\widetilde{\psi},\widetilde{\psi}\right)_{H^2}, \tag{10}$$

where $H^2 = \bigoplus_{m=1}^{2} H, G_k(\gamma;i)$ is a matrix obtained from $G_k(\gamma)$ removing the $i+1$-th column and row, $\widetilde{\psi} = \left(\widetilde{\psi}_{r_j} = A^{3-r_j-1/2} \frac{d^{r_j}u(0)}{dt^{r_j}}\right)_{j=0}^{1}, r_j \neq i$. The truth of equality (10) directly follows from theorem 3.

The following lemma show's that the numbers $a_{i,k}, k = 1;2$ may equal $q_k^{1/2}$, $k = 1;2$.

Lemma 3. *For* $a_{i,k} = q_k^{1/2}$ *it is necessary and sufficient the matrix* $G_k(\gamma;i)$ *be positive for any* $\gamma \in \left[0; q_k^{-1}\right)$.

Proof. Necessity. Let $a_{i,k} = q_k^{1/2}$. The from (10) for any vector-function $u(t) \in \overset{0}{W}{}_2^3(R_+;H;i)$ and $\gamma \in \left[0; q_k^{-1}\right)$ we have

$$\left\| F_k\left(\frac{d}{dt};\gamma;A\right) u \right\|_{L_2(R_+;H)}^2 + \left(G_k(\gamma;i)\widetilde{\psi},\widetilde{\psi}\right)_{H^2} \geq$$

$$\geq \left\| \frac{1}{\sqrt{\beta}} \frac{d^3u}{dt^3} + \sqrt{\alpha} A^3 u \right\|^2_{L_2(R_+;H)} \times \left(1 - \gamma a^2_{i,k}\right) > 0. \tag{11}$$

As the polynomial operator bundle $F_k(\lambda;\gamma;A)$ for $\gamma \in [0; q_k^{-1})$ has the form (see lemma 1)

$F_k(\lambda;\gamma;A) = \prod\limits_{n=1}^{3} \left(\frac{1}{\sqrt[6]{\beta}} \lambda E - \sqrt[6]{\alpha}\omega_{k,n}(\gamma) A \right)$, where $\operatorname{Re}\omega_{k,n}(\gamma) < 0, n = 1;2;3$, the Cauchy problem

$$F_k\left(\frac{d}{dt};\gamma;A\right) u(t) = 0, \tag{12}$$

$$\frac{d^i u(0)}{dt^i} = 0, \tag{13}$$

$$\frac{d^{r_j} u(0)}{dt^{r_j}} = A^{-(3-r_j-1/2)} \widetilde{\psi}_j, \ i \neq r_j, \widetilde{\psi}_j \in H, j = 0;1, \tag{14}$$

has a unique solution $u_\gamma(t) \in W_2^3(R_+;H)$, represented in the form

$$u_\gamma(t) = e^{\sqrt[6]{\alpha\beta}\omega_{k,1}(\gamma)tA}\varphi_0 + e^{\sqrt[6]{\alpha\beta}\omega_{k,2}(\gamma)tA}\varphi_1 + e^{\sqrt[6]{\alpha\beta}\omega_{k,3}(\gamma)tA}\varphi_2,$$

where $\varphi_0, \varphi_1, \varphi_2 \in D\left(A^{5/2}\right)$, subjected to unique determination from the conditions in zero of (13), (14). Thus, writing inequality (11) for the vector-function $u_\gamma(t)$, we get for $\gamma \in [0; q_k^{-1})$ $\left(G_k(\gamma;i)\widetilde{\psi}, \widetilde{\psi}\right)_{H^2} > 0$.

Sufficiency. If for any $\gamma \in [0; q_k^{-1})$ the matrix $G_k(\gamma;i)$ is positive, the equality (10) implies that for all $u(t) \in \overset{0}{W}{}_2^3(R_+;H;i)$ and $\gamma \in [0; q_k^{-1})$

$$\left\| \frac{1}{\sqrt{\beta}} \frac{d^3u}{dt^3} + \sqrt{\alpha} A^3 u \right\|^2_{L_2(R_+;H)} \geq \gamma \left\| A^{3-k} \frac{d^k u}{dt^k} \right\|^2_{L_2(R_+;H)}.$$

Here we pass to limit as $\gamma \to q_k^{-1}$, get $a_{i,k} \leq q_k^{1/2}$, and have $a_{i,k} = q_k^{1/2}$. The lemma is proved.G

If should be noted that for specially chosen i and k there may be $a_{i,k} > q_k^{1/2}$.

Lemma 4. $a_{i,k} > q_k^{1/2}$ *if and only if the equation* $\det G_k(\gamma;i) = 0$ *would have a solution from the interval* $(0; q_k^{-1})$, *moreover, the least of these roots equals* $a_{i,k}^{-2}$.

Proof. Let $a_{i,k} > q_k^{1/2}$. Then $a_{i,k}^{-2} \in (0; q_k^{-1})$ .From (10) for $\gamma \in \left(0; a_{i,k}^{-2}\right)$ we have:

$$\left\| F_k\left(\frac{d}{dt};\gamma;A\right) u \right\|^2_{L_2(R_+;H)} + \left(G_k(\gamma;i)\widetilde{\psi}, \widetilde{\psi}\right)_{H^2} \geq$$

$$\geq \left\| \frac{1}{\sqrt{\beta}} \frac{d^3u}{dt^3} + \sqrt{\alpha} A^3 u \right\|^2_{L_2(R_+;H)} \times \left(1 - \gamma a^2_{i,k}\right) > 0.$$

If in the last inequality we take the solution of problem (12), (13), (14), we find that the matrix $G_k(\gamma;i)$ is positive for $\gamma \in \left[0; a_{i,k}^{-2}\right)$, and thereby the last eigenvalue $\mu_1(\gamma)$ of the matrix $G_k(\gamma;i)$ is greater than zero for any $\gamma \in \left[0; a_{i,k}^{-2}\right)$. From definition of $a_{i,k}$ we have that for $\gamma \in \left(a_{i,k}^{-2}; q_k^{-1}\right)$ there exists such a vector-function $\nu_\gamma(t) \in \overset{0}{W}{}_2^3(R_+; H; i)$ that

$$\left\| \frac{1}{\sqrt{\beta}} \frac{d^3\nu_\gamma}{dt^3} + \sqrt{\alpha} A^3 \nu_\gamma \right\|^2_{L_2(R_+;H)} < \gamma \left\| A^{3-k} \frac{d^k \nu_\gamma}{dt^k} \right\|^2_{L_2(R_+;H)}.$$

Taking the last inequality in (10) into account we find

$$\left\| F_k\left(\frac{d}{dt}; \gamma; A\right) \nu_\gamma \right\|^2_{L_2(R_+;H)} + \left(G_k(\gamma; i)\,\widetilde{\psi}_\gamma, \widetilde{\psi}_\gamma\right)_{H^2} < 0,$$

where

$$\widetilde{\psi}_\gamma = \left(A^{3-r_j-1/2} \frac{d^{r_j}\nu_\gamma(0)}{dt^{r_j}}\right)^1_{j=0}, \quad r_j \neq i.$$

By the same there exists such a vector $\widetilde{\psi}_\gamma \in H^2$ that for $\gamma \in \left(a_{i,k}^{-2}; q_k^{-1}\right)$ $\left(G_k(\gamma; i)\,\widetilde{\psi}_\gamma, \widetilde{\psi}_\gamma\right)_{H^2} < 0$, i.e. the least eigen-value $\mu_1(\gamma)$ of the matrix $G_k(\gamma; i)$ is negative for $\gamma \in \left(a_{i,k}^{-2}; q_k^{-1}\right)$. As the $\mu_1(\gamma)$ is a continuous function of the argument $\gamma \in \left[0; q_k^{-1}\right)$ we get $\mu_1\left(a_{i,k}^{-2}\right) = 0$ and this means that the equation det $G_k(\gamma; i) = 0$ has even if one root from the interval $\left(0; q_k^{-1}\right)$. Inversely if det $G_k(\gamma; i) = 0$ has a root from the interval $\left(0; q_k^{-1}\right)$ it means that for any $\gamma \in \left[0; q_k^{-1}\right)$ the matrix $G_k(\gamma; i)$ may not be positive. Therefore, by lemma 3 $a_{i,k} > q_k^{1/2}$. Denoting the least root of the equation det $G_k(\gamma; i) = 0$ by $\lambda_{i,k}$, we see that $a_{i,k}^{-2} \leq \lambda_{i,k}$ by virtue of the fact that from the proof of the lemma we get that for $\gamma \in \left[0; a_{i,k}^{-2}\right)$ the matrix $G_k(\gamma; i)$ is positive. And as det $G_k\left(a_{i,k}^{-2}; i\right) = 0$ we find that $a_{i,k}^{-2} = \lambda_{i,k}$. The lemma is proved.

Combining the last two lemmas we get the following theorem being one of the main results of the paper.

Theorem 4. *The following equality is true:*

$$a_{i,k} = \begin{cases} q_k^{1/2} & \text{for} \quad \det G_k(\gamma; i) \neq 0, \ \gamma \in \left(0; q_k^{-1}\right), \\ \lambda_{i,k}^{-1/2} & \text{in the contrary case.} \end{cases}$$

Now, considering the cases i and k we get the statement

Theorem 5. $a_{0,1} = a_{1,1} = q_1^{1/2}$; $a_{2,1} = \frac{\beta^{1/6}}{2^{1/3}\alpha^{1/3}}$; $a_{0,1} = \frac{\beta^{1/3}}{2^{1/3}\alpha^{1/6}}$; $a_{1,2} = a_{2,2} = q_2^{1/2}$.

Proof. Considering the above mentioned ones to find the numbers $a_{0,1}$ we have to solve system (3) for $k = 1$ together with equation $\det G_1(\gamma; 0) = 0$. There det $G_1(\gamma; 0) = 0$ has only the solution $\gamma = 0 \notin \left(0; q_1^{-1}\right)$. Therefore $a_{0,1} = q_1^{1/2}$ .To find the numbers $a_{1,1}$ we have to solve system (3) for $k = 1$. Together with equation $\det G_1(\gamma; 1) = 0$. Here det $G_1(\gamma; 1) = 0$ has no solution from the interval $\left(0; q_1^{-1}\right)$ i.e., $\gamma = 0 \notin \left(0; q_1^{-1}\right)$ as well. Therefore $a_{1,1} = q_1^{1/2}$. Considering system (3) for $k = 1$ with equation $\det G_1(\gamma; 2) = 0$ we have $d_{1,1}(\gamma) = \frac{2^{2/3}\alpha^{1/6}}{\beta^{1/3}}$, $d_{2,1}(\gamma) =$

$\frac{2^{1/3}\alpha^{1/3}}{\beta^{1/6}}$, consequently $\gamma = \frac{2^{2/3}\alpha^{2/3}}{\beta^{1/3}} \in \left(0; q_1^{-1}\right)$. Therefore $a_{2,1} = \frac{\beta^{1/6}}{2^{1/3}\alpha^{1/3}}$. To find the number $a_{0,2}$ we have to solve system (3) for $k = 2$ together with equation $\det G_2(\gamma; 0) = 0$. And we have $d_{1,2}(\gamma) = \frac{2^{1/3}\alpha^{1/6}}{\beta^{1/3}}$, $d_{2,2}(\gamma) = \frac{2^{2/3}\alpha^{1/3}}{\beta^{1/6}}$, consequently $\gamma = \frac{2^{2/3}\alpha^{1/3}}{\beta^{2/3}} \in \left(0; q_2^{-1}\right)$. Therefore $a_{0,2} = \frac{\beta^{1/3}}{2^{1/3}\alpha^{1/6}}$. To find the numbers $a_{1,2}$ and $a_{2,2}$ we have do solve system (3) for $k = 2$ together with, equations $\det G_2(\gamma; 1) = 0$ and $\det G_2(\gamma; 2) = 0$, respectively. And such of these equations has only the solution $\gamma = 0 \notin \left(0; q_2^{-1}\right)$. Therefore $a_{1,2} = a_{2,2} = q_2^{1/2}$. The theorem is proved.

In sequel, by obtaining conditions of regular solvability of initial boundary value problems of the form (1), (2) we need the following coercive inequalities.

Lemma 5. *For any vector-function* $u(t) \in \overset{0}{W}{}_2^3(R_+; H; i)$ *the inequalities*

$$\left\| \frac{1}{\sqrt{\beta}} \frac{d^3u}{dt^3} + \sqrt{\alpha} A^3 u \right\|_{L_2(R_+;H)} \leq \alpha^{-1/2} \left\| Q_0^{(i)} u \right\|_{L_2(R_+;H)}, \ i = 0; 1; 2, \tag{15}$$

hold.

Proof. Let $i = 0$. In the paper [4] it is established that if $u(t) \in \overset{0}{W}{}_2^3(R_+; H; 0)$, then

$$\left\| \rho^{-1/2}(t) \frac{d^3u}{dt^3} \right\|_{L_2(R_+;H)}^2 + \left\| \rho^{1/2}(t) A^3 u \right\|_{L_2(R_+;H)}^2 + \left\| A^{3/2} \frac{du(0)}{dt} \right\|_H^2 \leq \alpha^{-1} \left\| Q_0^{(0)} u \right\|_{L_2(R_+;H)}^2 .$$

Whence we have

$$\frac{1}{\beta} \left\| \frac{d^3u}{dt^3} \right\|_{L_2(R_+;H)}^2 + \alpha \left\| A^3 u \right\|_{L_2(R_+;H)}^2 + \left\| A^{3/2} \frac{du(0)}{dt} \right\|_H^2 \leq \alpha^{-1} \left\| Q_0^{(0)} u \right\|_{L_2(R_+;H)}^2 . \tag{16}$$

Applying integration by parts and considering that $u(t) \in \overset{0}{W}{}_2^3(R_+; H; 0)$ we get

$$\left\| \frac{1}{\sqrt{\beta}} \frac{d^3u}{dt^3} + \sqrt{\alpha} A^3 u \right\|_{L_2(R_+;H)}^2 =$$

$$= \frac{1}{\beta} \left\| \frac{d^3u}{dt^3} \right\|_{L_2(R_+;H)}^2 + \alpha \left\| A^3 u \right\|_{L_2(R_+;H)}^2 + \frac{\sqrt{\alpha}}{\sqrt{\beta}} \left\| A^{3/2} \frac{du(0)}{dt} \right\|_H^2 . \tag{17}$$

Comparing (16) and (17) we have

$$\left\| \frac{1}{\sqrt{\beta}} \frac{d^3u}{dt^3} + \sqrt{\alpha} A^3 u \right\|_{L_2(R_+;H)} \leq \alpha^{-1/2} \left\| Q_0^{(0)} u \right\|_{L_2(R_+;H)} .$$

The truth of inequalities (15) for $i = 1$ and $i = 2$ is established similarly. The lemma is proved.

The investigations carried out up to now allow to establish sufficient conditions of regular solvability of initial boundary problems of the form (1), (2). The following main theorem is true.

Theorem 6. *Let* $A = A^* > cE$, $c > 0$, *the operators* $A_k(t) A^{-k}$, $k = 1; 2$, *be restricted in* H *and it is fulfilled the inequality*

$$b_{i,1} \sup_t \left\| A_1(t) A^{-1} \right\|_{H \to H} + b_{i,2} \sup_t \left\| A_2(t) A^{-2} \right\|_{H \to H} < 1,$$

where the numbers $b_{i,1}$ *and* $b_{i,2}$ $(i=0;1;2)$ *are determined in the following way:*

$$b_{0,1}=\frac{\beta^{1/3}}{2^{1/3}\alpha^{2/3}};\ b_{1,1}=b_{2,1}=\alpha^{-1/2}q_2^{1/2};\ b_{0,2}=b_{1,2}=$$

$$=\alpha^{-1/2}q_1^{1/2};b_{2,2}=\frac{\beta^{1/6}}{2^{1/3}\alpha^{5/6}}.$$

Then each of initial boundary value problems (1), (2) $(i=0;1;2)$ *is regularly solvable.*

Proof. Obviously we can represent the initial boundary value problems (1), (2) in the form of the following operator equations:

$$Q_0^{(i)}u\left(t\right)+Q_1^{(i)}u\left(t\right)=f\left(t\right)\quad\left(i=0;1;2\right),$$

where

$$f\left(t\right)\in L_2\left(R_+;H\right),\ u\left(t\right)\in \overset{0}{W}{}_2^3\left(R_+;H;i\right)\quad\left(i=0;1;2\right).$$

It follows from theorem 1 that the operators $Q_0^{(i)}$ $(i=0;1;2)$ have restricted inverse operators $Q_0^{(i)^{-1}}$ $(i=0;1;2)$ that act from the space $L_2\left(R_+;H\right)$ to the space $\overset{0}{W}{}_2^3\left(R_+;H;i\right)$ $\quad(i=0;1;2)$, respectively.

After substitution $u\left(t\right)=Q_0^{(i)^{-1}}\nu\left(t\right)$ $\left(i=0;1;2\right)$, where $\nu\left(t\right)\in L_2\left(R_+;H\right)$ we get the following equations

$$\left(E+Q_1^{(i)}Q_0^{(i)^{-1}}\right)\nu\left(t\right)=f\left(t\right)\ \left(i=0;1;2\right).$$

Now we can show that by fulfilling the conditions of the theorem the norm of the operators $Q_1^{(i)}Q_0^{(i)^{-1}}$ $(i=0;1;2)$ is less than unit. Taking into account theorem 5 and lemma 5 we have

$$\left\|Q_1^{(i)}Q_0^{(i)^{-1}}\nu\right\|_{L_2(R_+;H)}=\left\|Q_1^{(i)}u\right\|_{L_2(R_+;H)}\le\left\|A_1\left(t\right)\frac{d^2u}{dt^2}\right\|_{L_2(R_+;H)}+$$

$$+\left\|A_2\left(t\right)\frac{du}{dt}\right\|_{L_2(R_+;H)}\le\sup_t\left\|A_1\left(t\right)A^{-1}\right\|_{H\to H}\left\|A\frac{d^2u}{dt^2}\right\|_{L_2(R_+;H)}+$$

$$+\sup_t\left\|A_2\left(t\right)A^{-2}\right\|_{H\to H}\left\|A^2\frac{du}{dt}\right\|_{L_2(R_+;H)}\le$$

$$\le\left(b_{i,1}\sup_t\left\|A_1\left(t\right)A^{-1}\right\|_{H\to H}+b_{i,2}\sup_t\left\|A_2\left(t\right)A^{-2}\right\|_{H\to H}\right)\left\|Q_0^{(i)}u\right\|_{L_2(R_+;H)}=$$

$$=\left(b_{i,1}\sup_t\left\|A_1\left(t\right)A^{-1}\right\|_{H\to H}+b_{i,2}\sup_t\left\|A_2\left(t\right)A^{-2}\right\|_{H\to H}\right)\left\|\nu\right\|_{L_2(R_+;H)}.$$

As a result we have

$$\left\|Q_1^{(i)}Q_0^{(i)^{-1}}\right\|_{L_2(R_+;H)\to L_2(R_+;H)}\le$$

$$\le b_{i,1}\sup_t\left\|A_1\left(t\right)A^{-1}\right\|_{H\to H}+b_{i,2}\sup_t\left\|A_2\left(t\right)A^{-2}\right\|_{H\to H}<1.$$

When these inequalities are true the operators $E+Q_1^{(i)}Q_0^{(i)^{-1}}$ $(i=0;1;2)$ have the inverse in the space $L_2(R_+;H)$ and we can determine $u(t)$ by the following formulae:

$$u(t)=Q_0^{(i)^{-1}}\left(E+Q_1^{(i)}Q_0^{(i)^{-1}}\right)^{-1}f(t),$$

moreover

$$\|u\|_{W_2^3(R_+;H)}\le\left\|Q_0^{(i)^{-1}}\right\|_{L_2(R_+;H)\to W_2^3(R_+;H)}\times$$

$$\times\left\|\left(E+Q_1^{(i)}Q_0^{(i)^{-1}}\right)^{-1}\right\|_{L_2(R_+;H)\to L_2(R_+;H)}\|f\|_{L_2(R_+;H)}\le const\,\|f\|_{L_2(R_+;H)}.$$

The theorem is proved.

Remark. Notice that the results of the paper essentially improve the corresponding results of the paper [4], refine the results of the paper [5] and of these for $\rho(t)\equiv 1,\ t\in R_+$ we can obtain some results of [7].

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