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BEHAVIOR OF SOLUTION OF THE SECOND ORDER QUASILINEAR PARABOLIC EQUATION IN UNBOUNDED DOMAIN

Abstract

In this paper we consider the second order quasilinear parabolic equations of nondivergent form in an unbounded domain whose supplement contains rotation funnel or conical domain. For the generalized solutions of the first boundary-value problem, the Phragmen-Lindelöf theorem is proved.

Introduction. Let E_{n+1} be an $(n+1)$ dimensional Euclidean space of the points $(t,x) = (t, x_1, ...x_n)$, $D = \Omega \times (0,t) \subset E_{n+1}$ be an unbounded domain, $\Omega \subset E_n$, ∂D and $\Gamma(D)$ be boundary and parabolic boundary of the domain D, respectively, $C_R = C^{t_1,t_2}_{x^0;R}$ be a cylinder $\{t_1 < t < t_2,\ |x - x^0| < R\}$, $A_{4^m} = \{(\tau,\xi)|\ F_{s,\beta}(t-\tau, x-\xi) \geq 4^{-ms}\}$, $A_{4^m,\nu_m} = A_{4^m} \cap \{t \leq -\nu_m\}$, $\tilde{A}_{4^m,\nu_m} = A_{4^m,\nu_m} \cap \{t \leq -\nu_m\}$, $m = 1, 2, ...$; $\gamma_{s,\beta}(E)$ be parabolic (s,β)-capacity of the set $E \subset E_{n+1}$ generated by the kernel

$$F_{s,\beta}(t,x) = \begin{cases} t^{-s}\exp\left(-\dfrac{|x|^2}{4\beta t}\right), & \text{for } t > 0, \\ 0, & \text{for } t \leq 0. \end{cases}.$$

Let $V_f = \left\{ x : \left(\sum\limits_{i=1}^{n-1} x_i^2\right)^{1/2} < f(x_n),\ 0 < x_n < \infty \right\}$ be a rotation funnel , where $f(t)$ is a positive, continuous and nondecreasing function of t such that the function $f(t)/t$ is bounded, and doesn't increase with respect to t, $t \in (0,\infty)$ or conical domain.

In the domain Ω whose supplement contains funnel rotation or conical domain V_f we consider the following first boundary-value problem

$$Lu = \sum_{i,j=1}^{n} a_{ij}(t,x,u,u_x)\, u_{ij} + b(t,x,u,u_x) - u_t = 0, \tag{1}$$

$$u|_{\Gamma(D)} = 0, \tag{2}$$

where $u_t = \dfrac{\partial u}{\partial t}$, $u_x = \left(\dfrac{\partial u}{\partial x_1}, ..., \dfrac{\partial u}{\partial x_n}\right)$, $u_i = \dfrac{\partial u}{\partial x_i}$, $u_{ij} = \dfrac{\partial^2 u}{\partial x_i \partial x_j}$, $i,j = 1,...,n$, $\|a_{ij}(t,x,z,\upsilon)\|$ is a real symmetrical matrix with elements measurable in D. Let $u(t,x)$ be a solution of equation (1) and for all $(t,x) \in D$, $z \in E_1^+$, $\upsilon \in E_n$, $\xi \in E_n$ the following conditions be fulfilled

$$\gamma |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(t,x,z,\upsilon)\, \xi_i \xi_j \leq \gamma^{-1} |\xi|^2, \tag{3}$$

$A_{ij}(t,x) = a_{ij}(t,x,u,u_x)$ and $|A_{ij}(t,x) - A_{ij}(\tau,y)| \le C_1\varphi\left(|x| + \sqrt{t}\right)$; $C_1-const$,

$$\varphi(z) = |\ln|\ln z||^{-1}, \ A_{ij}(0,0) = \delta_{ij} = \begin{cases} 1, \text{ if } i = j, \\ 0, \text{ if } i \neq j, \end{cases} \tag{4}$$

$$b(t,x,z,v) \le b_0 |v|^2 + (B(t,x), v) + c(t,x) z. \tag{5}$$

Here $E_1^+ = \{z : z \in E_1, z \ge 0\}$, $\gamma \in (0,1]$, $b_0 \ge 0$ are constants,
$B(t,x) = (b_1(t,x), ..., b_n(t,x))$ and $b_1(t,x), ..., b_n(t,x)$, $c(t,x) \in L_\infty^{loc}(D)$.

We'll assume that

$$\left(B(t,x), \left(x - x^0\right)\right) \le 0, \tag{6}$$

$$c(t,x) \le 0, \tag{7}$$

for all $(t,x) \in D$, $x^0 \in \overrightarrow{l}$, where $\overrightarrow{l}$ is some ray starting from the origin of coordinate.

The goal of our paper is to obtain the Phragmen-Lindelöf type theorem for nonnegative solutions of boundary-value problem (1)-(2). Under solution of the mentioned problem we'll understand its generalized solution from the space $W_p^{(2,1)}(D)$. Here we denote by $W_p^{(2,1)}(D)$ a Banach space of the functions $u(t,x)$, where norm is defined as

$$\|u\|_{W_p^{2,1}(D)} = \left(\int_D \left(|u|^p + \sum_{i=1}^n |u_i|^p + \sum_{i,j=1}^n |u_{ij}|^p + |u_t|^p \right) dtdx \right)^{1/p},$$

where $p \in (1,\infty)$.

Everywhere in the sequel, notation $C(...)$ denotes that the positive constant C depends only on contents of parentheses.

For the second order elliptic equations of nondivergent form such type results have been obtained in the papers [1-5]. Concerning divergent elliptic equations we refer to the papers [6-7]. In the papers [8-9] the behavior of solutions of parabolic equations have been considered. We also note the papers [10-11] wherein the Phragmen-Lindelöf type theorems for quasilinear elliptic equations have been obtained.

Lemma 1. *Let $u(t,x)$ be a fixed nonnegative solution of equation (1) and $A_{ij}(t,x) = a_{ij}(t,x,u(t,x),u_x(t,x))$, $w(t,x) = e^{\tau u(t,x)} - 1$, where $\tau = \frac{b_0}{\gamma}$. Then if conditions (3),(5),(6),(7) are satisfied, then*

$$L_1 w = \sum_{i,j=1}^n A_{ij}(t,x) w_{ij} + \sum_{i=1}^n b_i(t,x) w_i - w_t \ge 0,$$

for $(t,x) \in D$.

Proof. It's clear that $w_i = \tau e^{\tau u} u_i$, $w_{ij} = \tau^2 e^{\tau u} u_i u_j + \tau e^{\tau u} u_{ij}$. Allowing for conditions (3), (5), (6), (7), we have

$$\sum_{i,j=1}^n A_{ij}(t,x) w_{ij} = \tau^2 e^{\tau u} \sum_{i,j=1}^n a_{ij}(t,x,u,u_x) u_i u_j + \tau e^{\tau u} \sum_{i,j=1}^n a_{ij}(t,x,u,u_x) u_{ij} \ge$$

$$\geq \tau^2 e^{\tau u}\gamma\left|u_x\right|^2-\tau e^{\tau u}b\left(t,x,u,u_x\right)+\tau e^{\tau u}u_t\geq \tau^2 e^{\tau u}\gamma\left|u_x\right|^2-$$

$$-\tau e^{\tau u}\left[\left(B\left(t,x\right),u_x\right)+b_0\left|u_x\right|^2+c\left(t,x\right)u\right]+\tau e^{\tau u}u_t=$$

$$=\tau e^{\tau u}\left|u_x\right|^2\left(\tau\gamma-b_0\right)-\tau e^{\tau u}\sum_{i=1}^{n}b_i\left(t,x\right)u_i-\tau e^{\tau u}c\left(t,x\right)u+\tau e^{\tau u}u_t.$$

Since $\tau=\dfrac{b_0}{\gamma}$, then it follows from the last inequality that

$$\sum_{i,j=1}^{n}A_{ij}\left(t,x\right)w_{ij}-w_t\geq-\tau e^{\tau u}\sum_{i=1}^{n}b_i\left(t,x\right)u_i+\tau e^{\tau u}u_t=-\sum_{i=1}^{n}b_i\left(t,x\right)w_i+w_t,$$

or

$$L_1w=\sum_{i,j=1}^{n}A_{ij}\left(t,x\right)w_{ij}+\sum_{i=1}^{n}b_i\left(t,x\right)w_i-w_t\geq 0.$$

The lemma is proved.

Lemma 2. *Let with respect to the coefficients of the operator L_1 conditions (3)-(7) be satisfied and there exist the constants $R_1=R_1\left(n,C_1\right),\ C_2=C_2\left(n,C_1\right)$ and $C_3=C_3\left(n,C_1\right)$ such that at $R\leq R_1,\ s=\dfrac{n}{2}+C_2\varphi\left(R\right),\ \beta=1-C_3\varphi\left(R\right),$ then for $\left(x,t\right)\in D\cap C_R,\ \left(y,\tau\right)\in CD\cap C_R$*

$$L_1F_{s,\beta}\left(t-\tau,x-y\right)\geq 0 \tag{8}$$

holds.

Proof. For $t>\tau$

$$L_1F_{s,\beta}=F_{s,\beta}\left\{\frac{1}{4\beta^2\left(t-\tau\right)^2}\sum_{i,j=1}^{n}a_{ij}\left(t,x,u,u_x\right)\left(x_i-y_i\right)\left(x_j-y_j\right)-\right.$$

$$\left.-\frac{1}{2\beta\left(t-\tau\right)}\sum_{i,j=1}^{n}a_{ii}\left(t,x,u,u_x\right)+\frac{s}{t-\tau}-\frac{\left|x-y\right|^2}{4\beta\left(t-\tau\right)^2}-\left(B\left(x\right),x-x^0\right)\right\}\geq$$

$$\geq F_{s,\beta}\left\{\frac{1}{4\beta^2\left(t-\tau\right)^2}\sum_{i,j=1}^{n}\left[a_{ij}\left(t,x,u,u_x\right)-a_{ij}\left(0,0,u,u_x\right)\right]\left(x_i-y_i\right)\left(x_j-y_j\right)+\right.$$

$$+\frac{\left|x-y\right|^2}{4\beta^2\left(t-\tau\right)^2}-\frac{1}{2\beta\left(t-\tau\right)}\times$$

$$\left.\times\sum_{i=1}^{n}\left[a_{ii}\left(t,x,u,u_x\right)-a_{ii}\left(0,0,u,u_x\right)\right]-\frac{n}{2\beta\left(t-\tau\right)}+\frac{s}{t-\tau}-\frac{\left|x-y\right|^2}{4\beta\left(t-\tau\right)^2}\right\}\geq$$

$$\geq F_{s,\beta}\left\{\frac{\left|x-y\right|^2}{4\beta\left(t-\tau\right)^2}-\left[\frac{1-nC_1\varphi\left(R\right)}{\beta}-1\right]+\frac{1}{t-\tau}\left[s-\frac{n+nC_1\varphi\left(2R\right)}{2\beta}\right]\right\}. \tag{9}$$

We choose R_1 so small that for $R \leq R_1$ $\varphi(2R) \leq 2\varphi(R)$, $1 - 2nC_1\varphi(R) > \frac{1}{2}$. Then assuming $\beta = 1 - 2nC_1\varphi(R)$, $s \geq \frac{n + 2nC_1\varphi(R)}{2\beta}$ from (9) we obtain the required estimate. Now it sufficies to fix $C_2 = 2nC_1(1+n)$, $C_3 = 2nC_1$, and the lemma is proved.

In [8] the analogous results have been obtained for linear equations.

Let $s > 0$, $\beta > 0$ and $a > 5$ be given.

Consider the cylinders

$$C_m = C^{-4^m,0}_{0,2a\sqrt{\frac{\beta s}{a}4^m}}, \quad m = 1, 2, \ldots\ .$$

Denote by S_m the lateral surface of the cylinder C_m. Here and further we'll assume that the following conditions are satisfied

$$\frac{4^{m+1}}{\nu_m} \ln \frac{4^m}{\nu_m} \overset{m\to\infty}{\longrightarrow} 0, \quad 4^{m+1} < \nu_m < 4^m, \ m = 1, 2, \ldots\ . \tag{10}$$

Let (τ, ξ) be an arbitrary point belonging to $E_m = A_{l_m,\nu_m} \backslash D$.

Definition. *Let $s > 0$ and $\beta > 0$ be given, and E be a Borel set in E_{n+1}. The measure μ is called feasible on E if*

$$\int\limits_E F_{s,\beta}(t-\tau, x-y)\, d\mu(\tau, y) \leq 1 \ \text{ for } \ (t,x) \notin \overline{E}.$$

The number

$$\gamma_{s,\beta}(E) = \sup \mu(E),$$

where the exact upper bound is taken on all feasible measures, is called (s,β)-capacity of the set E.

Lemma 3. *Let $s > 0, \beta > 0, a > 5$ be given. Let $D \subset E_{n+1}$ be a domain with the proper boundary Γ and $C_{m+1} \cap D = \oslash$. Let Γ_m be such part of proper boundary D which is strictly in C_m. Let the operator L_1 be determined in D, for this operator the conditions*

$$\beta \leq \gamma, \ s \geq \frac{\gamma^{-1}}{2\beta} \tag{11}$$

be satisfied.

Let $w(t,x)$ be a subparabolic function for this operator continuous in $\overline{D}$, positive in D and vanishing in Γ_m. Then if condition (10) is satisfied, then

$$\sup_{D\cap C_m} w \geq (1+\eta) \sup_{D\cap C_{m+1}} w. \tag{12}$$

Proof. Let's fix m and give arbitrary $\varepsilon > 0$ and let measure μ be defined on E_m such that

$$U(t,x) = \int\limits_{E_m} F_{s,\beta}(t-\tau, x-y)\, d\mu(\tau, y) \leq 1$$

out of $\overline{E}_m$ and

$$\mu(E_m) > \gamma_{s,\beta}(E_m) - \varepsilon.$$

Denote $\sup\limits_{D\cap C_m} w = M$ and consider the auxiliary function

$$v(t,x) = M\left[1 - U(t,x) + \sup_{S_m} U(t,x)\right].$$

By equality (12) the function U is subparabolic, and therefore v is also subparabolic. Everywhere on proper boundary of the domain D we have

$$u(t',x') \leq \lim_{(t,x)\to(t',x')} v(t,x).$$

Actually the proper boundary contains $\overline{\Gamma}_m$ and points on S_m, and in lower base of C_m.

Since $U \leq 1$ out of $\overline{E}_m$, then

$$\lim_{(t,x)\to(t',x')\in\overline{\Gamma}_m} v(t,x) \geq 0,$$

when $u|_{\overline{\Gamma}_m} = 0$.

By the maximum principle $u \leq v$ in D

$$\sup_{D\cap C_{m+1}} w \leq \sup_{D\cap C_{m+1}} v \leq M\left[1 - \left(\inf_{D\cap C_{m+1}} G(x) - \sup_{S_m} G(x)\right)\right].$$

Since [9]

$$\sup_{(t,x)\in S_m} U \leq 4^{-ms} e^{-\frac{s(a-1)^2}{e}} \mu(E_m),$$

$$\inf_{(t,x)\in C_{m+1}} U \leq 4^{-ms} e^{-\frac{s}{e}} \mu(E_m),$$

we obtain

$$\sup_{D\cap C_{m+1}} w \leq M\left[1 - 4^{-ms} e^{-\frac{s}{e}} \mu(E_m) + 4^{-ms} e^{-\frac{s(a-1)^2}{e}} \mu(E_m)\right] \leq$$

$$\leq M\left[1 - 4^{-ms}\left(e^{-\frac{s}{e}} - e^{-\frac{s(a-1)^2}{e}}\right)\left(\gamma_{s,\beta}(E_m) - \varepsilon\right)\right].$$

Thus we obtain

$$\sup_{D\cap C_R} w \geq \left(1 + \eta_1 4^{-ms} \gamma_{s,\beta}(E_m)\right) \sup_{D\cap C_{R+1}} w,$$

where $\eta_1 = e^{-\frac{s}{e}} - e^{-\frac{s(a-1)^2}{e}}$.

According to [2]

$$\sup_{D\cap C_m} w \geq \left(1 + \eta_1 4^{-ms} \gamma_{s,\beta}(E_m)\right) \sup_{D\cap C_{m+1}} w =$$

$$= \left(1 + \eta_1 \frac{4^{-ms}}{2^s} \gamma_{s,\beta}(E_m)\right) \sup_{D\cap C_{m+1}} w =$$

$$= \left(1 + \eta_1 4^{-ms}\right) \sup_{D\cap C_{m+1}} w = (1+\eta) \sup_{D\cap C_{m+1}} w,$$

where $\eta = \eta_1 4^{-ms}$ is a constant.

The lemma is proved.

Theorem. *Let the nonnegative solution $u(t,x)$ of boundary-value problem (1),(2) be defined in the domain D, moreover with respect to conditions of the operator L conditions (3)-(7) be satisfied. Then either $u(t,x) \equiv 0$ in D or*

$$\varliminf_{r\to\infty} \frac{M(r)}{r^\delta} > 0, \tag{13}$$

where $M(r) = \sup_{D\cap C_r} |u|$, $\delta = \delta(\gamma, n, l)$ is a positive constant.

Proof. Let $u(t,x) \neq 0$. Then there exists a point $y \in D$, in which $u(t,y) = a \neq$ $\neq 0$. Assume that $a > 0$. Denote by D^+ a set $\{(t,x) : (t,x) \in D,\ u(t,x) > 0\}$, and by D' connected component of D^+ containing the point y. By the maximum principle D' is an unbounded domain.

Let m_0 be the least natural number for which $y \in C_{4^{m_0}}$. We fix an arbitrary sufficiently large number r. Denote by m a natural number satisfying the inequalities

$$4^m \le r < 4^{m+1}. \tag{14}$$

It follows from (14) that

$$m > \frac{\ln r}{\ln 4} - 1.$$

We'll take r so large that $m > m_0$ and $\frac{\ln r}{\ln 4} - 1 \ge \frac{\ln r}{2\ln 4}$. Then

$$m > \frac{\ln r}{2\ln 4}. \tag{15}$$

Let $M(r) = \sup_{D\cap C_R} u$. Applying subsequentially inequality (12) and allowing for (14)-(15), we obtain

$$M(r) \ge M(4^m) \ge (1+\eta) M\left(4^{m-1}\right) \ge ... \ge (1+\eta)^{m-m_0} M(4^{m_0}) \ge$$

$$\ge (1+\eta)^m \frac{a}{(1+\eta)^{m_0}} \ge (1+\eta)^{\frac{\ln r}{2\ln 4}} \frac{a}{(1+\eta)^{m_0}} = \eta_2^{\ln r} a_1, \qquad (16),$$

where $\eta_2 = (1+\eta)^{\frac{1}{2\ln 4}}$, $a_1 = \frac{a}{(1+\eta)^{m_0}}$. From (16) we conclude that

$$M(r) \ge a_1 r^\delta,$$

where $\delta = \ln \eta_2$.

If $a < 0$, then we multiply the solution $u(x)$ by -1, and we lead analogous reasoning. Thus we showed that for sufficiently large r

$$M(r) \geq a_1 r^{\delta},$$

where a is a positive constant independent of r. Hence the required limit equality (13) follows. The theorem is proved.

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