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BEHAVIOR OF SOLUTIONS OF FIRST MIXED PROBLEM FOR DIVERGENT PARABOLIC EQUATIONS

Abstract

The behavior of solutions of the first mixed problem for divergent parabolic equations is investigated.

In the paper the questions on stabilization of solutions of the first mixed problem for divergent parabolic equations with the lowest coefficients are considered. Conditions that connect the coefficients of equation with the geometry of the domain, providing the stabilization of solutions are found. At that, class of domains, for which the estimation defined by the geometry of domain is established, is distinguished. The lower and upper estimations of the solutions, which show an exact order of the solution growth are established. The Harnack inequality under the definite conditions on the coefficients, connected with the geometry of domain is proved. The Cauchy problem is considered.

1. Let $\Omega \subset R^n, n \geq 2$ be an arbitrary unbounded domain, $x = (x_1, ..., x_n)$ be a point of this space. In the cylindrical domain $D = \Omega \times (t > 0)$ consider the first mixed problem for the second order linear parabolic equation

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}(x,t) u_{x_j} \right) + \sum_{i=1}^{n} b_i(x,t) \frac{\partial u}{\partial x_i} + c(x,t) u \tag{1}$$

$$u|_{\partial\Omega} = 0 \tag{2}$$

$$u|_{t=0} = \varphi(x). \tag{3}$$

Concerning the coefficients, it is supposed, that they are measurable functions, satisfying for almost all $(t,x) \in D$ and all $\xi \in R^n$ the conditions

$$a_0 |\xi|^2 \leq a_{ij}(x,t) \xi_i \xi_j \leq A_0 |\xi|^2 \tag{4}$$

$$|b(x,t)| \leq m, \quad c(x,t) \leq -c_0, \tag{5}$$

where a_0, A_0, m, c_0 are positive constants.

With respect to the initial function we'll suppose, that $\varphi(x)$ belongs to the space $L_2(\Omega)$ and has the compact support, i.e. $\varphi(x) = 0$ for all

$$x \in \{|x| > R_0\}.$$

The questions of stabilization for solutions of the first mixed problem for linear divergent equations were studied by Yu.N.Cheromnykh [1], F.Kh.Mukminov [2], for second mixed problem by A.K.Gushin [3], [4], for the third mixed problem in noncylindrical domain by V.I.Ushakov [5], for high order linear divergent equations by F.Kh.Mukminov, I.M. Bikkulov [6], for high order nonlinear equations by A.F.Tedeev[7].

For example, in the A.K.Gushin paper the following result for solution of mixed problem in the wide classes of the domain Ω has been obtained:

$$|u(t,x)| \leq \frac{c\,\|\varphi\|_{L_2(\Omega)}}{V(\sqrt{t})},$$

where $V(r)$is a volume of the intersection $\Omega_r = B_r \cap \Omega,\ B_r = \{x \in R^n : |x| < r\}$.

For the second mixed problem the speed of stabilization of solution decreases during the decrease of ‘opening of domain Ω at the infinity", but for the first mixed problem the situation is opposite. Really, let the domain $\Omega \subset R^2$ contain angular point with the opening α. Then the solution of the first mixed problem for the heat equation with nonnegative finite initial function decreases as $t \to \infty$ at $t^{-(\pi/\alpha+1)}$.

Let $\Omega_r = \Omega \cap B_r, \quad B_r = \{x \in R^n, |x| < r\}$, and denote by $\lambda(r), r > 0$ the basic frequency of the set Ω_r

$$\lambda(r) = \inf \left(\int\limits_{\Omega_r} |\nabla \eta|^2 \, dx \right) \left(\int\limits_{\Omega_r} \eta^2 dx \right)^{-1},$$

where the lower bound is taken by all functions $\eta \in \overset{0}{W}{}_2^1(\Omega_r)$. $\lambda(r)$is the first eigen value of the Direchlet problem for the Laplace operator in Ω_r. $\lambda(r)$ is a continuous monotone nonincreasing function.

With respect to the domain suppose that the following conditions are fulfilled

$$\lim_{r\to\infty} r^2 \lambda(r) = \infty, \qquad \lim_{r\to\infty} \lambda(r) = 0. \tag{6}$$

It is possible to indicate estimations of the function $\lambda(r)$, characterizing its behavior as $r \to \infty$. For example, if Ω is the convex unbounded domain

$$\Omega = \left\{x \in R^n, x = (x_1, x'), \left|x'\right| < f(x_1), x_1 > 0\right\},$$

where f is a continuous monotone increasing function, then for all $r \geq 1$ the inequality $c^{-1} f^{-2}(r) \leq \lambda(r) \leq c f^{-2}(r)$ is true.

It is possible to introduce the concept of nonlinear basic frequency, which is applied during investigation of nonlinear equations. The results for nonlinear equations will be reduced by us in the other papers. Mark, that the properties of nonlinear basic frequency were studied in V.M.Miklukov paper [8], and many important properties for different classes of domains in G.Polea, P.Sege paper [9].

With the function $\lambda(r)$ we'll also consider the function $\mu(r)$, for which the lower bound is taken by all functions $\eta \in \overset{0}{W}{}_2^1(\Omega)$. Note, that $\mu(r)$ and $\lambda(r)$ as $r \to \infty$ behave equally, namely

$$\beta(\lambda)(r-1) \leq \mu(r) \leq \lambda(r), \beta = const > 0.$$

Before passing to the main result, we'll represent some auxiliary statements.

Lemma 1. *Let* $u(x) \in \overset{0}{W}{}_2^1(\Omega)$ *and* $g(r)$ *be measurable locally bounded in* Ω_r *functions. Then the following estimation*

$$\int\limits_{\Omega_r} |u|^2 \, g(r) dx \leq \int\limits_{\Omega_r} \lambda^{-2}(r) \, |\nabla u|^2 \, g(r) dx \tag{7}$$

is true.

The proof of the lemma follows from the A.S.Kronrod, Y.M.Landis paper [10].

The generalized solution of problem (1)-(3) in $D_T = \Omega \times (0, T)$ is called the function $u(t,x) \in \overset{0}{W}{}_2^{1,1}$, satisfying the integral identity

$$\int_{D_T} u_t v dx dt + \int_{D_T} \left(\sum_{i,j=1}^n a_{ij}(x,t) u_{x_i} v_{x_j} + \sum_{i=1}^n b_i(x,t) u_{x_i} v + c(x,t) uv \right) dx dt = 0. \quad (8)$$

and initial condition (3) for all $v(t,x) \in \overset{0}{W}{}_2^{1,1}(D_T)$. The space $\overset{0}{W}{}_2^{1,1}(D_T)$ is a completion by the norm

$$\|u\|_{W_2^{1,1}(D_T)} = \left(\int_{D_T} \left(\sum_{i=1}^n |u_{x_i}|^2 + u_t^2 + u^2 \right) dx dt \right)^{\frac{1}{2}}$$

of the set of infinitely differentiable functions from $C^\infty(D_T)$, vanishing at the neighbourhood of $\partial\Omega \times (0, T)$. The function $u(t,x)$ is a solution of problem (1)-(3) in D, if at all $T > 0$ it is the solution of the same problem in D_T.

We'll suppose, that the solution of problem (1)-(3) exists and unique. It is proved by the standard way under some conditions on smoothness of coefficients.

Lemma 2. *Let $u(t,x) \in \overset{0}{W}{}_2^{1,1}(D)$ be a solution of problem (1)-(3) in D. Then $u(t,x)$ belongs to $C([0,\infty) \to L_2(\Omega))$ and for almost all $t \geq 0$ and*

$$\int_\Omega u^2(t,x) dx = \int_\Omega \varphi^2 dx -$$

$$-2 \int_0^t \int_\Omega \left(\sum_{i,j=1}^n a_{ij}(x,\tau) u_{x_i} u_{x_j} + \sum_{i=1}^n b_i(x,\tau) u_{x_i} u + c(x,\tau) u^2 \right) dx d\tau. \quad (9)$$

Proof. For proof it is sufficient to substitute to integral identity (8) the test function $v = u(t,x), T = t$, and to integrate by parts the obtained expression.

Note, that from lemma 2 it follows, that the function

$$\int_\Omega u^2(t,x) dx$$

is absolutely continuous and the equality

$$\frac{d}{dt} \int_\Omega u^2(t,x) dx =$$

$$= -2 \int_\Omega \left(\sum_{i,j=1}^n a_{ij}(x,t) u_{x_i} u_{x_j} + \sum_{i=1}^n b_i(x,t) u_{x_i} u + c(x,t) u^2 \right) dx \quad (10)$$

is fulfilled.

Denote $E_R(t)=\int\limits_{\Omega\backslash\Omega_R} u^2(t,x)dx+\overline{a_0}\int\limits_0^t\int\limits_{\Omega\backslash\Omega_R}\sum\limits_{i=1}^n |u_{x_i}|^2\,dxd\tau$, $\overline{a_0}$ is a constant, whose estimation will be given below.

Lemma 3. *Let $u(t,x)\in \overset{0}{W}{}_2^{1,1}(D)$ be a solution of problem (1)-(3) in D. With respect to coefficients suppose, that conditions (4), (5) and condition*

$$a_0>\frac{m}{2}\left(1+\sup_{0<r<R}\lambda^{-2}(r)\right)-c_0\sup_{0<r<R}\lambda^{-2}(r) \tag{11}$$

are fulfilled. Moreover, suppose, that $u(t,x)$is bounded, $\sup\limits_{\overline{D}}|u(t,x)|\le M$. Then the estimation

$$E_R(t)\le c_1\,\|\varphi\|^2_{L_2(\Omega)}\exp\left(-\gamma_1\left[\frac{(R-R_0)^2}{t}\right]\right). \tag{12}$$

holds for any $R>R_0$, where c_1,γ_1 are positive constants, independent of R and t.

Proof. Let us consider the following patch function $\eta_r(\xi),\xi\in(0,\infty)$ is continuously differentiable function such that $\eta_r(\xi)=0$ as $0<\xi\le R$, $0\le\eta_r(\xi)\le 1$ as $R<\xi<R+r$, $\eta_r(\xi)=1$ as $R+r<\xi<\infty$ for any $R>R_0, r>0$, and $\left|\frac{d\eta_r}{d\xi}\right|\le Kr^{-1}$.

To the integral identity (8) as a test function we'll substitute $v=u(t,x)\eta_r^s(|x|)$, $s>2$. Then we have

$$\frac{1}{2}\int\limits_\Omega u^2(t,x)\eta_r^s dx-\frac{1}{2}\int\limits_\Omega \varphi^2(x)\eta_r^s dx+\int\limits_0^t\int\limits_\Omega\left(\sum_{i,j=1}^n a_{ij}(x,t)u_{x_i}u_{x_j}\eta_r^s\right)dxd\tau=$$

$$=\int\limits_0^t\int\limits_\Omega\left(\sum_{i,j=1}^n a_{ij}(x,t)u_{x_i}u\eta_{r_{x_i}}\eta_r^{s-1}+\sum_{i=1}^n b_i(x,t)u_{x_i}u\eta_r^s+c(x,t)u^2\eta_r^s\right)dxd\tau \tag{13}$$

From (13), taking into account the conditions on coefficients, we have

$$\int\limits_{\Omega\backslash\Omega_{R+r}} u^2(t,x)dx+2a_0\int\limits_0^t\int\limits_\Omega |u_{x_i}|^2\eta_r^s dxd\tau\le A_0\int\limits_0^t\int\limits_\Omega |u_{x_i}|\,|u|\left|\eta_{r_{x_i}}\right|\eta_r^{s-1}dxd\tau+$$

$$+m\int\limits_0^t\int\limits_\Omega |u_{x_i}|\,|u|\,\eta_r^s dxd\tau-c_0\int\limits_0^t\int\limits_\Omega u^2(t,x)\eta_r^s dxd\tau \tag{14}$$

Estimate the conditions on the right-hand side of (14). First of all we'll estimate the last two integrals using lemma 1 and applying Young inequality with $\epsilon=1$, then we have

$$m\int\limits_0^t\int\limits_\Omega |u_{x_i}|\,|u|\,\eta_r^s dxd\tau\le\frac{m}{2}\int\limits_0^t\int\limits_\Omega |u_{x_i}|^2\eta_r^s dxd\tau+\frac{m}{2}\int\limits_0^t\int\limits_\Omega |u|^2\eta_r^s dxd\tau\le$$

$$\le\frac{m}{2}\int\limits_0^t\int\limits_\Omega |u_{x_i}|^2\eta_r^s dxd\tau+\frac{m}{2}\int\limits_0^t\int\limits_\Omega \lambda^{-2}(r)\,|u_{x_i}|^2\eta_r^s dxd\tau\le\frac{m}{2}\int\limits_0^t |u_{x_i}|^2\eta_r^s dxd\tau+$$

$$+\frac{m}{2}\sup_{0<r<R}\lambda^{-2}(r)\int_0^t\int_\Omega |u_{x_i}|^2\eta_r^s dxd\tau \tag{15}$$

$$-c_0\int_0^t\int_\Omega u^2(t,x)\eta_r^s dxd\tau \le -c_0\int_0^t\int_\Omega \lambda^{-2}(r)|u_{x_i}|^2\eta_r^s dxd\tau \le$$

$$\le -c_0\sup_{0<r<R}\lambda^{-2}(r)\int_0^t\int_\Omega |u_{x_i}|^2\eta_r^s dxd\tau \tag{16}$$

Pass to estimation of the first integral in (14). For that we'll apply the Young inequality $ab\le\frac{\epsilon}{2}a^2+\frac{1}{2\epsilon}b^2$ with $\epsilon=\frac{a_0}{2A_0}$ we have taken into account, that $\left|\eta_{r_{x_i}}\right|\le k\eta_r^{s-1}r^{-1}$

$$A_0\int_0^t\int_\Omega |u_{x_i}||u|\eta_{r_{x_i}}\eta_r^{s-1}dxd\tau\le$$

$$\le a_0\int_0^t\int_\Omega |u_{x_i}|^2\eta_r^s dxd\tau + c_1\int_0^t\int_\Omega |u|^2\left|\eta_{r_{x_i}}\right|^2\eta_r^{s-2}dxd\tau\le$$

$$\le a_0\int_0^t\int_\Omega |u_{x_i}|^2\eta_r^s dxd\tau + c_1\int_0^t\int_{\Omega\backslash\Omega_R}|u|^2 r^{-2}dxd\tau, \tag{17}$$

where $c_1=A_0^2/a_0$.

Taking into account estimations (15), (16), (17) in (14) and denoting

$$\overline{a_0}=a_0-\frac{m}{2}\left(1+\sup_{0<r<R}\lambda^{-2}(r)\right)-c_0\sup_{0<r<R}\lambda^{-2}(r),$$

we have

$$\int_{\Omega\backslash\Omega_{R+r}}u^2(t,x)dx+\overline{a_0}\int_0^t\int_{\Omega\backslash\Omega_{R+r}}|u_{x_i}|^2dxd\tau\le c_1\int_0^t\int_{\Omega\backslash\Omega_{R+r}}|u|^2r^{-2}dxd\tau \tag{18}$$

Estimate the integral on the right-hand side of (18) using the Nurenberg-Galyardo inequality

$$\int_0^t\int_{\Omega\backslash\Omega_R}|u|^2r^{-2}dxd\tau\le$$

$$\le r^{-2}\left(\epsilon t\int_0^t\int_{\Omega\backslash\Omega_R}|u_{x_i}|^2dxd\tau+\epsilon^{-1}c_2\int_0^t\int_{\Omega\backslash\Omega_R}u^2dxd\tau\right), \tag{19}$$

where c_2, ϵ are positive constants. Taking into account (19) in (18), we'll obtain

$$E_{R+r}(t) \le c_3 r^{-2} \left(\epsilon t E_R(t) + \epsilon^{-1} \int_0^t E_R(\tau) d\tau \right), \tag{20}$$

$c_3 = \max\{1, c_2\}$.

Suppose, that $(R - R_0)^2 t^{-1} > \gamma$, if it is true the inverse inequality, then the required estimation (12) is obvious. Let the parameters r and ϵ satisfy the inequality $r^{-2} t\epsilon < c_4$. Then inequality (20) can be written in the form

$$E_{R+r}(t) \le c_5 \left(\epsilon t r^{-2} E_R(t) + \epsilon^{-1} r^{-2} \int_0^t E_R(\tau) d\tau \right), \tag{21}$$

where $c_5 = c_3 \max\{1, c_4\}$.

Assume in (21) $\epsilon = 1, R = R_0$. As $E_{R_0+r}(t) < E_{R_0}(t)$, then $E_{R+r}(t) \le \|\varphi\|^2_{L_2(\Omega)}$. Suppose, that

$$E_{R_0+kr}(t) \le \frac{\|\varphi\|^2_{L_2(\Omega)} \cdot c_6^k \cdot r^{-2k} \cdot t^k}{(k+1)!}, \tag{22}$$

where $c_6 = 2c_5$.

In (21) assume $\epsilon = (k+1)^{-1}$, then we have

$$E_{R_0+(k+1)r}(t) \le c_5 r^{-2} \left(\frac{t \cdot c_6^k r^{-2k} t^k \|\varphi\|^2_{L_2(\Omega)}}{(k+1)\{(k+1)!\}} + \frac{\|\varphi\|^2_{L_2(\Omega)} (k+1) \cdot c_6^k t^{(k+1)}}{(k+1)\{(k+1)!\}} \right) \le$$

$$\le \frac{\|\varphi\|^2_{L_2(\Omega)} \cdot c_6^{k+1} r^{-2(k+1)} t^{(k+1)}}{\{(k+2)!\}}$$

So, inequality (22) is proved at any $k \ge 1$. Let us take $r = (R - R_0)/k$. Then from the Stirling formula we have $(k+1)! \ge \lambda_0 k^k \exp(-k)$, here λ_0 is some absolute constant. Therefore inequality (22) can be rewritten in the form

$$E_R(t) \le c_7 \|\varphi\|^2_{L_2(\Omega)} \cdot \lambda_0^{-1} \exp\left\{ -k \ln \left[\frac{(R - R_0)^2}{c_6 \cdot t \cdot k \exp(1)} \right] \right\}$$

Now assuming, that k is equaled to the whole part of the expression

$$(R - R_0)^2 \cdot c_6^{-1} \cdot t^{-1} \exp(-1),$$

from the last inequality we'll obtain

$$E_R(t) \le c_8 \cdot \|\varphi\|^2_{L_2(\Omega)} \exp\left\{ -\frac{1}{c_6 \exp(2)} \left[(R - R_0)^2 \cdot t^{-1} \right] \right\},$$

That proves estimation (12).

Introduce the function $F(r) = r/(\lambda(r))^{\frac{1}{2}}, r > 0$. This function is a monotone increasing continuous function. Denote by $r(t), t > 0$ the inverse to $F(r)$ function. Note, that $r(t)/\sqrt{\lambda(r(t))} = t$, and, hence,

$$\frac{r^2(t)}{t} = \lambda(r(t)) \cdot t = r(t)\sqrt{\lambda(r(t))} \tag{23}$$

It is possible to indicate the estimations, characterizing behavior of the functions $\lambda(r), F(r)$ as $r \to \infty$. In particular, if is the convex unbounded domain

$$\Omega = \left\{x \in R^n, x = (x_1, x') : \left|x'\right| < f(x_1), x_1 > 0\right\},$$

where f is continuous monotone increasing on $[0, +\infty)$ function, for all $r \geq 1$ the inequalities

$$c^{-1} f^{-2}(r) \leq \lambda(r) \leq c f^{-2}(r) \tag{24}$$

are true.

Conditions (6) in this case will take the form

$$\lim_{r \to \infty} \frac{r}{f(r)} = \infty, \ \lim_{r \to \infty} f(r) = \infty$$

And function $F(r)$ by virtue of (24) satisfies the inequality

$$c^{-\frac{1}{2}} r f(r) \leq F(r) \leq c^{\frac{1}{2}} r f(r), r \geq 1,$$

As for any $r > 0$ and any $\vartheta > 1$ the inequality $\vartheta r f(r) \leq r f(\vartheta r)$ is true, and consequently the inverse to $r f(r)$ the function $\tilde{r}(t)$for any $t > 0, \vartheta > 1$ satisfies the inequality $\tilde{r}(\vartheta t) \leq \vartheta \tilde{r}(t)$, then for the function $r(t)$ we have the estimations

$$c^{-\frac{1}{2}} \tilde{r}(t) \leq \tilde{r}(t \cdot c^{-\frac{1}{2}}) \leq r(t) \leq \tilde{r}(c^{\frac{1}{2}} t) \leq c^{\frac{1}{2}} \tilde{r}(t), \ t \geq F(1)$$

Theorem 1. *Let $u(t,x) \in \overset{0}{W}{}_2^{1,1}(D)$ be a solution of problem (1)-(3) in D. Suppose that with respect to coefficients, conditions (4), (5), and (11), and with respect to domain, condition (6) are fulfilled, and $\varphi = 0$ at $|x| > R_0$. Then at sufficiently large $t > T$*

$$\int_\Omega u^2(t,x)dx \leq C_2 \exp\left\{-\gamma_2 \left[\frac{r^2(t)}{t}\right]\right\} \tag{25}$$

where the constants C_2, γ_2 are independent of t.

Theorem 2. *Let $u(t,x)$be bounded solution of problem (1)-(3) in the domain*

$$D = \left\{x \in R^n, \left|x'\right| < x_1^\alpha, 0 < \alpha < (p+2)/(2p + (p-2)(n-1))\right\} \times \{t > 0\},$$

where

$$p > 2, \varphi(x) = 0 \quad at \ \ |x| > R_0.$$

With respect to coefficients we'll require the fulfillment of conditions (4), (5), (11). Then there is a positive constant $\beta = \beta(n, \alpha - p)$ such that, at sufficiently large t it holds the inequality

$$\int_\Omega u^2(t,x)dx \leq C_3 t^{-\beta} \tag{26}$$

where $C_3 = const > 0$.

Remark 1. *Let in theorem 1 the domain Ω have the form*

$$\Omega = \left\{x \in R^n, \left|x'\right| < x_1^\alpha, 0 < \alpha < 1\right\}.$$

Then estimation (25) has the form

$$\int\limits_{\Omega} u^2(t,x)dx \le C_2 \exp\left\{-\gamma_2 t^{\frac{(1-\alpha)}{(1+\alpha)}}\right\}. \qquad (27)$$

Proof. Theorems 1 and 2 are proved similarly. Therefore let us prove one of them. Prove theorem 2.

Denote $r(t) = t^{\vartheta}$, where ϑ satisfies the conditions

$$\frac{1}{2} < \vartheta < \frac{2}{(p-2)\left[\alpha(2p/(p-2)+(n-1))+1\right]}, \qquad (28)$$

where $p > 2$.

Using definition of $\mu(r)$ we have

$$\int\limits_{\Omega_{r(\tau)}} u^2(\tau,x)dx \le \mu^{-1}(r(t)) \int\limits_{\Omega_{r(\tau)}} |u_{x_i}|^2 dx. \qquad (29)$$

Now taking into account lemmas 2, 3 and inequality (29) at sufficiently large t we have

$$\int\limits_{\Omega} u^2(t,x)dx = \int\limits_{\Omega_{r(t)}} u^2(\tau,x)dx + \int\limits_{\Omega\setminus\Omega_{r(t)}} u^2(\tau,x)dx \le$$

$$\le \Phi(t)\left(-\frac{d}{d\tau}\int\limits_{\Omega} u^2(\tau,x)dx\right) + \epsilon(t), \qquad (30)$$

where

$$\Phi(t) = \left(\frac{A_0}{2a_0}\right)^2 \mu^{-1}(r(t)); \epsilon(t) = c_2 \exp\left\{-\gamma_2 \cdot \frac{r^2(t)}{t}\right\}.$$

Integrating inequality (30) from 0 to $t/2$, we'll have

$$\int\limits_{\Omega} u^2(t,x)dx \le c_9\left(t^{\vartheta[\alpha(2p/(p-2)+(n-1))]-2/(p-2)}\right) + \epsilon(t)$$

Hence taking into account (28) we obtain (26). Theorem is proved.

2. Let us consider the Cauchy problem in the half-space $R_+^{n+1} = R^n \times \{t > 0\}$, i.e. problem (1), (3). With respect to the initial function $\varphi(x)$suppose, that the conditions

$$\varphi(x) \ge 0,\ \ \varphi(x) \in L_2(R^n),\ \ \varphi(x) = 0 \quad for\ \ |x| > R_0 \qquad (31)$$

are fulfilled.

Under the solution of problem (1), (3) we'll understand the function

$$u(t,x) \in W_2^{1,1}\left(R_+^{n+1}\right),$$

satisfying integral identity (8) with $D = R^n \times (0,T)$ at all $T > 0$ and $v(t,x) \in W_2^{1,1}(R_+^{n+1})$, and to initial condition (3).

The existence and uniqueness of solution of problem (1), (3) under some smoothness conditions follows from the known results.

Theorem 3. *Let $u(t,x) \in W_2^{1,1}(R_+^{n+1})$ be a nonnegative and bounded solution of problem (1), (3) in R_+^{n+1}. Suppose, that with respect to the coefficients conditions (4), (5), (11), and with respect to the initial function condition (31) are fulfilled. Then for all $t > 0$ it holds the estimation*

$$\int\limits_{R^n} u^2(t,x)dx \le C\left(\|\varphi\|_{L_1(R^n)}\right)\cdot t^{-n/2} \tag{32}$$

Proof. For solutions of problem (1), (3) lemmas 2, 3 are similarly proved with substitution of the domain Ω by R^n. At that equality (9) will be in the form

$$\frac{d}{dt}\int\limits_{R^n} u^2(t,x)dx =$$

$$= -2\int\limits_{R^n}\left(\sum_{i,j=1}^{n} a_{ij}(x,\tau)u_{x_i}u_{x_j} + \sum_{i=1}^{n} b_i(x,\tau)u_{x_i}u + c(x,\tau)u^2\right)dx \tag{33}$$

and estimation (12) takes the form

$$E_R(t) \le c_{11}\exp\left\{-\gamma_3\left[\frac{(R-R_0)^2}{t}\right]\right\}, \forall R > R_0, \tag{34}$$

where

$$E_R(t) = \int\limits_{R^n\setminus B_R} u^2(t,x)dx + \overline{a_0}\int\limits_0^t\int\limits_{R^n\setminus B_R}\left|u_{x_i}^2\right| dxd\tau$$

By virtue of estimation (34) let prove the inequality

$$\int\limits_{R^n} u(t,x)dx \le \int\limits_{R^n}\varphi(x)dx, \tag{35}$$

for almost all $t > 0$. For this reason let us substitute the test function $v(t,x) = \xi_R(|x|)$ to integral identity (8) with $D_T = R^n \times (0,t)$, where $\xi_R(|x|) = 1$ at $|x| \le R$, $0 \le \xi_R(|x|) \le 1$ at $R < |x| < R+1$, $\xi_R(|x|) = 0$ at $|x| \ge R+1$ and $\xi_R(\tau), \tau > 0$ is continuously differentiable and $\left|\frac{d\xi_R}{d\tau}\right| \le K_1$.

After this substitution we'll obtain

$$\int\limits_0^t\int\limits_{R^n} u_t\xi_R dxd\tau = -\int\limits_0^t\int\limits_{R^n}\left(\sum_{i,j=1}^{n} a_{ij}u_{x_i}\xi_{R_{x_i}} + \sum_{i=1}^{n} b_i u_{x_i}\xi_R + cu\xi_R\right)dxd\tau. \tag{36}$$

Using conditions on coefficients (4), (5), conditions on the patch function ξ_R, we'll obtain

$$\int\limits_0^t\int\limits_{R^n} u_t\xi_R dxd\tau \le$$

$$c_{12}\int_0^t\int_{R^n}\left(\sum_{i,j=1}^n A_0\left|u_{x_i}\right|\xi_{R_{x_i}}+\sum_{i=1}^n m\left|u_{x_i}\right|\xi_R+c_0\left|u\right|\xi_R\right)dxd\tau \qquad (37)$$

Applying to the first and second member of the right-hand side of inequality (37) the Holder inequality, and to the third member first of all Holder inequality, and then imbedding theorem

$$\overset{0}{W}{}_2^1(B_{R+1}\backslash B_R)\to L_2(B_{R+1}\backslash B_R)$$

we'll get

$$\int_0^t\int_{R^n}u_t\xi_R dxd\tau\le c_{13}t^{\frac12}\left(mes\left(B_{R+1}\backslash B_R\right)\right)^{\frac12}\left(\int_0^t\int_{R^n\backslash B_r}\left|u_{x_i}\right|^2dxd\tau\right)^{\frac12}, \qquad (38)$$

where the coefficient c_{13} depends on A_0, m, c_0.
In inequality (38) tending $R\to\infty$ and using estimation (34), we'll obtain (35).
Now using inequality (35) and the multiplicative Nirenberg-Galyardo, inequality, we'll obtain

$$\|u\|_{L_2(R^n)}\le B\left\|u_{x_i}\right\|_{L_2(R^n)}^{\vartheta}\cdot\|u\|_{L_1(R^n)}^{1-\vartheta}\le B\left\|u_{x_i}\right\|_{L_2(R^n)}^{\vartheta}\cdot\|\varphi\|_{L_1(R^n)}^{1-\vartheta}, \qquad (39)$$

where $\vartheta=\frac{n}{(n+2)}$.

Let us return to (33). Estimating the right-hand side of (33), using conditions (4), (5), applying the Holder inequality and imbedding theorem and taking into account (39), we'll obtain

$$\frac{d}{dt}\int_{R^n}u^2(t,x)dx\le-2c_{15}\int_{R^n}\left|u_{x_i}\right|^2dx\le-c_{16}\left(\int_{R^n}u^2dx\right)^{\frac1\vartheta}, \qquad (40)$$

where $c_{16}=2c_{15}\cdot B^{-\frac{2}{\vartheta}}\cdot\|\varphi\|_{L_1(R^n)}^{-\frac{(1-\vartheta)}{\vartheta}}$, and c_{15} is a constant dependent on A_0, m, c_0.
Integrating inequality (40) from 0 to t, we'll obtain inequality (32).

Remark 2. *The statements of theorems 1 and 2 are true for* $vrai\max\limits_{\Omega}u(x)$.

Really, under the conditions on coefficients of equation it is possible to obtain the following type estimations

$$vrai\max_{\Omega}u(t,x)\le C_{17}\int_{\Omega}u^2(t,x)dx \qquad (41)$$

Although the proof of estimation (41) requires the separate consideration, but from the reason don't make the paper more complicated we'll take the truth of estimation (41). For example, this estimation can be proved similar to the paper [11]. Then estimation (25) of theorem 1 will take the form

$$\sup_{\Omega}u(t,x)\le C_2\exp\left\{-\gamma_2\left[\frac{r^2(t)}{t}\right]\right\}, \qquad (42)$$

And estimation (26) of theorem 2 will take the form

$$\sup_{\Omega} u(t,x) \leq C_3 t^{-\beta}. \qquad (43)$$

So, it is possible to do such statement

Supposition 1. *Under the condition of theorems 1, 2, taking into account remark 2, estimations (42) and (43) are true.*

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