Sabir S. MIRZOYEV, Rovshan G. GASANOV

ON THE EXISTENCE OF GENERALIZED SOLUTION OF SOME BOUNDARY-VALUE PROBLEM FOR OPERATOR-DIFFERENTIAL EQUATIONS OF THE FOURTH ORDER ON THE SEGMENT

Abstract

In the paper the conditions providing the existence and uniqueness of some boundary-value problems are obtained for elliptic type operator-differential equations of the fourth order on the segment. There conditions are expressed only by the coefficients of the given equations.

In the separable Hilbert space H we consider the boundary-value problem

$$P(d/dt) \equiv \frac{d^4u}{dt^4} + A^4u + \sum_{j=0}^{4} A_j u^{4-j} = 0, \qquad t \in (0,1)\,, \tag{1}$$

$$u^{(j)}(0) = \varphi_j,\ u^{(j)}(1) = \psi_j, \quad j = 0,1, \tag{2}$$

where u(t) is a vector-valued function with the values in $H,\ \varphi_j,\ \psi_j,\ (j=0,1)$ are vectors from $H,\ A$ and $A_j\ \left(j=\overline{0,4}\right)$ are linear operators acting in H, all derivatives are understood in sense of theory of distributions [1].

In future we assume the fulfilment of the following conditions:

a) A is a positive definite selfadjoint operator;

b) The operators $B_j = A_j A^{-j}\ \ (j=0,1,2)$ and $D_j = A^{-2}A_j A^{2-j}\ \ (j=3,4)$ are bounded operators in H.

In the present paper we'll give the determination of generalized solution of problem (1), (2) and we'll prove the theorem on the existence and uniqueness of generalized solution of problem (1), (2).

Note that in infinite domain the similar problems are considered for example, in the papers [2-4]. In finite domain at completely continuous A^{-1} and at another conditions the solvability of problem (1), (2) is indicated in the paper [4], moreover these conditions are expressed in some abstract terms "conditions of solvability". In the present paper all conditions of solvability are expressed by the coefficients of equation (1) and are easily verifiable in concrete problems.

Let $H_\gamma\ (\gamma \geq 0)$ be scale of Hilbert spaces generated by the operators A, i.e. $H_\gamma = D(A^\gamma)\,,\ (x,y)_\gamma = (A^\gamma x, A^\gamma y)\,,\ x,y \in H_\gamma$.

Denote by $D([0,1];H)$ linear set of infinitely-differentiable vector-functions (by the norm H) with the values in H_4, having compact support in $[0,1]$. The set $D([0,1];H)$ provided with the norm

$$\|u\|^2_{W_2^2} = \left\|u''\right\|^2_{L_2} + \left\|A^2u\right\|^2_{L_2}\,,$$

is a pre-Hilbert space whose completion we denote by $W_2^2((0,1);H)$ (see [1], p.23). Note that here the Hilbert space

$$L_2\left((0,1);H\right)=\left\{f:\int\limits_0^1\|f(t)\|^2\,dt=\|f(t)\|^2_{L_2}<\infty\right\}.$$

Note we determine the following sub-space of the space $D\left([0,1];H\right)$:

$$D_0\left([0,1];H\right)=\left\{g:g\in D\left([0,1];H\right),\ g^{(j)}(0)=g^{(j)}(1)=0,\ j=0,1\right\}.$$

From the trace theorem the space

$$\overset{\circ}{W}{}_2^2\left((0,1);H\right)=\left\{u:u\in W_2^2\left((0,1);H\right),\ u^{(j)}(0)=u^{(j)}(1)=0,\ j=0,1\right\}$$

is a Hilbert (complete) space [1, p.133] and $D_0\left([0,1];H\right)$ everywhere is a dense set in $\overset{\circ}{W}{}_2^2\left((0,1);H\right)$.

For determination of generalized solution of problem (1), (2) we need the following

Lemma 1. *Let conditions a) and b) be fulfilled. Then the bilinear functional*

$$\mathcal{P}(u,g)=\left(P(d/dt)\,u,g\right)_{L_2}$$

continues by the continuity on the space $W_2^2\left((0,1);H\right)\times\overset{\circ}{W}{}_2^2\left((0,1);H\right)$ *as functional*

$$\mathcal{P}(u,g)=(u,g)_{W_2^2}+\mathcal{P}_1(u,g) \tag{3}$$

from the space $D\left([0,1];H\right)\times D_0\left([0,1];H\right)$. *Here*

$$(u,g)_{W_2^2}=\left(u'',g''\right)_{L_2}+\left(A^2u,A^2g\right)_{L_2}$$

and

$$\mathcal{P}_1(u,g)=\sum_{j=0}^{2}\left(A_ju^{(2-j)},g''\right)_{L_2}+\sum_{j=3}^{4}\left(A_ju^{(4-j)},g\right)_{L_2}. \tag{4}$$

Proof. Since $u\in D\left([0,1];H\right),\ g\in D_0\left([0,1];H\right)$, then integrating by parts we find:

$$\left(P(d/dt)\,u,g\right)_{L_2}=\left(P_0(d/dt)\,u,g\right)_{L_2}+\left(P_1(d/dt)\,u,g\right)_{L_2}=$$

$$=\left(\frac{d^4u}{dt^4}+A^4u,g\right)_{L_2}+\left(\sum_{j=0}^{4}A_ju^{(4-j)},g\right)_{L_2}=\left(\frac{d^2u}{dt^2},\frac{d^2g}{dt^2}\right)_{L_2}+\left(A^2u,A^2g\right)_{L_2}+$$

$$+\left(\sum_{j=0}^{2}A_ju^{(2-j)},g''\right)_{L_2}+\left(\sum_{j=3}^{4}A_ju^{(4-j)},g\right)_{L_2}=(u,g)_{W_2^2}+\mathcal{P}_1(u,g). \tag{5}$$

Since the operators B_j $\left(j=\overline{0,2}\right)$ and D_j $(j=3,4)$ are bounded, using the theorem on intermediate derivatives [1, p.28] we have:

$$\left|\left(A_j u^{(2-j)}, g''\right)_{L_2}\right| = \left|\left(A_j A^{-j} A^j u^{(2-j)}, g''\right)_{L_2}\right| \leq$$

$$\leq \|B_j\| \left\|A^j u^{(2-j)}\right\|_{L_2} \|g''\|_{L_2} \leq \|B_j\| \|u\|_{W_2^2} \|g\|_{W_2^2}, \qquad j=\overline{0,2}, \tag{6}$$

and again by the theorem of intermediate derivatives we obtain:

$$\left|\left(A_j u^{(4-j)}, g\right)_{L_2}\right| = \left|\left(A^{-2} A_j A^{2-j} A^{j-2} u^{(4-j)}, A^2 g\right)_{L_2}\right| \leq$$

$$\leq \|D_j\| \left\|A^{j-2} u^{(4-j)}\right\| \left\|A^2 g\right\|_{L_2} \leq \|D_j\| \|u\|_{W_2^2} \|g\|_{W_2^2}, \qquad (j=3,4). \tag{7}$$

allowing for (6) and (7) in (5) we complete the proof of the lemma.

Definition. *The vector-function $u(t) \in W_2^2((0;1);H)$ is called generalized solution of problem (1), (2) if for any vector-function $g(t) \in \overset{\circ}{W}{}_2^2((0,1);H)$ the identity*

$$P(u,g) \equiv (u,g)_{L_2} + \mathcal{P}_1(u,g) = 0$$

is fulfilled and for any φ_j, $\psi_j \in H_{2-j-1/2}$ $(j=0,1)$ it holds the equality:

$$u^{(j)}(0) = \varphi_j, \; u^{(j)}(1) = \psi_j, \quad j=0,1.$$

At first we consider the elementary problem

$$P_0(d/dt)\, u(t) = \frac{d^4 u}{dt^4} + A^4 u = 0, \tag{8}$$

$$u^{(j)}(0) = \varphi_j, \qquad u^{(j)}(1) = \psi_j, \quad \varphi_j, \psi_j \in H_{2-j-1/2} \qquad (j=0,1) \tag{9}$$

It holds

Theorem 1. *Problem (8), (9) has generalized solution.*

Proof. Denote by $\omega_1, \omega_2, \omega_3, \omega_4$ the roots of the equation $\lambda^4+1=0$, moreover $\operatorname{Re}\omega_1<0$, $\operatorname{Re}\omega_2<0$, $\operatorname{Re}\omega_3>0$, $\operatorname{Re}\omega_4>0$. Then the general solution of equation (8) from the class $W_2^2((0;1);H)$ has the form

$$u_0(t) = e^{\omega_1 t A} c_1 + e^{\omega_2 t A} c_2 + e^{\omega_3 (1-t) A} c_3 + e^{\omega_4 (1-t) A} c_4, \tag{10}$$

where $c_1, c_2, c_3, c_4 \in H_{3/2}$. It is obvious that each addend from equality (10) satisfies the relation $(u_0, g)_{W_2^2} = 0$. Now we select the coefficients c_j $\left(j=\overline{1,4}\right)$ such that equalities (9) be fulfilled. Then for the determination of φ_j and ψ_j $(j=0,1)$ we obtain the system of equations

$$\begin{cases} c_1 + c_2 + e^{\omega_3 A} c_3 + e^{\omega_4 A} c_4 = \varphi_1, \\ \quad \omega_1 c_1 + \omega_2 c_2 + \omega_3 e^{\omega_3 A} c_3 + \omega_4 e^{\omega_4 A} c_4 = A^{-1}\varphi_2, \\ e^{\omega_1 A} c_1 + e^{\omega_2 A} c_2 + c_3 + c_4 = \psi_1, \\ \quad \omega_1 e^{\omega_1 A} c_1 + \omega_2 e^{\omega_2 A} c_2 + \omega_3 c_3 + \omega_4 c_4 = A^{-1}\psi_2. \end{cases} \tag{11}$$

Since the basis determinant-matrix is convertible in H^4 and the vectors in the right part of the system of equations belongs to the space $H_{3/2}$, hence we easily conclude that the vectors c_1, c_2, c_3, c_4 belong to the space $H_{3/2}$ and they are uniformly determined by the vectors $\varphi_1, \varphi_2, \psi_1$ and ψ_2.

Thus, problem (8), (9) has a unique generalization solution. The theorem is proved.

Now, we'll prove the existence and uniqueness of generalized solution of problem (1), (2).

Theorem 2. *Let conditions a) and b) be fulfilled and it hold the inequality*

$$\alpha = \sum_{j=0}^{2} m_j \|B_j\| + \sum_{j=3}^{4} m_j \|D_j\| < 1,$$

where $m_0 = m_4 = 1;\ m_2 = \frac{1}{2},\ m_1 = m_3 = \frac{1}{\sqrt{2}}$.

Then problem (1), (2) has generalized solution.

Proof. According to theorem 1 problem (8), (9) has a unique solution $\upsilon(t) \in W_2^2((0,1);H)$. We'll search the generalized solution of problem (1), (2) in the form $u = \upsilon + \theta$. Then it is obvious that $\theta \in W_2^2((0,1);H)$. Since at $g \in \overset{\circ}{W}{}_2^2((0,1);H)$ it holds the equality

$$(\upsilon, g)_{W_2^2((0,1);H)} = 0,$$

then substituting the vector-function $u = \upsilon + \theta$ in to the identity $\mathcal{P}(u,g) = 0$ we obtain:

$$\langle \theta, g \rangle \equiv (\theta, g)_{W_2^2} + \mathcal{P}_1(\theta, g) = -\mathcal{P}_0(\upsilon, g), \tag{12}$$

where $g \in \overset{\circ}{W}{}_2^2((0,1);H)$.

Show that at $g \in \overset{\circ}{W}{}_2^2((0,1);H)$ it holds the inequality

$$|\mathcal{P}(g,g)| \geq (1-\alpha)\|g\|_{W_2^2}^2. \tag{13}$$

Really, at $g \in \overset{\circ}{W}{}_2^2((0,1);H)$ we have

$$|\mathcal{P}(g,g)| \geq \|g\|_{W_2^2}^2 - |\mathcal{P}_1(g,g)|. \tag{14}$$

Since (see formula (4))

$$\mathcal{P}_1(g,g) = \sum_{j=0}^{2} \left(A_j g^{(2-j)}, g''\right)_{L_2} + \sum_{j=3}^{4} \left(A_j g^{(4-j)}, g\right)_{L_2}, \tag{15}$$

then we'll estimate each addend in equality (15).

It is obvious that at $j = 0$

$$\left|\left(A_0 g'', g''\right)_{L_2}\right| \leq \|A_0\| \left|g''\right|_{L_2}^2 = \|B_0\| \|g\|_{W_2^2}. \tag{16}$$

At $j = 1$ we have

$$\left|\left(A_1 g', g''\right)_{L_2}\right| \leq \left|\left(A_1 A^{-1} A g', g''\right)_{L_2}\right| \leq \|B_1\| \left\|A g'\right\|_{L_2} \left\|g''\right\|_{L_2}. \tag{17}$$

On the other hand, integrating by parts we obtain:

$$\left\|A g'\right\|_{L_2}^2 = \int_0^1 \left(A g', A g'\right) dt = -\int_0^1 \left(A^2 g, g''\right) dt \leq$$

$$\leq \left\|A^2 g\right\|_{L_2} \left\|g''\right\|_{L_2} \leq \frac{1}{2}\left(\left\|A^2 g\right\|_{L_2}^2 + \left\|g''\right\|_{L_2}^2\right) = \frac{1}{2}\|g\|_{W_2^2}^2. \tag{18}$$

Allowing for inequality (18) in (17) we obtain that

$$\left|\left(A_1 g', g''\right)_{L_2}\right| \leq \frac{1}{\sqrt{2}}\|B_1\| \|g\|_{W_2^2}. \tag{19}$$

At $j = 2$ we obtain:

$$\left|\left(A_2 g', g''\right)_{L_2}\right| = \left|\left(A_2 A^{-2} A^2 g, g''\right)_{L_2}\right| \leq \|B_2\| \left\|A^2 g\right\| \left\|g''\right\|_{L_2} \leq$$

$$\leq \frac{1}{2}\|B_2\| \left(\left\|A^2 g\right\|_{L_2}^2 + \left\|g''\right\|_{L_2}^2\right) = \frac{1}{2}\|B_2\| \|g\|_{W_2^2}^2. \tag{20}$$

Further using inequality (18) at $j = 3$ we have:

$$\left|\left(A_3 g', g\right)_{L_2}\right| = \left|\left(A^{-2} A_3 A^{-1} A g', A^2 g\right)_{L_2}\right| \leq \|D_3\| \left\|A g'\right\|_{L_2} \left\|A^2 g\right\|_{L_2} \leq$$

$$\leq \|D_3\| \frac{1}{\sqrt{2}} \|g\|_{W_2^2} \left\|A^2 g\right\|_{L_2} = \frac{1}{\sqrt{2}} \|D_3\| \|g\|_{W_2^2}^2. \tag{21}$$

And at $j = 4$ we obtain analogously:

$$\left\|\left(A_4 g, g\right)_{L_2}\right\| = \left|\left(A^{-2} A_4 A^{-2} A^2 g, A^2 g\right)_{L_2}\right| \leq$$

$$\leq \|D_4\| \left\|A^2 g\right\|_{L_2}^2 \leq \|D_4\| \|g\|_{W_2^2}^2. \tag{22}$$

Allowing for inequalities (16), (17) and (19)-(22) in equality (15) we find that

$$\left|\mathcal{P}_1 (g, g)_{L_2}\right| \geq \alpha \|g\|_{W_2^2}^2$$

(α is determined from the condition of the theorem).

Then

$$\left|\mathcal{P}_1 (g, g)_{L_2}\right| \geq (1 - \alpha) \|g\|_{W_2^2}^2.$$

Now using this fact that the right-hand side of equality (12) determines the continuous functional in the space $W_2^2((0,1);H) \oplus \overset{\circ}{W}{}_2^2((0,1);H)$, and the left-hand side $\langle \theta, g \rangle$ at $\theta = g$ satisfies the condition

$$|\langle g, g\rangle| \geq (1-\alpha)\, \|g\|_{W_2^2},$$

we obtain by Lax-Milgram theorem [5, p.188] that there exists a unique vector-function $\theta(t) \in \overset{\circ}{W}{}_2^2((0,1);H)$ which satisfies (12). Thus, $u = v + \theta$ will be desired generalized solution.

The Theorem is proved.

References

[1]. Lions J.-L., E.Majenes. *Non-homogeneous boundary-value problems and their applications.* M.: "Mir", 1971,371p. (Russian)

[2]. Ustinov Yu.A., Yudovich V.I. *On completeness of the system of elementary solutions of biharmonic equaiton in half-string.* Prikl. mat. i mech., 1973, v.37, pp.706-714. (Russian)

[3]. Orazov M.B. *On completeness of the systems of elementary solutions for some operator differential equations on semi-axis and on segment.* DAN SSSR, 1979, v.245, No4, pp.788-792. (Russian)

[4]. Mirzoyev S.S. *On generalized solutions of boundary-value problems for operator-differential equations.* Applied problems of functional analysis. Baku, 1987, pp.71-80. (Russian)

[5]. Bers L., John F., Shekhter M. *Partial equations.* M.: "Mir", 1966, 350p. (Russian)

Sabir S. Mirzoyev, Rovshan G. Gasanov

Institute of Mathematics and Mechanics of NAS of Azerbaijan
9, F.Agayev str., AZ1141, Baku, Azerbaijan
Tel.: (99412) 439 47 20 (off.)

Received September 05, 2005; Revised October 28, 2005.
Translated by Mamedova V.A.