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ON APPLICATION OF DIFFERENTIAL TRANSFORMATIONS TO N-DIMENSIONAL PROBLEMS FOR DIFFERENTIAL EQUATIONS ON A GRID

Abstract

The differential transformation is one of numeric methods for solving of differential equations. Advantage of this method is transformation of the differential equation into algebraic. Application of differential transformation method to various problems gives more effective errors. In this investigation we use this method for numerical solution on n-dimensional grid of boundary value problems for evolution differential equation. Researches have shown the efficiency of application of a method to a considered class of problems.

1. Introduction

The majority of the problem of physics and engineering fall naturally into one of three physical categories: equilibrium problems, eigenvalue problems and propagation problems [1-3]. Each of this kind of problems, from mathematical point of view, is described by application of some linear and/or nonlinear differential equations. A variety of methods are applied to find the solution of considered problems [1-8]. However, in the most case, these differential equations may be too complicated to solve analytically. In this case, and at desire to get mare practical result, it is used some numerical method to find a solution of considered problem.

Integral transform such as Laplace and Fourier transform are commonly used to solve differential equations in physics and engineering problems. The decided superiority of these transforms lies in their ability to transform differential equations into algebraic ones. However, using the integral transform in nonlinear problems leads to increasing of complexity. Not withstanding what modern computers are capable to solve complex problems, there was a requirement for less complex approaches. Such as a numerical method for solving differential equations was first proposed in Zhou [5] and its main application therein is solve both linear and nonlinear initial value problems in electric circuit analysis. This method, called the differential transformation (DT) method, serves for constructs an analytical solution in the form of a polynomial.

Different aspects of DT method have been analyzed and developed in many researchers [6-14]. Initial value problems were solved in [6], the application of DT to transient adjective dispersive transport equation were investigate in [7], the DT approximation for the system of ordinary differential equations were studied in [8] and ets. In the works [9,10] two- and n-dimensional DT methods were developed and their applications to differential equations were considered. The DT method to find a numeric solution of differential equation in one- dimensional grid point were

[1]This research was partially supported by TUBITAK PC-B Program (Turkey).

considered in [11] and using the DT and central finite difference methods simultaneously, were studied in [7].

In this work more general approach to the notion of differential transformation method is considered. One- and n-dimensional DT are investigated. Using the DT and finite difference methods for numeric solution of differential equations on n-dimensional grid points is studied.

2. Preliminaries

In this section we consider some notions and descriptions corresponding to the differential transformation methods.

2.1. One dimensional case

Let $I = (a, b) \subset R$ is an arbitrary real interval and $u : I \to R$ is an analytic function in I.

Using the definition of analytic function [15], for an arbitrary $x_0 \in I$ we can write the power series representation of function $u(x)$ as follow

$$u(x) = \sum_{k=0}^{\infty} C_k (x - x_0)^k \tag{1}$$

This series convergences to the function $u(x)$in some neighborhood of point x_0.

Referring on properties of power series and analytic function we can affirm that the function $u(x)$ is infinity continuously differentiable and

$$C_k = \frac{u^{(k)}(x_0)}{k!}$$

so

$$u(x) = \sum_{k=0}^{\infty} \frac{u^{(k)}(x_0)}{k!} (x - x_0)^k \tag{2}$$

Introducing a denotation

$$U(x_0, k) = u^{(k)}(x_0) = \left[u^{(k)}(x) \right]\Big|_{x=x_0} \tag{3}$$

the expression (2) can be rewritten in the following form

$$u(x) = \sum_{k=0}^{\infty} \frac{(x - x_0)^k}{k!} U(x_0, k) \tag{4}$$

Again, referring on properties of analytic functions we can consider the biunivocal mapping

$$u(x) \leftrightarrow \{U(x_0, k)\}_{k=0}^{\infty}$$

about point x_0.

Analyticity of the function $u(x)$ in the I allows us to differentiate both side of (4). In this way we get

$$u'(x) = \left[\sum_{k=0}^{\infty} \frac{(x-x_0)^k}{k!} U(x_0,k) \right]' =$$
$$= \sum_{k=0}^{\infty} \frac{k(x-x_0)^{k-1}}{k!} U(x_0,k) = \sum_{k=0}^{\infty} \frac{(x-x_0)^k}{k!} U(x_0,k+1) \tag{5}$$

From (5) we can consider the biunivocal mapping

$$u'(x) \leftrightarrow \{U(x_0,k+1)\}_{k=0}^{\infty} \tag{6}$$

about point x_0.

Similar reasoning allows us to get the following relation

$$u^{(m)}(x) \leftrightarrow \{U(x_0,k+m)\}_{k=0}^{\infty} \tag{7}$$

Taking into account considered above we introduce the following

Definitions 1. *If* $u(x)$ *is analytic in the domain* I, *then the sequence* $\{U(x_0,k)\}_{k=0}^{\infty}$ *will be called the spectrum of function* $u(x)$ *at the point* $x = x_0$.

Property 1. *If* $u(x)$ *is analytic in the domain* I, *then the spectrum of* m*-th order* $(m \geq 0)$ *derivatives of the function* $u(x)$ *at the point* $x = x_0$ *is the sequence* $\{U(x_0,k+m)\}_{k=0}^{\infty}$.

Property 2. *If* $u(x)$ *is analytic function in the domain* I *and* α *is an arbitrary number from* R^1, *then the spectrum of the function* $\alpha\, u(x)$ *at the point* $x = x_0$ *is the sequence* $\{\alpha\, U(x_0,k\}_{k=0}^{\infty}$.

Property 3. *If* $u(x)$ *and* $v(x)$ *are analytic functions in the domain* I, *then the spectrum of* m*-th order* $(m \geq 0)$ *derivatives of the function* $u(x) \pm v(x)$ *at the point* $x = x_0$ *is the sequence* $\{U(x_0,k+m) \pm V(x_0,k+m)\}_{k=0}^{\infty}$.

Property 4. *If* $u(x)$ *and* $v(x)$ *are analytic functions in the domain* I, *then the spectrum of the function* $u(x)v(x)$ *at the point* $x = x_0$ *is the sequence*

$$\left\{ \sum_{\lambda=0}^{k} U(x_0,\,\lambda\,)\,V(x_0,k-\lambda) \right\}_{k=0}^{\infty}.$$

Similarly as in works [6-14] we can consider the weight differential transformations of analytic functions and get the same table of correspondence of functions and their spectrums (See Table 1).

In the last denotations the fundamental mathematical operations reformed by n-dimensional differential transformation can readily be obtained and are listed in Table 1 ([6, 9]).

Table 1. Operations for differential transformation

Original function	Transformed function
$z(x) = u(x) \pm v(x)$	$Z(x,k) = U(x,k) \pm V(x,k)$
$z(x) = \alpha u(x) \quad \alpha = cons\tan s$	$Z(x,k) = \alpha U(x,k)$
$z(x) = \dfrac{du(x)}{dx}$	$Z(x,k) = (k+1)U(x,k+1)$
$z(x) = \dfrac{d^m u(x)}{dx^m}$	$Z(x,k) = (k+1)(k+2)...(k+n)U(x,k+n)$
$z(x) = u(x)v(x)$	$Z(x,k) = U(x,k)\otimes V(x,k) = \sum\limits_{l=0}^{k} U(x,l)V(x,k-l)$
$z(x) = 1$	$Z(x,k) = \delta(x,k)$
$z(x) = x^n$	$Z(x,k) = \delta(x,k-n) \quad ,\delta(x,k-n) = \begin{cases} 1 & if\ k=n \\ 0 & if\ k \neq n \end{cases}$
$z(x) = \exp(\lambda x)$	$Z(x,k) = \frac{\lambda^k}{k!}$

At the end of this section we note that use of the differential transformation allows to represent the analytic functions $u(x)$ by finite-term Taylor series plus a remainder, as

$$\begin{aligned} u(x) &= \tfrac{1}{q(x)} \sum_{k=0}^{N} \left(\tfrac{U(x_0,k)}{M(k)} \tfrac{(x-x^0)^k}{k!} \right) + R_{\lambda+1}(x) \\ &= \sum_{k=0}^{N} \left(U(x_0,\ k) \tfrac{(x-x^0)^k}{k!} \right) + R_{\lambda+1}(x) \end{aligned} \tag{8}$$

and this kind of representation will help to transform the differential equations in the domain of interest to an algebraic equation.

3. Applied the differential transformation to the PDEs on the grid.

3.1. The objective of this section is to find the solution on grid points of the following initial value problem

$$u_{x_1}(x) = f(x,\ u(x)) \quad x \in (0,\ 1)^n \tag{9}$$

$$u(0,\ x') = u_0(x') \quad x' = (x_1, ..., x_n) \in (0,\ 1)^{n-1} \tag{10}$$

by using the differential transformation method.

Let us N_i be an arbitrary natural number and $h_i = \frac{1}{N_i}$. Then for an each coordinate x_i we denote $x_i^{\lambda} = \lambda h_i$, where $\lambda = 0, 1, ..., N_i$. The domain of interest $(0,\ 1)^n$is divided into $\prod\limits_{i-1}^{n} N_i$ sub-domains. Our main aim is the construction of solution of the problem (9)-(10) in the each sub-domain. Thus, we will give a solution of considered problem in the whole domain of interest.

Applying the differential transformation with finite-term spectrum to the problem (9)-(10) in the arbitrary point form $(0,\ 1)^n$ we get

$$(k+1)U(x_0,\ k+1) = F(x_0, U(x_0,\ k)) \tag{9'}$$

$$U((0,\ x_0^{'}), 0) = U_0(x_0^{'}) \tag{10'}$$

where $x_0^{'} = \left(x_2^0, ..., x_n^0\right)$.

At the one of the sub-domain of first layer the solution can be approximately represented in term of its m-th order Taylor polynomial about $\left(x_1^0, x_2^{i_2}, ..., x_n^{i_n}\right)$ where $(i_2, ..., i_n) \in \prod\limits_{i=2}^{n} [0,\ N_i]$.

$$u(x) = \sum_{\lambda=0}^{m} U\left(\left(x_1^0, x_2^{i_2}, ..., x_n^{i_n}\right),\ \lambda\right)\ (x_1)^{\lambda} \tag{11}$$

Coefficients in the right hand side of the equality (16) are founded by using relations (9′) and (10′).

After the construction of solutions for all sub-domains of first layer we can pass to consideration of solutions in the sub-domains of second layer. For this purpose we need the initial value for second layer.

Once the Taylor polynomial for the each sub-domain of first layer is obtained $u\left(x_1^1, x_2^{i_2}, ..., x_n^{i_n}\right)$ can be evaluated as

$$u\left(x_1^1, x_2^{i_2}, ..., x_n^{i_n}\right) \approx \sum_{\lambda=0}^{m} U\left(\left(x_1^0, x_2^{i_2}, ..., x_n^{i_n}\right),\ \lambda\right)(x_1^1)^{\lambda}$$
$$= \sum_{\lambda=0}^{m} U\left(\left(x_1^0, x_2^{i_2}, ..., x_n^{i_n}\right),\ \lambda\right) h_1^{\lambda}$$

Using last values we can represent the function $u(x)$ in the each sub-domain of second layer in the following way:

$$u\,(x) = \sum_{\lambda=0}^{m} U\left(\left(x_1^1, x_2^{i_2}, ..., x_n^{i_n}\right),\ \lambda\right)\ \left(x_1 - x_1^1\right)^{\lambda}$$

In a similar manner we can give the representation of the function $u(x)$ on the each layer, up to the end.

Remark. Similarly as above, high order initial value problem can be considered.

3.2. This sub-section is devoted to the investigation of application of the differential transformation method for the initial boundary value problem and its numerical solution on the grid. We will show that some type of problems can be transformed to problems such as considered in sub-section 2.1.

Consider the following differential equations

$$\frac{\partial^m u(t,\ x)}{\partial t^m} = f\left(t,\ x,\ u,\ \frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n}\right) \qquad (t,\ x) \in (0,\ 1)^{n+1} \tag{12}$$

with initial values

$$\left.\frac{\partial^{\lambda} u(t,\, x)}{\partial t^{l}}\right|_{t=0} = u_{\lambda}(x) \qquad \lambda = 0,\, m-1 \;\; x \in (0,1)^{\mathrm{n}} \tag{13}$$

and corresponding boundary conditions

$$u|_{\Gamma} = \varphi\,(t,\, x) \qquad\qquad x \in (0,1)^{\mathrm{n}} \tag{14}$$

where Γ is a boundary of the considered domain at points $t=0$ and $t=1$.

To use the reasoning of sub-section 2.1 in this situation we need to represent the right hand side of (12) in the form depending on variables $(t,\, x)$ and solution u only. For achievement of this purpose it is possible to use one of the finite difference schemes of the representation of derivatives [1-3].

It is not difficult to see that considered strategy is applicable in the case when the right hand side of (12) consists of derivatives of the high order also (of course, the corresponding boundary value must be considered instead of (14)).

4. Numerical illustrations

The basic purpose of this section is to show on examples results submitted in previous sections.

Example 1. Consider the following differential equation

$$u_t(t,x) = -u(t,x) + x^2 + t + 1 \qquad\qquad (t,x) \in [0,1] \times [0,1] \tag{15}$$

with initial value

$$u(0,\, x) = x^2 \qquad x \in [0,1] \tag{16}$$

and boundary conditions

$$u(t,\, 0) = t \qquad t \in [0,1] \tag{17}$$

$$u(t,\, 1) = t + 1 \qquad t \in [0,1] \tag{18}$$

Let's take any natural number N_1 and N_2, and divide the considered domain $[0,1] \times [0,1]$ into $N_1 \times N_2$ sub-domains. The grid nodes of our domain we denote as $P_{ij} = (t_i,\, x_j) = (ih_t,\, jh_x)$, $\quad h_t = \frac{1}{N_1}$, $h_x = \frac{1}{N_2}$.

Using differential transformation from previous sections, we get following relations:

$$(k+1)u((t_i,x_j),k+1) = -u((t_i,x_j),k) + \varphi\,((t_i,x_j),k) + \psi\,((t_i,x_j),k) + h(k) \tag{15$'$}$$

$$u((0,x_j),0) = x_j^2 \delta((0,x_j),0) \tag{16$'$}$$

$$u((t_i,0),\, k) = \delta((t_i,0),k-1) \tag{17$'$}$$

$$u((t_i,1),\, k) = \delta((t_i,1),k-1) + \delta((t,1),k) \tag{18$'$}$$

where

$$\varphi(\,(t_i,x_j),k) = \begin{cases} x_j^2 & k=0 \\ 0 & k \neq 0 \end{cases} \quad , \quad \psi(\,(t_i,x_j),k) = \begin{cases} 1 & k=1 \\ 0 & otherwise \end{cases} ,$$

$$h\,(k) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}, \quad \delta(\,(t_i, x_j), k) = \begin{cases} 1 & k = 1 \\ 0 & otherwise \end{cases}$$

and $i = 0, 1, 2, ..., N_1$, $j = 0, 1, 2, ..., N_2$.

Remark. Note, that we have applied the differential transforms on variable t only.

For calculation of values of the approximated solution in grid nodes we use the following scheme:

1. Taking into acount the initial value condition (relation$(16')$) we calculate values points $P_{0j} = (0,\, x_j),\ \ j = 0, 1, 2, ..., N_2$.
2. Using the boundary conditions $(17')$and $(18')$we calculate values of solution in points $P_{i0} = (t_i, 0)$ *and* $P_{iN_2} = (t_i, x_{N_2})$, $i = 0, 1, 2, ..., N_2$.
3. Using the finite term representation of the solution in the neighbourhood of the point $P_{10} = (t_1, 0)$and taking into account relations $(15')$ and $(17')$ we calculate the approximated value of the solution in the point $P_{11} = (t_1, x_1)$.
4. Repeating step 3 for calculation of approximated value of the solution in the next points $P_{1j} = (t_1,\, x_j),\ \ j = 2, ..., N_2$ using instead of initial value of solution the value calculated in the previous step.
5. After finish of the calculation at the considered level (t_i) we pass to following level t_{i+1} and we repeat operations of steps 3 and 4.

Results of calculations for the considered problem (15)-(18) are represented in Table 2.

It is not difficult to see, that the function $u(t,\, x) = x^2 + t$ is the analytic solution of considered problem.

Table 2. Errors table ($N_1 = 5$,$N_2 = 5$, $M = 5$)

x, t	x_0	x_1	x_2	x_3	x_4
t_0	0	0	0	0	0
t_1	0	1,00E-18	0	1,00E-17	0
t_2	-9,90E-15	-9,85E-15	-9,69E-15	-9,58E-15	-9,64E-15
t_3	-2,97E-14	-2,76E-14	-2,45E-14	-2,54E-14	-2,37E-14
t_4	0	0	0	0	0

Comparisons of the approximated values and corresponding values calculated for the analytic solution show that the error of the method for this example has the order $10^{-14} - 10^{-17}$.

Example 2. This example devoted to the consideration of the following differential equation

$$u_t(t, x) = u_x(t, x) + (1 - t)e^x \qquad (t, x) \in [0, 1] \times [0, 1] \tag{19}$$

with initial value

$$u(0,\,x)=0 \qquad x\in[0,1] \tag{20}$$

and boundary conditions

$$u(t,\,0)=t \qquad t\in[0,1] \tag{21}$$

$$u(t,\,1)=t\,e \qquad t\in[0,1] \tag{22}$$

Similarly to the example 1, for a given natural numbers N_1 and N_2 we divide the domain $[0,1]\times[0,1]$ into $N_1\times N_2$ sub-domains and grid's nodes of this domain we denote as $P_{ij}=(t_i,\,x_j)=(ih_t,\,jh_x)$, $\quad h_t=\frac{1}{N_1},\, h_x=\frac{1}{N_2}$.

In this example we use differential transforms and backforward difference scheme for representation of derivate simultaneously. Applying this approach to the problem (19)-(22) we get relations .

$$(k+1)u((t_i,x_j),k+1)=\frac{1}{h}[u((t_i,x_j),k)-u((t_i,x_{j-1}),k)]$$

$$+\varphi\,((t_i,x_j),k)+\psi(\,(t_i,x_j),k) \tag{19'}$$

$$u((0,x_j),0)=0 \tag{20'}$$

$$u((t_i,0),\,k)=h_1(t_i,k) \tag{21'}$$

$$u((t_i,1),\,k)=h_2(t_i,k) \tag{22'}$$

where

$$\varphi\,((t_i,x_j),k)=\begin{cases} e^{x_j} & k=0 \\ 0 & k\neq 0 \end{cases} \quad , \quad \psi\,((t_i,x_j),k)=\begin{cases} e^{x_j} & k=1 \\ 0 & otherwise \end{cases}$$

$$h_1\,(t_i,k)=\begin{cases} 1 & k=1 \\ 0 & otherwise \end{cases} \quad , \quad h_2\,(t_i,k)=\begin{cases} e & k=1 \\ 0 & otherwise \end{cases}$$

and $i=0,1,2,...,N_1$, $\; j=0,1,2,...,N_2$.

For the solution of this example we use the same scheme as in example 1. However, on the step 3 at calculation of the approximation value of the solution in the grid node (t_i,x_j), coefficients of decomposition at the corresponding point $(t_i,x_{j-1})$are necessary.

The analytic solution of the problem (19)-(22) is a function $u(t,\,x)=\,t\,e^x$.

Comparisons of the approximated values and corresponding values calculated for the analytic solution show that the error of the method for this example has the order $10^{-12}-10^{-15}$.

5. Conclusions

Some aspects of one dimensional differential transformation method are investigated. Using differential transformation solvability of initial boundary value problems for some classes of differential equations on grid are studied. Effective application of suggested method to numeric solution of considered class problems are shown.

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Received October 10, 2003; Revised February 09, 2004.
Translated by Authors.