Telman B. GASYMOV, Shakhrizad J. MAMMADOVA

ON CONVERGENCE OF SPECTRAL EXPANSIONS FOR ONE DISCONTINUOUS PROBLEM WITH SPECTRAL PARAMETER IN THE BOUNDARY CONDITION

Abstract

In this paper basic properties of eigenfunctions of discontinuous spectral problem for the second order differential operator with spectral parameter in the boundary condition in spaces $L_p \oplus \mathbb{C}$ and L_p, $1 < p < \infty$, are studied.

1.Introduction. Consider a spectral problem

$$y''(x) + \lambda y(x) = 0, \quad x \in (-1,0) \cup (0,1), \tag{1}$$

$$\left.\begin{array}{l} y(-1) = y(1) = 0, \\ y(-0) = y(+0), \\ y'(-0) - y'(+0) = \lambda m y(0), m \neq 0, \end{array}\right\} \tag{2}$$

which arises by solving the problem on vibrations of a loaded string with the fixed ends and with a load placed in the middle of a string [1].

In the present paper we research convergence of expansions in eigenfunctions of problem (1), (2). For this purpose, at first, in p.2 formulas for eigenvalues, eigenfunctions and Green's functions of the problem (1), (2) are derived. Then, in p.3 the linearizing operator L acting in space $L_p(-1,1) \oplus \mathbb{C}$, where $\mathbb{C}$ is a complex plane, is constructed and properties of operator L and it's resolvent are studied. On the basis of previous items results in p.4 the theorem about basicity of eigenfunctions of operator L in space $L_p(-1,1) \oplus \mathbb{C}$, $1 < p < \infty$, is proved. Finally, in p.5 we solve the issue how we can get a basis of the space $L_p(-1,1)$, $1 < p < \infty$, if we eliminate some eigenfunction from the system of eigenfunctions of problem (1), (2).

Similar questions for the problem on vibrations of a loaded string when the load is fixed in one or two ends of a string, are investigated by other methods in the papers[2-6].

2. Eigenvalues, eigenfunctions and Green's function of problem (1), (2). Let's put $\lambda = \rho^2$ and introduce the following designation for boundary forms (2)

$$U_\nu(y) = U_{\nu 1}(y) + U_{\nu 2}(y), \quad \nu = \overline{1,4}, \tag{3}$$

where $U_{11}(y) = y(-1)$, $U_{12}(y) \equiv 0$, $U_{21}(y) \equiv 0$, $U_{22}(y) = y(1)$, $U_{31}(y) = y(-0)$, $U_{32}(y) = -y(+0)$, $U_{41}(y) = y'(-0)$, $U_{42}(y) = -y'(+0) - \rho^2 m y(0)$.

Lemma 1. *Spectral problem (1), (2) has two series of simple eigenvalues:* $\lambda_{1,n} = (\pi n)^2$, $n = 1,2,...,$ *and* $\lambda_{2,n} = (\rho_{2,n})^2$, $n = 0,1,2,...,$ *where* $\rho_{2,n}$ *has asymptotic form*

$$\rho_{2,n} = \pi n + \frac{2}{\pi m n} + O\left(\frac{1}{n^2}\right). \tag{4}$$

The eigenfunctions $u_n(x)$, $n = 0,1,...,$ *prescribed by formula*

$$u_{2n-1}(x) = \sin \pi n x, \quad n = 1,2,...,$$
$$u_{2n}(x) = \begin{cases} \sin \rho_{2,n}(1+x) & \text{at} \quad x \in [-1,0], \\ \sin \rho_{2,n}(1-x) & \text{at} \quad x \in [0,\ 1], \quad n = 0,1,..., \end{cases} \tag{5}$$

correspond to them.

Proof. Take $\varphi_{11}(x) = \sin \rho(1+x)$, $\varphi_{12}(x) = \cos \rho(1+x)$ for $x \in [-1,0]$ and $\varphi_{21}(x) = \sin \rho x$, $\varphi_{22}(x) = \cos \rho x$ for $x \in [0,1]$ as linear - independent solutions of the equation (1). We shall search eigenfunctions of the problem (1), (2) in the form of

$$u(x,\rho) = \begin{cases} c_{11}\varphi_{11}(x) + c_{12}\varphi_{12}(x) & \text{at} \quad x \in [-1,0], \\ c_{21}\varphi_{21}(x) + c_{22}\varphi_{22}(x) & \text{at} \quad x \in [0,1]. \end{cases} \tag{6}$$

Let's demand, that function $u(x,\rho)$ satisfy boundary conditions (2). Then for definition of numbers c_{jk} we receive the system of the linear homogeneous equations

$$c_{11}U_{\nu 1}(\varphi_{11}) + c_{12}U_{\nu 1}(\varphi_{12}) + c_{21}U_{\nu 2}(\varphi_{21}) + c_{22}U_{\nu 2}(\varphi_{22}) = 0, \qquad \nu = \overline{1,4}, \tag{7}$$

whose determinant is

$$\Delta(\rho) = \det \left\| U_{\nu j}\left(\varphi_{jk}\right) \right\|_{\nu=\overline{1,4};\ \ j,k=1,2}.$$

Taking into consideration (3), for values of forms $U_{\nu j}\left(\varphi_{jk}\right)$ we have:

$$\begin{array}{llll} U_{11}(\varphi_{11}) = 0, & U_{11}(\varphi_{12}) = 0, & U_{12}(\varphi_{21}) = 0, & U_{12}(\varphi_{22}) = 0, \\ U_{21}(\varphi_{11}) = 0, & U_{21}(\varphi_{12}) = 0, & U_{22}(\varphi_{21}) = \sin \rho, & U_{22}(\varphi_{22}) = \cos \rho, \\ U_{31}(\varphi_{11}) = \sin \rho, & U_{31}(\varphi_{12}) = \cos \rho, & U_{32}(\varphi_{21}) = 0, & U_{32}(\varphi_{22}) = -1, \\ U_{41}(\varphi_{11}) = \cos \rho, & U_{41}(\varphi_{12}) = -\rho \sin \rho, & U_{42}(\varphi_{21}) = -\rho, & U_{42}(\varphi_{22}) = -\rho^2 m. \end{array} \tag{8}$$

Opening determinant $\Delta(\rho)$ with the account (8), we shall receive

$$\Delta(\rho) = -\rho \sin \rho \left(\rho m \sin \rho - 2\cos \rho\right). \tag{9}$$

From the formula (9) it is obvious, that function $\Delta(\rho)$ has two series of zeroes, the first of which consists of zeros of function $\sin \rho$, i.e. $\rho_{1,n} = \pi n$, and the second series $\rho_{2,n}$ consists of zeros of function $\rho m \sin \rho - 2\cos \rho$. Reasoning as in [7, p.20], we receive that for $\rho_{2,n}$ it is true the asymptotic formula

$$\rho_{2,n} = \pi n + \delta_n,$$

where δ_n satisfy the relation

$$\sin \delta_n = \frac{2}{m\rho_{2,n}} \cos \delta_n.$$

From the last relation we have

$$\delta_n = \frac{2}{\pi m n} + O\left(\frac{1}{n^2}\right),$$

that proves the truth of (4).

Substituting $\rho = \pi n$ in (7) subject to (8), we find $c_{12} = 0$, $c_{22} = 0$, $c_{21} = (-1)^n c_{11}$. Therefore, we choose $c_{11} = 1$, we receive from (6) eigenfunction $u_{2n-1}(x)$ corresponding to eigenvalue $\lambda_{1,n} = (\pi n)^2$ in the form of

$$u_{2,n-1}(x) = \sin \pi n x, \qquad n = 1, 2, \ldots .$$

Similarly at $\rho = \rho_{2,n}$ from (7), (8) we conclude that

$$c_{12} = 0, \qquad c_{12} \sin \rho_{2n} + c_{22} \cos \rho_{2n} = 0, \qquad c_{11} \sin \rho_{2n} = c_{22}.$$

From here taking into consideration that $\sin \rho_{2n} \neq 0$, we receive

$$c_{21} = -c_{11} \cos \rho_{2,n}.$$

Now choosing $c_{11} = 1$ we get from (6) eigenfunction $u_{2n}(x)$ corresponding to eigenvalue $\lambda_{2,n} = \left(\rho_{2,n}\right)^2$ in the following form

$$u_{2n}(x) = \begin{cases} \sin \rho_{2,n}(1 + x), & \text{at} \quad x \in [-1, 0], \\ \sin \rho_{2,n}(1 - x), & \text{at} \quad x \in [0, 1], \quad n = 0, 1, 2, \ldots . \end{cases}.$$

The lemma is proved.

Now let's pass to construction of Green function of problem (1), (2). It is defined as a kernel of integral representation for solution of the corresponding non-homogeneous problem

$$y''(x) + \rho y(x) = f(x), \tag{10}$$

satisfying boundary conditions (2). We shall search the solution of problem (10), (2) in the form of

$$y(x) = \begin{cases} y_1(x) & \text{at} \quad x \in [-1, 0], \\ y_1(x) & \text{at} \quad x \in [0, 1], \end{cases} \tag{11}$$

where

$$\begin{aligned} y_1(x) &= c_{11}\varphi_{11}(x) + c_{12}\varphi_{12}(x) + \int_{-1}^{0} g(x, \xi, \rho) f(\xi) \, d\xi, \\ y_2(x) &= c_{21}\varphi_{21}(x) + c_{22}\varphi_{12}(x) + \int_{0}^{1} g(x, \xi, \rho) f(\xi) \, d\xi; \end{aligned} \tag{12}$$

$$g(x, \xi, \rho) = \begin{cases} -\frac{1}{2\rho} \sin(x - \xi) & \textit{at} \quad \xi < x, \\ \frac{1}{2\rho} \sin(x - \xi) & \textit{at} \quad x \leq \xi. \end{cases} \tag{13}$$

Let's demand, function (11) satisfy the boundary conditions (2). Then for definition of numbers c_{jk} we receive the system of algebraic equations

$$\begin{aligned} U_\nu(y) &= c_{11}U_{\nu 1}(\varphi_{11}) + c_{12}U_{\nu 1}(\varphi_{12}) + c_{21}U_{\nu 2}(\varphi_{21}) + c_{22}U_{\nu 2}(\varphi_{22}) + \\ &+ \int_{-1}^{0} U_{\nu 1}(g) f(\xi) \, d\xi + \int_{0}^{1} U_{\nu 2}(g) f(\xi) \, d\xi = 0, \quad \nu = \overline{1, 4}. \end{aligned} \tag{14}$$

Let's define numbers c_{jk} from (14) and we shall substitute their values in (12). Then for solving of problem (10), (2) we shall get the formula

$$\begin{aligned} y_1(x) &= \int_{-1}^{0} G_{11}(x, \xi, \rho) f(\xi) \, d\xi + \int_{0}^{1} G_{12}(x, \xi, \rho) f(\xi) \, d\xi, \qquad x \in [-1, 0], \\ y_2(x) &= \int_{-1}^{0} G_{21}(x, \xi, \rho) f(\xi) \, d\xi + \int_{0}^{1} G_{22}(x, \xi, \rho) f(\xi) \, d\xi, \qquad x \in [0, 1]; \end{aligned} \tag{15}$$

where

$$G_{11}(x,\xi,\rho)=\frac{1}{\Delta(\rho)}\begin{vmatrix} g & \varphi_{11} & \varphi_{12} & 0 & 0 \\ U_{\nu 1}(g) & U_{\nu 1}(\varphi_{11}) & U_{\nu 1}(\varphi_{12}) & U_{\nu 2}(\varphi_{21}) & U_{\nu 2}(\varphi_{22}) \\ \cdots & & & & \\ \nu=\overline{1,4} & & & & \end{vmatrix},$$

$$G_{12}(x,\xi,\rho)=\frac{1}{\Delta(\rho)}\begin{vmatrix} 0 & \varphi_{11} & \varphi_{12} & 0 & 0 \\ U_{\nu 2}(g) & U_{\nu 1}(\varphi_{11}) & U_{\nu 1}(\varphi_{12}) & U_{\nu 2}(\varphi_{21}) & U_{\nu 2}(\varphi_{22}) \\ \cdots & & & & \\ \nu=\overline{1,4} & & & & \end{vmatrix},$$

$$G_{21}(x,\xi,\rho)=\frac{1}{\Delta(\rho)}\begin{vmatrix} 0 & 0 & 0 & \varphi_{21} & \varphi_{22} \\ U_{\nu 1}(g) & U_{\nu 1}(\varphi_{11}) & U_{\nu 1}(\varphi_{12}) & U_{\nu 2}(\varphi_{21}) & U_{\nu 2}(\varphi_{22}) \\ \cdots & & & & \\ \nu=\overline{1,4} & & & & \end{vmatrix},$$

$$G_{22}(x,\xi,\rho)=\frac{1}{\Delta(\rho)}\begin{vmatrix} g & 0 & 0 & \varphi_{21} & \varphi_{22} \\ U_{\nu 2}(g) & U_{\nu 1}(\varphi_{11}) & U_{\nu 1}(\varphi_{12}) & U_{\nu 2}(\varphi_{21}) & U_{\nu 2}(\varphi_{22}) \\ \cdots & & & & \\ \nu=\overline{1,4} & & & & \end{vmatrix},$$

and $\Delta(\rho)$ is a determinant from (7).

Calculating values of forms $U_{\nu k}(g)$ where g is defined by (13), we shall find that

$$\begin{array}{ll} U_{11}(g)=-\frac{1}{2\rho}\sin\rho(1+\xi), & U_{12}(g)=0, \\ U_{21}(g)=0, & U_{22}(g)=-\frac{1}{2\rho}\sin\rho(1-\xi), \\ U_{31}(g)=\frac{1}{2\rho}\sin\rho\xi, & U_{32}(g)=\frac{1}{2\rho}\sin\rho\xi, \\ U_{41}(g)=-\frac{1}{2\rho}\cos\rho\xi, & U_{42}(g)=-\frac{1}{2}\cos\rho\xi+\frac{1}{2}\rho m\sin\rho\xi, \end{array} \tag{16}$$

Let's substitute (16), (8) and (9) in determinants of formulae for $G_{k,j}(x,\xi,\rho)$. Transforming the received determinants similar to [8, page 95] and then opening them, we shall receive formulae for Green function components. We'll formulate it as a lemma.

Lemma 2. *For Green function components $G_{kj}(x,\xi,\rho)$ the following expressions*

$$G_{11}(x,\xi,\rho)=\begin{cases} -\dfrac{1}{\rho}\sin\rho(x-\xi)+\dfrac{1}{\Delta(\rho)}\sin\rho(1+x)\sin\rho(1+\xi)- \\ \qquad -\dfrac{1}{\rho\sin\rho}\sin\rho(1+x)\sin\rho\xi, \quad -1\le\xi<x\le 0, \\ \dfrac{1}{\rho}\sin\rho(x-\xi)+\dfrac{1}{\Delta(\rho)}\sin\rho(1+x)\sin\rho(1+\xi)- \\ \qquad -\dfrac{1}{\rho\sin\rho}\sin\rho x\sin\rho(1+\xi), \quad -1\le x\le\xi\le 0; \end{cases} \tag{17}$$

$$G_{22}(x,\xi,\rho)=\begin{cases} -\dfrac{1}{\rho}\sin\rho(x-\xi)+\dfrac{1}{\Delta(\rho)}\sin\rho(1-x)\sin\rho(1-\xi)+ \\ \qquad +\dfrac{1}{\rho\sin\rho}\sin\rho x\sin\rho(1-\xi), \quad 0\le\xi<x\le 1, \\ \dfrac{1}{\rho}\sin\rho(x-\xi)+\dfrac{1}{\Delta(\rho)}\sin\rho(1-x)\sin\rho(1-\xi)- \\ \qquad -\dfrac{1}{\rho\sin\rho}\sin\rho(1-x)\sin\rho\xi, \quad 0\le x\le\xi\le 1; \end{cases} \tag{18}$$

$$G_{12}(x,\xi,\rho)=\frac{1}{\Delta(\rho)}\sin\rho(1+x)\sin\rho(1-\xi), \quad x\in[-1,0], \ \xi\in[0,1]; \tag{19}$$

$$G_{21}(x,\xi,\rho)=\frac{1}{\Delta(\rho)}\sin\rho(1-x)\sin\rho(1+\xi), \quad x\in[0,1], \ \xi\in[-1,0]; \tag{20}$$

are true.

3. Construction of linearizing operator. By $W_p^k(-1,0)\cup(0,1)$ we denote a space functions whose contractions on segments $[-1,0]$ and $[0,1]$ belong correspondingly to Sobolev spaces $W_p^k(-1,0)$ and $W_p^k(0,1)$. Let's define the operator L in $L_p(-1,1)\oplus\mathbb{C}$ space as follows:

$$D(L)=\left\{\hat{u}\in L_p(-1,1)\oplus\mathbb{C}: \ \hat{u}=(u;mu(0)), \ u\in W_p^2(-1,0)\cup(0,1),\right.$$

$$\left. u(-1)=u(1)=0, \quad u(-0)=u(+0)\right\}, \tag{21}$$

and for $\hat{u}\in D(L)$

$$L\hat{u}=\left(-u'';u'(-0)-u'(+0)\right). \tag{22}$$

Lemma 3. *Operator defined by formulae (21), (22) is a linear closed operator with dense definitional domain in $L_p(-1,1)\oplus\mathbb{C}$. Eigenvalues of the operator L and problem (1), (2) coincide, and $\{\hat{u}_k\}$ are eigenvectors of the operator L, where $\hat{u}_{2n-1}=(u_{2n-1}(x);\ 0)$, $\hat{u}_{2n}=\left(u_{2n}(x);\ m\sin\rho_{2,n}\right)$.*

Proof. To prove the first part of the lemma we shall take $\hat{u}=(u;a)\in L_p(-1,1)\oplus\mathbb{C}$ and define functional $F(\hat{u})$ as follows:

$$F(\hat{u})=mu(+0)-\alpha.$$

Let us assume

$$U_\nu(\hat{u})=U_\nu(u), \quad \nu=1,2,3.$$

Then F, U_ν , $\nu=1,2,3$, are bounded linear functionals on $W_p^2(-1,0)\cup(0,1)\oplus\mathbb{C}$, but unbounded on $L_p(-1,1)\oplus\mathbb{C}$. Therefore (see for example [9, p.27-29]) the set

$$D(L)=\left\{\hat{u}=(u,\alpha), \ u\in W_p^2(-1,0)\cup(0,1), \ F(\hat{u})=U_\nu(\hat{u})=0, \ \nu=1,2,3\right\}$$

is everywhere dense in $L_p(-1,1)\oplus\mathbb{C}$, and L is a closed operator as contraction of corresponding closed maximal operator.

The second part of the lemma is verified directly.

The lemma is proved.

For construction resolvent of operator L, we shall consider the equation

$$L\hat{u}-\lambda\hat{u}=\hat{f}, \tag{23}$$

where $\hat{u}\in D(L)$, $\hat{f}=(f;\beta)\in L_p(-1,1)\oplus\mathbb{C}$. We can rewrite equation (23) in the form of

$$\begin{cases} -u''=\lambda u+f, \\ u'(0)-u'(+0)-\lambda mu(0)=\beta, \\ U_\nu(u)=0, \quad \nu=1,2,3. \end{cases} \tag{24}$$

Lemma 4. *For solution $\hat{u} = (u;\ mu(0))$ of equation (23) it holds the following representations*

$$u(x,\rho) = \frac{\beta \sin\rho(1+x)}{\rho(2\cos\rho - \rho m \sin\rho)} - \frac{1}{\rho}\int_{-1}^{x} f(\xi)\sin\rho(x-\xi)\,d\xi +$$

$$+\frac{1}{\rho}\int_{x}^{0} f(\xi)\sin\rho(x-\xi)\,d\xi + \frac{1}{\Delta(\rho)}\int_{-1}^{0} f(\xi)\sin\rho(1+x)\sin\rho(1+\xi)\,d\xi -$$

$$-\frac{1}{\rho\sin\rho}\int_{-1}^{x} f(\xi)\sin\rho(1+x)\sin\rho\xi d\xi - \frac{1}{\rho\sin\rho}\int_{x}^{0} f(\xi)\sin\rho x\sin\rho(1+\xi)\,d\xi +$$

$$+\frac{1}{\Delta(\rho)}\int_{0}^{1} f(\xi)\sin\rho(1+x)\sin\rho(1-\xi)\,d\xi, \tag{25}$$

if $x \in [-1,0]$;

$$u(x,\rho) = \frac{\beta \sin\rho(1-x)}{\rho(2\cos\rho - \rho m \sin\rho)} - \frac{1}{\rho}\int_{0}^{x} f(\xi)\sin\rho(x-\xi)\,d\xi +$$

$$+\frac{1}{\rho}\int_{x}^{1} f(\xi)\sin\rho(x-\xi)\,d\xi + \frac{1}{\Delta(\rho)}\int_{0}^{1} f(\xi)\sin\rho(1-x)\sin\rho(1-\xi)\,d\xi +$$

$$+\frac{1}{\rho\sin\rho}\int_{0}^{x} f(\xi)\sin\rho x\sin\rho(1-\xi)\,d\xi + \frac{1}{\rho\sin\rho}\int_{x}^{1} f(\xi)\sin\rho(1-x)\sin\rho\xi d\xi +$$

$$+\frac{1}{\Delta(\rho)}\int_{-1}^{0} f(\xi)\sin\rho(1-x)\sin\rho(1+\xi)\,d\xi, \tag{26}$$

if $x \in [0,1]$;

$$u(0,\rho) = \frac{1}{\rho(2\cos\rho - \rho m\sin\rho)}\times$$

$$\times\left[\beta\sin\rho + \int\limits_{-1}^{0} f(\xi)\sin\rho(1+\xi)\,d\xi + \int\limits_{0}^{1} f(\xi)\sin\rho(1-\xi)\,d\xi\right]. \tag{27}$$

Proof. Solution of the equation (24) will be found as

$$u(x,\rho) = \begin{cases} c_{11}\varphi_{11}(x) + c_{12}\varphi_{12}(x) + y_1(x) & at \quad x \in [-1,0], \\ c_{21}\varphi_{21}(x) + c_{22}\varphi_{22}(x) + y_2(x) & at \quad x \in [0,1]; \end{cases} \tag{28}$$

where $y_1(x)$ and $y_2(x)$ are defined by (15). Since $y(x)$ defined by (11) satisfies boundary conditions (2), then

$$U_\nu(y) = 0, \quad \nu = \overline{1,4}. \tag{29}$$

Let's demand the function $u(x,\rho)$ satisfy boundary conditions $U_\nu(u) = 0$, $\nu = 1,2,3$, $U_4(u) = \beta$. Then allowing for (29) from (28) we shall receive

$$\begin{cases} c_{11}U_{\nu 1}(\varphi_{11}) + c_{12}U_{\nu 1}(\varphi_{12}) + c_{21}U_{\nu 2}(\varphi_{21}) + c_{22}U_{\nu 2}(\varphi_{22}) = 0, \quad \nu = 1,2,3; \\ c_{11}U_{\nu 1}(\varphi_{11}) + c_{12}U_{4}(\varphi_{12}) + c_{21}U_{42}(\varphi_{21}) + c_{22}U_{42}(\varphi_{22}) = \beta. \end{cases}$$

Solving this system with respect to unknowns c_{kj}, we shall receive

$$c_{11}=-\frac{\beta}{\Delta(\rho)}U_{11}(\varphi_{12})\left[U_{22}(\varphi_{21})U_{32}(\varphi_{22})-U_{22}(\varphi_{22})U_{32}(\varphi_{21})\right],$$

$$c_{12}=\frac{\beta}{\Delta(\rho)}U_{11}(\varphi_{11})\left[U_{22}(\varphi_{21})U_{32}(\varphi_{22})-U_{22}(\varphi_{22})U_{32}(\varphi_{21})\right],$$

$$c_{21}=\frac{\beta}{\Delta(\rho)}U_{22}(\varphi_{22})\left[U_{11}(\varphi_{11})U_{31}(\varphi_{12})-U_{11}(\varphi_{12})U_{31}(\varphi_{11})\right],$$

$$c_{22}=-\frac{\beta}{\Delta(\rho)}U_{22}(\varphi_{21})\left[U_{11}(\varphi_{11})U_{31}(\varphi_{12})-U_{11}(\varphi_{21})U_{31}(\varphi_{11})\right].$$

Substituting the received values of coefficient c_{kj} in (28) and taking into account formulae (8), (17) - (20), we shall receive validity of formulas (25) and (26). And formula (27) results from (26) (or from (25)) by substitution $x=0$.

The lemma is proved.

4. Basicity of eigenfunctions in spaces $L_p(-1,1)\oplus\mathbb{C}$.

Theorem 1. *Eigen vectors of operator L form basis in space $L_p(-1,1)\oplus\mathbb{C}$, $1<p<\infty$, and at $p=2$ this basis is Riesz basis.*

Proof. At first we shall prove completeness of a system of eigen functions of operator L in space $L_p(-1,1)\oplus\mathbb{C}$. For this purpose we shall receive estimation of the resolvent of the operator L at great values of $|\rho|$. We shall use the following known inequalities

$$|\sin\rho|\le ce^{|\rho|\sin\varphi},\qquad |\cos\rho|\le ce^{|\rho|\sin\varphi},\tag{30}$$

where $\rho=re^{i\varphi}$, $0\le\varphi\le\pi$. Besides outside of circles of the same radius δ with centers in zero of $\sin\rho$ the following estimation is true

$$|\sin\rho|\ge m_\delta e^{r\sin\varphi}.\tag{31}$$

From estimations (30), (31) and from the formula (9) it follows that at great values of $|\rho|$ outside circles of radius δ with the centers in zero of $\Delta(\rho)$ the following estimation is true

$$|\Delta(\rho)|\ge M_\delta re^{2r\sin\varphi}.\tag{32}$$

From representation (25), (26) allowing for of inequalities (30), (31), (32) we shall receive the inequality

$$|u(x,\rho)|\le\frac{C_\delta}{|\rho|},$$

which fairly uniform on $x\in[-1,1]$. From the last estimation it follows that for resolvent operator $R(\lambda)=(L-\lambda I)^{-1}$ of the operator L outside of the above-stated circles the following estimation is true

$$\left\|R\left(\rho^2\right)\right\|\le\frac{C_\delta}{|\rho|},\quad |\rho|\ge r_0.\tag{33}$$

Having estimation (33) by a standard method (see for example [10], p. 445) we receive that eigenfunctions of operator L form complete system in space $L_p(-1,1)\oplus\mathbb{C}$.

Let $\Gamma_{2n-1} = \left\{\rho : |\rho - \rho_{1,n}| = \frac{1}{2\pi mn}\right\}$, $\Gamma_{2n} = \left\{\rho : |\rho - \rho_{2,n}| = \frac{1}{2\pi mn}\right\}$, $C_n = \left\{\rho : |\rho| = \pi\left(n+\frac{1}{2}\right),\ 0 \le \arg\rho \le \pi\right\}$, by Γ'_n and C'_n we denote the images of Γ_n and C_n, respectively at mapping $\lambda = \rho^2$. Let's research convergence of series

$$\sum_{k=0}^{\infty} E_k \hat{f}, \tag{34}$$

where

$$E_{2n-1}\hat{f} = \frac{1}{2\pi i}\int_{\Gamma'_{2n-1}} R(\lambda)\hat{f}d\lambda = \frac{1}{\pi i}\int_{\Gamma_{2n-1}} \rho R\left(\rho^2\right)\hat{f}d\rho,$$

$$E_{2n}\hat{f} = \frac{1}{2\pi i}\int_{\Gamma'_{2n}} R(\lambda)\hat{f}d\lambda = \frac{1}{\pi i}\int_{\Gamma_{2n}} \rho R\left(\rho^2\right)\hat{f}d\rho.$$

From asymptotic formulae (4) it follows that at great values of n between contours C'_n and C'_{n+1} there are exactly two eigen values of the operator L . Let $S_n\left(\hat{f}\right)$are partial sums of series (34). Then

$$S_{2n+1}\left(\hat{f}\right) = \frac{1}{2\pi i}\int_{C'_n} R(\lambda)\hat{f}d\lambda = \frac{1}{\pi i}\int_{C_n}\rho R\left(\rho^2\right)\hat{f}d\rho = \frac{1}{\pi i}\int_{C_n}\rho\hat{u}(x,\rho)\,d\rho =$$

$$= \frac{1}{\pi i}\left(\int_{C_n}\rho u(x,\rho)\,d\rho;\ \int_{C_n} mu(0,\rho)\,d\rho\right), \tag{35}$$

where $\hat{u}(x,\rho) \in L_p(-1,1)\oplus\mathbb{C}$, whose first component of is defined by (25), (26), and the second component is $mu(0)$, where $u(0)$ is defined by equality (27). Let's estimate norm $S_{2n+1}\left(\tilde{f}\right)$. We shall carry out the estimation for each component from (35), and for each addend of representations (25), (26), (27). From (25) at $x \in [-1,0]$ we have

$$\frac{1}{\pi i}\int_{C_n}\rho u(x,\rho)\,d\rho = \frac{\beta}{\pi i}\int_{C_n}\frac{\sin\rho(1+x)}{2\cos\rho - \rho m\sin\rho}d\rho - \frac{1}{\pi i}\int_{C_n}\int_{-1}^{x} f(\xi)\sin\rho(x-\xi)\,d\xi d\rho +$$

$$+\frac{1}{\pi i}\int_{C_n}\int_{x}^{0} f(\xi)\sin\rho(x-\xi)\,d\xi d\rho + \frac{1}{\pi i}\int_{C_n}\frac{\rho}{\Delta(\rho)}\int_{0}^{1} f(\xi)\sin\rho(1+x)\sin\rho(1-\xi)\,d\xi d\rho +$$

$$+\frac{1}{\pi i}\int_{C_n}\frac{\rho}{\Delta(\rho)}\int_{-1}^{0} f(\xi)\sin\rho(1+x)\sin\rho(1+\xi)\,d\xi d\rho -$$

$$-\frac{1}{\pi i}\int_{C_n}\frac{1}{\sin\rho}\int_{-1}^{x} f(\xi)\sin\rho(1+x)\sin\rho\xi d\xi d\rho -$$

$$-\frac{1}{\pi i}\int\limits_{C_n}\frac{1}{\sin\rho}\int\limits_{-1}^{x}f(\xi)\cdot\sin\rho x\sin\rho(1+\xi)\,d\xi d\rho=$$

$$=J_{n,1}(x)+J_{n,2}(x)+J_{n,3}(x)+J_{n,4}(x)+J_{n,5}(x)+J_{n,6}(x)+J_{n,7}(x) \qquad (36)$$

Taking into account estimations (30), (31) we shall receive

$$|J_{n,1}(x)|=\left|\frac{\beta}{\pi i}\int\limits_{C_n}\frac{\sin\rho(1+x)}{2\cos\rho-\rho m\sin\rho}\,d\rho\right|\le\frac{|\beta|}{\pi}C_\delta\int\limits_{|\rho|=\pi\left(n+\frac{1}{2}\right)}\frac{e^{|\rho|x\sin\varphi}}{|\rho|}\,|d\rho|=$$

$$=\frac{|\beta|}{\pi}C_\delta\int\limits_{0}^{\pi}e^{\pi\left(n+\frac{1}{2}\right)x\sin\varphi}d\varphi\le|\beta|\,C_\delta. \qquad (37)$$

Moreover $J_{n,2}(x)=J_{n,3}(x)\equiv0$, because subintegral functions are regular functions with respect to λ. Let's estimate $J_{n,4}(x)$. Allowing for (32) we shall receive

$$|J_{n,4}(x)|=\left|\frac{1}{\pi i}\int\limits_{C_n}\frac{\rho}{\Delta(\rho)}\int\limits_{-1}^{x}f(\xi)\sin\rho(1+x)\sin\rho(1+\xi)\,d\xi d\rho\right|\le$$

$$\le\frac{1}{\pi}M_\delta\int\limits_{C_n}e^{-2|\rho|\sin\varphi}\int\limits_{-1}^{0}|f(\xi)|\,e^{|\rho|(2+x+\xi)\sin\varphi}d\xi\,|d\rho|=$$

$$=\frac{M_\delta\left(n+\frac{1}{2}\right)}{\pi}\int\limits_{0}^{\pi}\int\limits_{-1}^{0}|f(\xi)|\,e^{\left(n+\frac{1}{2}\right)(x+\xi)\sin\varphi}d\xi d\varphi\le$$

$$\le\frac{M_\delta(2n+1)}{\pi}\int\limits_{-1}^{0}|f(\xi)|\int\limits_{0}^{\frac{\pi}{2}}e^{\left(n+\frac{1}{2}\right)(x+\xi)\sin\varphi}d\varphi d\xi\le$$

$$\le\frac{M_\delta(2n+1)}{\pi}\int\limits_{-1}^{0}|f(\xi)|\int\limits_{0}^{\frac{\pi}{2}}e^{\left(n+\frac{1}{2}\right)(x+\xi)\frac{\varphi}{\pi}}d\varphi d\xi=$$

$$=2M_\delta\int\limits_{-1}^{0}\frac{|f(\xi)|}{x+\xi}\left[e^{\frac{1}{2}\left(n+\frac{1}{2}\right)(x+\xi)}-1\right]d\xi\le2M_\delta\int\limits_{-1}^{0}\frac{|f(\xi)|}{-x-\xi}d\xi.$$

Applying Riesz theorem on boundedness of Hilbert transformation in L_p to the integral in the right hand side of the last inequality , we shall receive

$$|J_{n,4}(x)|\le C\,\|f\|_{L_p}.3 \qquad (38)$$

Furher, taking into account that for function $\Delta_1(\rho)=2\cos\rho-\rho m\sin\rho$ it holds place the estimation $|\Delta_1(\rho)|\ge M_\delta\,|\rho|\,e^{|\rho|\sin\varphi}$ that is true outside of circles identical radius δ with the centers in zeros of $\Delta_1(\rho)$, for integrals $J_{n,5}(x)$, $J_{n,6}(x)$ and $J_{n,7}(x)$ in the same way we get the analogous estimation

$$|J_{n,5}(x)| \le C\|f\|_{L_p}, \quad |J_{n,6}(x)| \le C\|f\|_{L_p}, \quad |J_{n,7}(x)| \le C\|f\|_{L_p}. \tag{39}$$

From inequalities (37), (38), (39) it follows that

$$\left| \frac{1}{\pi i} \int\limits_{C_n^1} \rho u(x,\rho) \right| \le C \left\| \hat{f} \right\|_{L_p \oplus \mathbb{C}} \tag{40}$$

The last inequality is fulfilled uniformly on $x \in [-1,0]$. Operating similarly, we receive from representation (26) that the inequality (40) is true at $x \in [0;1]$. Moreover from (40), we receive also the estimation of the second components from (35):

$$\left| \frac{1}{\pi i} \int\limits_{C_n^1} \rho u(0,\rho) \right| \le C \left\| \hat{f} \right\|_{L_p \oplus \mathbb{C}}. \tag{41}$$

From (40) and (41) we receive estimation

$$\left\| S_{2n+1}\left(\hat{f}\right) \right\|_{L_p \oplus \mathbb{C}} \le C \left\| \hat{f} \right\|_{L_p \oplus \mathbb{C}}. \tag{42}$$

To receive estimation of norm $S_{2n}\left(\hat{f}\right)$, we shall represent it in the following form

$$S_{2n}\left(\hat{f}\right) = S_{2n-1}\left(\hat{f}\right) + \left\langle \hat{f},\ \hat{\vartheta}_{2n-1} \right\rangle \hat{u}_{2n-1} \tag{43}$$

where $\left\{ \hat{\vartheta}_n \right\}$ is a system biorthogonally adjoint to $\{\hat{u}_n\}$: $\hat{\vartheta}_n = (\vartheta_n(x),\ \overline{m}\vartheta_n(0))$. Here $\vartheta_n(x)$, $n = 0,1,...$, are eigenfunctions of the adjoint spectral problem

$$\vartheta''(x) + \lambda\vartheta(x) = 0, \quad \vartheta(-1) = \vartheta(1) = 0,$$

$$\vartheta(-0) = \vartheta(+0), \quad \vartheta'(-0) - \vartheta'(+0) = \lambda\overline{m}\vartheta(0).$$

Therefore for $\vartheta_n(x)$ the formulae similar (5) are true. From these formulas it follows that

$$\|\hat{u}_n\|_{L_p \oplus \mathbb{C}} \le C, \quad \left\| \hat{\vartheta}_n \right\|_{L_q \oplus \mathbb{C}} \le C. \tag{44}$$

From (42), (43) and (44) we shall receive

$$\left\| S_{2n}\left(\hat{f}\right) \right\|_{L_p \oplus \mathbb{C}} \le \left\| S_{2n-1}\left(\hat{f}\right) \right\|_{L_p \oplus \mathbb{C}} +$$

$$+ \left\| \hat{f} \right\|_{L_p \oplus \mathbb{C}} \left\| \hat{\vartheta}_{2n-1} \right\|_{L_q \oplus \mathbb{C}} \left\| \hat{U}_{2n-1} \right\|_{L_p \oplus \mathbb{C}} \le C \left\| \hat{f} \right\|_{L_p \oplus \mathbb{C}}. \tag{45}$$

Let's note, that in various places by C we denote, generally speaking, various constants. Inequalities (42) and (45) show, that the partial sums of series (34) are iuniformly bounded and therefore (see [11]) eigenfunctions of the operator L form basis in space $L_p(-1,1) \oplus \mathbb{C}$, $1 < p < \infty$.

For the proof of Riesz basicity for system $\{\hat{u}_n\}$ in space $L_2(-1,1)\oplus\mathbb{C}$, we shall show that integral on the circle Γ_{2n-1}, Γ_{2n} from each addend from representations (25) and (26) are expressed as $a_n\varphi_n(x)$, where $\{a_n\}\in l_2$, and system $\{\varphi_n(x)\}$ is besselian. We shall show it, for example, for the addend from (25) of the form

$$\frac{1}{\Delta(\rho)}\int\limits_{-1}^{0} f(\xi)\sin\rho(1+x)\sin\rho(1+\xi)\,d\xi. \tag{46}$$

Let's represent function $\dfrac{\rho}{\Delta(\rho)}$ in the following form

$$\frac{\rho}{\Delta(\rho)}=\frac{1}{2}\left[\frac{\cos\rho}{\sin\rho}+\frac{2\sin\rho+\rho m\cos\rho}{2\cos\rho-\rho m\sin\rho}\right].$$

The first function in the right hand side of (46) has simple poles at the points $\rho_{1,n}=\pi n$,and the second function has simple poles at the points $\rho_{2,n}$. Then

$$\frac{1}{\pi i}\int_{\Gamma_{2n-1}}\frac{\rho}{\Delta(\rho)}\int_{-1}^{0} f(\xi)\sin\rho(1+x)\sin\rho(1+\xi)\,d\xi d\rho=$$

$$=\frac{1}{2\pi i}\int_{\Gamma_{2n-1}}\frac{\cos\rho}{\sin\rho}\int_{-1}^{0} f(\xi)\sin\rho(1+x)\sin\rho(1+\xi)\,d\xi d\rho=$$

$$=\sin\rho_{1,n}(1+x)\int_{-1}^{0} f(\xi)\sin\rho_{1,n}(1+\xi)\,d\xi=c_{1,n}\varphi_{1,n}(x),$$

$$\frac{1}{\pi i}\int_{\Gamma_{2n}}\frac{\rho}{\Delta(\rho)}\int_{-1}^{0} f(\xi)\sin\rho(1+x)\sin\rho(1+\xi)\,d\xi d\rho=\frac{1}{2\pi i}\int_{\Gamma_{2n}}\frac{2\sin\rho+\rho m\cos\rho}{2\cos\rho-\rho m\sin\rho}\times$$

$$\times\int_{-1}^{0} f(\xi)\sin\rho(1+x)\sin\rho(1+\xi)\,d\xi d\rho = \sin\rho_{2,n}(1+x)\times$$

$$\times\int_{-1}^{0} f(\xi)\sin\rho_{2,n}(1+\xi)\,d\xi = c_{2,n}\varphi_{2,n}(x),$$

where $c_{1,n}=\int\limits_{-1}^{0} f(\xi)\sin\pi n(1+\xi)\,d\xi$, $c_{2,n}=\int\limits_{-1}^{0} f(\xi)\sin\rho_{2,n}(1+\xi)\,d\xi$, $\varphi_{1,n}(x)=\sin\pi n(1+x)$, $\varphi_{2,n}(x)=\sin\rho_{2,n}(1+x)$ .It is obvious, that $\{c_{1,n}\}$, $\{c_{2,n}\}\in l_2$, and systems $\left\{\varphi_{1,n}(x)\right\}$ and $\left\{\varphi_{21,n}(x)\right\}$ are besselian. Hence it follows that series

$$\sum_{n=1}^{\infty} c_{1,n}\varphi_{1,n}(x) \qquad \text{and} \qquad \sum_{n=0}^{\infty} c_{2,n}\varphi_{2,n}(x)$$

unconditionally converge (see [12] or [10, p. 420]).

For other addens from (25) and (26) it it is get similarly. Now we are to investigate convergence of series

$$\sum_{n=1}^{\infty}\int\limits_{\Gamma_{2n-1}}\rho u(0,\rho)\,d\rho \qquad \text{and} \qquad \sum_{n=0}^{\infty}\int\limits_{\Gamma_{2n}}\rho u(0,\rho)\,d\rho.$$

From representation (27) it is seen, that function $u(0,\rho)$ is regular inside Γ_{2n-1}. Therefore

$$\int\limits_{\Gamma_{2n-1}} \rho u(0,\rho)\, d\rho = 0.$$

Inside of circle Γ_{2n} the function $u(0,\rho)$ has only one simple pole, therefore

$$\frac{1}{\pi i}\int\limits_{\Gamma_{2n}} \rho u(0,\rho)\, d\rho = \frac{\beta}{2\pi i}\int\limits_{\Gamma_{2n}} \frac{\sin\rho}{2\cos\rho - \rho m \sin\rho} d\rho + \frac{1}{2\pi i}\int\limits_{\Gamma_{2n}} \frac{1}{2\cos\rho - \rho m\sin\rho} d\rho \times$$

$$\times \left[\int_{-1}^{0} f(\xi)\sin\rho(1+\xi)\, d\xi + \int_{0}^{1} f(\xi)\sin\rho(1-\xi)\, d\xi\right] d\rho =$$

$$= \frac{\beta \sin\rho_{2,n}}{-\sin\rho_{2,n} - \rho_{2,n} m\cos\rho_{2,n}} + \frac{1}{-\sin\rho_{2,n} - \rho_{2,n} m \cos\rho_{2,n}} \times$$

$$\times \left[\int_{-1}^{0} f(\xi)\sin\rho_{2,n}(1+\xi)\, d\xi + \int_{0}^{1} f(\xi)\sin\rho_{2,n}(1-\xi)\, d\xi\right] = a_n + b_n(c_n + d_n).$$

From asymptotic formula (4) for $\rho_{2,n}$ it follows that $a_n = 0\left(\frac{1}{n^2}\right)$, $b_n = 0\left(\frac{1}{n}\right)$. Moreover, $\{c_n\}, \{d_n\} \in l_2$. Hence, series

$$\sum_{n=0}^{\infty} \frac{1}{\pi i}\int_{\Gamma_n} \rho u(0,\rho)\, d\rho$$

absolutely converges. Thus, we have shown, that the series $\sum\limits_{n=0}^{\infty} \frac{1}{\pi i}\int_{\Gamma_n} \rho \hat{u}(x,\rho)\, d\rho$ unconditionally converges in the norm of the space norm $L_2(-1,1) \oplus \mathbb{C}$ that is equivalent to Riesz basicity of systems of eigenfunctions in this space. The theorem is proved.

5. Basicity in space $L_p(-1,1)$. From the results of the previous paragraph it follows that the system $\{u_n\}_{n=0}^{\infty}$ is overfull complete in space $L_p(-1,1)$. Therefore, in the given paragraph we shall answer to the question on how we can receive basis of space $L_p(-1,1)$ by eliminating some function from this system .

Theorem 2. *Excepting any n_0 even-numbered function from the system $\{u_n\}_{n=0}^{\infty}$ we get that received system will be basis in space $L_p(-1,1)$, $1 < p < \infty$ and be Riss basis in space $L_2(-1,1)$. If from this system to we eliminate any n_0 odd-numbered function $u_{n_0}(x)$ the received system will not be basis in space $L_p(-1,1) \oplus \mathbb{C}$.*

Proof. Let $\{\hat{u}_n\}_{n=0}^{\infty}$ be a system of eigenvectors of operator L and $\left\{\hat{\vartheta}_n\right\}_{n=0}^{\infty}$ be biorthogonal by adjoint system: $\hat{\vartheta}_n = (\vartheta_n(x), \overline{m}\vartheta_n(0))$. Since $\vartheta_n(x)$ are eigenfunctions of the spectral problem conjugate to (1), (2), then for them the following formula is true

$$\vartheta_{2n-1}(x) = \sin \pi n x, \quad n = 1, 2, ...,$$
$$\vartheta_{2n}(x) = \begin{cases} C_{2n}\sin\overline{\rho}_{2,n}(1+x) & \text{if} \quad x \in [-1,0], \\ C_{2n}\sin\overline{\rho}_{2,n}(1-x) & \text{if} \quad x \in [0,1]. \end{cases} \tag{47}$$

where C_{2n} are normalization numbers for which the following relation is true

$$C_{2n} = 1 + O\left(\frac{1}{n^2}\right). \tag{48}$$

Let's take any vector-function $\hat{f} \in L_p(-1,1) \oplus \mathbb{C}$, $\hat{f} \in (f(x);\ \beta)$ and expand it in biorthogonal series

$$\hat{f} = \sum_{n=0}^{\infty} < \hat{f},\ \ \hat{\vartheta}_n > \hat{u}_n = \sum_{n=0}^{\infty} \left((f, \vartheta_n) + \beta m \overline{\vartheta_n(0)} \right) \hat{u}_n. \tag{49}$$

Hence for the first component we shall receive

$$f = \sum_{n=0}^{\infty} \left((f, \vartheta_n) + \beta m \overline{\vartheta_n(0)} \right) u_n. \tag{50}$$

Let n_0 be an even number: $n_0 = 2k_0$. Then $\vartheta_{2k_0}(0) = C_{2k_0} \sin \overline{\rho_{2,k_0}} \neq 0$. We shall choose number β as follows:

$$\beta = -\frac{(f, \vartheta_{2k_0})}{m\vartheta_{2k_0}(0)}.$$

Then

$$f = \sum_{\substack{n=0 \\ n \neq n_0}}^{\infty} \left(f, \vartheta_n - \frac{\vartheta_n(0)}{\vartheta_{2k_0}(0)} \vartheta_{2k_0} \right) u_n = \sum_{\substack{n=0 \\ n \neq n_0}}^{\infty} (f, \vartheta_n^*)\, u_n, \tag{51}$$

where

$$\vartheta_n^* = \vartheta_n - \frac{\vartheta_n(0)}{\vartheta_{2k_0}(0)} \vartheta_{2k_0}, \quad n = 0, 1, ..., \quad n \neq n_0. \tag{52}$$

It is easy to check that $(u_j,\ \vartheta_n^*) = \delta_{jn}$, i.e. $\{\vartheta_n^*\}$ is a biorthogonally adjoint system to $\{u_n\}_{n=0,\ n\neq n_0}^{\infty}$, and any function $f \in L_p(-1,1)$ can be expanded in series (51). It means that the system $\{u_n\}_{n=0,\ n\neq n_0}^{\infty}$ is a basis of space $L_p(-1,1)$. If $p = 2$ then according to theorem 1 series (49) unconditionally converges in norm of space $L_p(-1,1) \oplus \mathbb{C}$. Then series (50) unconditionally converges in norm $L_p(-1,1)$. From formulas (5), (47), (48) and (52) it follows that

$$\|u_n\|_{L_p} \leq C, \qquad \|\vartheta_n^*\|_{Lq} \leq C.$$

Then the system $\{u_n\}_{n=0,\ n\neq n_0}^{\infty}$ is almost normalized and under Lorch-Gelfand's theorem [13] it is Riesz basis of space $L_2(-1,1)$.

If now $n_0 = 2k_0 + 1$, then $\vartheta_{2k_0+1}(0) = 0$ and

$$0 =< \hat{u}_n, \quad \hat{\vartheta}_{2k_0+1} >= (u_n, \vartheta_{2k_0+1}), \qquad n \neq n_0,$$

and we receive non-zero function $\vartheta_{2k_0+1}(x)$ which is orthogonal to all functions of system $\{u_n\}_{n=0,\ n\neq n_0}^{\infty}$, and it means, that the system $\{u_n\}_{n=0,\ n\neq n_0}^{\infty}$ is not complete in $L_p(-1,1)$ and all the more is not basis of this space.

The theorem is proved.

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Telman B. Gasymov
Institute of Mathematics and Mechanics of NAS of Azerbaijan
9, F.Agayev str., AZ1141, Baku, Azerbaijan
E-mail: telmankasumov@rambler.ru

Shakhrizad G. Mammadova
Kocatepe University, Afyon, Turkey

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