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ON REMOVABLE SETS OF SOLUTIONS OF A MIXED BOUNDARY-VALUE PROBLEM FOR SECOND ORDER NONDIVERGENT ELLIPTIC EQUATIONS

Abstract

In the present paper a mixed boundary-value problem is considered for the second order linear nondivergent elliptic equations and for some class of nonlinear elliptic equations. The sufficient condition of removability of a compact is proved.

Consider the following mixed boundary-value problem for the second order elliptic equation in the bounded domain $D \subset R^n,\ n \geq 3$

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x)\, u_{x_i x_j} + \sum_{i=1}^{n} b_i(x)\, u_{x_i} + c(x)\, u = 0, \tag{1}$$

$$u|_{\Gamma_1} = 0, \quad \left.\frac{\partial u}{\partial \nu}\right|_{\Gamma_2} = 0, \tag{2}$$

where Γ_1 and Γ_2 are such two sets that $\partial D \backslash E = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = \varnothing$, ∂D is a boundary of the domain D, E is some compact set lying on ∂D, and $\dfrac{\partial}{\partial \nu}$ is a derivative by conormal. Call the set E removable relative to the boundary-value problem for equation (1) in $C^{0,\lambda}(D),\ 0 < \lambda < 1$ if from

$$Lu = 0,\ x \in D \backslash E, \quad u|_{\Gamma_1} = 0, \quad \left.\frac{\partial u}{\partial \nu}\right|_{\Gamma_2} = 0, \quad u(x) \in C^{0,\lambda}(D), \tag{3}$$

it follows that $u(x) \equiv 0$ in D.

We find the solution of boundary-value problem (1), (2) from the classes $C^2(D) \cap C^0\left(\overline{D} \backslash E\right)$.

For the coefficients of equation (1) we assume that they are measurable and the following conditions are fulfilled

$$\gamma\, |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\, \xi_i \xi_j \leq \gamma^{-1} |\xi|^2, \tag{4}$$

$$|a_{ij}(x) - a_{ij}(y)| \leq k_1\, |x - y|, \tag{5}$$

$$|b_i(x)| \le b_0;\ -b_0 \le c(x) \le 0. \qquad (6)$$

Here $i,j=\overline{1,n},\ k_1,\ b_0>0$ are the constants and $\gamma\in(0,1]$.

In the paper the sufficient conditions of removability of a compact is proved for the problem (1)-(2) in the space $C^{0,\lambda}(D)$. The appropriate result has been obtained by L.Carleson [1] for the Laplace equation. In case of Neumann problem the questions on removability for Laplace equation in piecewise-smooth domains have been considered in [2], [3]. The questions of removability for solutions of the first boundary value problem for elliptic and parabolic equations have been considered in [4]. The removability conditions of a compact in space of continuous functions are constructed in [5]. Note also the paper [6].

Denote by $B_R(z)$ and $S_R(z)$ the ball $\{x:|x-z|<R\}$ and the sphere $\{x:|x-z|=R\}$ of radius R with the center at the point $z\in R^n$. $\dfrac{\partial u}{\partial \nu}$ is a derivative by the conormal

$$\frac{\partial u(x)}{\partial \nu}=\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial u(x)}{\partial x_i}\cos(n,x_j),$$

where $\cos(n,x_j)$, $j=\overline{1,n}$ are cosines of the external normal to the surface.

Denote by $m_H^s(A)$ the Hausdorff measure of the set A of order $s>0$.

Let's fix an arbitrary $\varepsilon>0$ and cover the set E by the system of balls $\left\{B_{r_i}\left(x^i\right)\right\}$, $i=1,2,...$, such that $r_i<\delta,\ E\subset\bigcup\limits_{i=1}^{\infty}B_{r_i}\left(x^i\right)$ and

$$\sum_{i=1}^{m} r_i^{n-2+\alpha}<\varepsilon, \qquad (7)$$

and $D_k=D\cap B_{2r_k}\left(x^k\right),\ k=1,2,...,$.

Let D_Σ be an open set situated in $D\backslash E$, whose boundary consists of the unification of Σ_i and Γ, where $\Sigma=\bigcup\limits_{k=1}^{\infty}\Sigma_k,\ \Gamma=\partial D\backslash\bigcup\limits_{k=1}^{\infty}D_k,\ D_k^+$ is a part of D_k remained after permutation of points, arranged between the Σ and $S_{2r_k}\left(x^k\right),\ k=1,2,...$. Denote by $D_{\Sigma'}$ an arbitrary connected component of $D_{\Sigma'}$, and consider the elliptic operator of a divergent structure

$$B=\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right).$$

Then according to the Green formula for any functions $z(x)$ and $W(x)$ we have

$$\int\limits_{D_{\Sigma'}} (zBW - WBz)\,dx = \int\limits_{\Gamma_1'\cup\Gamma_2'} \left(z\frac{\partial W}{\partial \nu} - W\frac{\partial z}{\partial \nu} \right) ds, \tag{8}$$

where $\partial D_{\Sigma'} = \Gamma_1' \cup \Gamma_2'$.

Consider the spheres $S_R(0)$ and $S_{2R}(0)$. By Landis-Gerver theorem for each i there exists the smooth surface Σ_i, dividing D_i and the spheres $S_{r_i}(x^i)$ and $S_{2r_i}(x^i)$. Denote by Σ_i the surface dividing the sphere of radius r_i and the sphere of radius $2r_i$ of the domain D and choosing the special points of ∂D as

$$\int\limits_{\Sigma_i} \left|\frac{\partial u}{\partial \nu}\right| ds \le C_1 \underset{r_i<r<2r_i}{osc} u \cdot r_i^{n-2}, \tag{9}$$

where C_1 depends on γ and n. The existence of such surface follows from [7].

As it is known from the conditions on the coefficients $u(x) \in C^1(\overline{D}_{\Sigma'})$ (see [8]). From (8) choosing the function z and W we have

$$\int\limits_{D_{\Sigma'}} B(u^2)\,dx = 2\int\limits_{\Gamma_1'\cup\Gamma_2'} u\frac{\partial u}{\partial \nu} ds.$$

Since we consider the bounded solution $|u(x)| \le M < \infty$, $x \in \overline{D}$, then according to (7) and

$$\int\limits_{D_{\Sigma'}} \left|\frac{\partial u}{\partial \nu}\right| ds \le C_1 \cdot r_k^{n-2+\alpha}, \quad k = 1, 2, ..., \tag{10}$$

we have

$$\int\limits_{D_{\Sigma'}} B(u^2)\,dx \le 2M\sum_{k=1}^{\infty}\int\limits_{\Sigma_k} \left|\frac{\partial u}{\partial \nu}\right| ds \le 2C_1\sum_{k=1}^{\infty} r_k^{n-2+\alpha} < C_2\varepsilon. \tag{11}$$

Besides

$$Bu = Lu + \sum_{i=1}^{n} d_i(x)\,u_{x_i} - c(x)\,u,$$

where

$$d_i(x) = \sum_{i,j=1}^{n} \frac{\partial a_{ij}(x)}{\partial x_j} - b_i(x), \; i = 1, ..., n.$$

Subject to

$$B\left(u^2\right)=2u\cdot Bu+2\sum_{i,j=1}^{n}a_{ij}\left(x\right)u_{x_i}u_{x_j},$$

and by virtue of conditions (4)-(6)

$$\left|d_i\left(x\right)\right|\le d_0<\infty;\quad i=1,...n,$$

from (11) we have

$$2\int\limits_{D_{\Sigma'}}u\sum_{i=1}^{n}d_i\left(x\right)u_{x_i}dx-2\int\limits_{D_{\Sigma'}}u^2c\left(x\right)dx+2\int\limits_{D_{\Sigma'}}\sum_{i,j=1}^{n}a_{ij}\left(x\right)u_{x_ix_j}dx<C_2\varepsilon.$$

Hence for any $\alpha>0$ applying Cauchy inequality we have

$$2\gamma\int\limits_{D_{\Sigma'}}\left|\nabla u\right|^2dx<2d_0\int\limits_{D_{\Sigma'}}\sum_{i=1}^{n}\left|u\right|\cdot\left|u_{x_i}\right|dx+C_2\varepsilon\le d_0\alpha\int\limits_{D_{\Sigma'}}\left|\nabla u\right|^2dx+$$

$$+\frac{d_0}{\alpha}\int\limits_{D_{\Sigma'}}\left|u\right|^2dx+C_2\varepsilon\le d_0\alpha\int\limits_{D_{\Sigma'}}\left|\nabla u\right|^2dx+\frac{d_0\cdot n\cdot M^2\cdot mes_nD}{\alpha}+C_2\varepsilon\le$$

$$\le d_0\alpha\int\limits_{D_{\Sigma'}}\left|\nabla u\right|^2dx\le\frac{d_0\cdot n\cdot M^2\cdot mes_nD}{\alpha}+C_2\varepsilon. \tag{12}$$

Choosing $\alpha=\dfrac{\gamma}{d_0}$ from (12) we have

$$\int\limits_{D_{\Sigma'}}\left|\nabla u\right|^2dx\le C_3, \tag{13}$$

where C_3 depends on $M,\ d_0,\ \gamma,\ mes_nD,\ n.$ Hence we have

$$\int\limits_{D}\left|\nabla u\right|^2dx\le C_4,$$

where C_4 depends on $C_3,E,D.$

Further, we have

$$\int\limits_{D_{\Sigma'}}u\sum_{i,j=1}^{n}a_{ij}\left(x\right)u_{x_ix_j}dx=-\int\limits_{D_{\Sigma'}}\frac{\partial}{\partial x_j}\left(ua_{ij}\left(x\right)\right)dx+\int\limits_{\Gamma_1'\cup\Gamma_2'}ua_{ij}\left(x\right)u_{x_j}ds, \tag{14}$$

since $|u|\le M$.

Using (14) we'll obtain

$$\int\limits_{D_{\Sigma'}} u\sum_{i,j=1}^{n} a_{ij}(x)\, u_{x_i x_j} dx \le M \int\limits_{D_{\Sigma'}} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\right) u_{x_j} dx +$$

$$+\int\limits_{D_{\Sigma'}} b_i(x)\, u_{x_i} dx + \int\limits_{D_{\Sigma'}} c(x)\, u dx. \qquad (15)$$

Using the conditions on the coefficients, conditions (7) and (8) and estimating the integrals on the right hand side of (15) we obtain

$$\int\limits_{D_{\Sigma'}} \sum_{i,j=1}^{n} a_{ij}(x)\, u_{x_i x_j} dx \le C_5 \varepsilon.$$

From the last inequality by virtue of (4) it follows that $u \equiv const$. So, the following theorem is proved.

Theorem 1. *Let D be a bounded domain in R^n, $n \ge 2$, $E \subset \overline{D}$ be some compact. For the coefficients of equation (1) conditions (4)-(6) be fulfilled. Then for removability of the compact E for problem (1), (2) it suffices that*

$$m_H^{n-2}(E) < \infty.$$

Consider the following equation with the boundary condition (3) in the domain $D \subset R^n$, $n \ge 2$

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{n} b_i(x)\, u_{x_i} + c(x)\, u + b(x, u, \nabla u) = 0. \qquad (16)$$

Here $a_{ij}(x)$ are measurable bounded functions satisfying condition (4),
$b_i(x)$, $c(x)$ satisfy condition (6) and

$$|b(x, u, \nabla u)| \le g(u)\, |\nabla u|^2, \quad \int\limits_0^a g^2(u)\, dx < k_1 < \infty, \ a < \infty. \qquad (17)$$

Consider the mixed problem for equation (16):

$$Lu = 0, \ x \in D\backslash E, \ u|_{\Gamma_1} = 0, \quad \left.\frac{\partial u}{\partial \nu}\right|_{\Gamma_2} = 0. \qquad (18)$$

We'll search a solution of problem (18) in the class $W_{2,0}^1(D) \cap C^{0,\lambda}\left(\overline{D}\right)$, $|u| \le k$, where $W_{2,0}^1(D)$ is a space of the functions turns to zero near Γ_1.

$$Bu = Lu + \sum_{i=1}^{n} d_i(x)\, u_{x_i} - c(x)\, u + b(x, u, \nabla u),$$

where

$$d_i(x) = \sum_{i,j=1}^{n} \frac{\partial a_{ij}(x)}{\partial x_j} - b_i(x), \ \ i = 1, ..., n.$$

Subject to

$$B\left(u^2\right) = 2uBu + 2\sum_{i,j=1}^{n} a_{ij}(x)\, u_{x_i x_j} + b(x, u, \nabla u)$$

and by virtue of (4)-(6)

$$|d_i(x)| \le d_0 < \infty; \quad i = 1, ..., n,$$

$$2\int\limits_{D_{\Sigma'}} u\sum_{i=1}^{n} d_i(x)\, u_{x_i} dx - 2\int\limits_{D_{\Sigma'}} u^2 c(x)\, dx + 2\int\limits_{D_{\Sigma'}} \sum_{i,j=1}^{n} a_{ij}(x)\, u_{x_i x_j} dx < C_2\varepsilon.$$

Applying the Cauchy inequality let's estimate nonlinear member in the right hand side

$$\int\limits_{D_{\Sigma'}} b(x, u, \nabla u)\, dx \le \int\limits_{D_{\Sigma'}} g(x)\, |\nabla u|\, dx \le \frac{1}{2\alpha}\int\limits_{D_{\Sigma'}} g^2(u)\, dx + \frac{\alpha}{2}\int\limits_{D_{\Sigma'}} |\nabla u|^2\, dx.$$

Applying the Cauchy inequality for any $\alpha > 0$ we have

$$2\gamma\int\limits_{D_{\Sigma'}} |\nabla u|^2\, dx < 2d_0\int\limits_{D_{\Sigma'}} |u|\, |u_{x_i}|\, dx + \int\limits_{D_{\Sigma'}} b(x, u, \nabla u)\, dx + C_2\varepsilon \le$$

$$\le d_0\alpha\int\limits_{D_{\Sigma'}} |u_{x_i}|^2\, dx + \frac{d_0}{\alpha}\int\limits_{D_{\Sigma'}} |u|^2\, dx + \int\limits_{D_{\Sigma'}} b(x, u, \nabla u)\, dx + C_2\varepsilon \le$$

$$\le k_1 + \int\limits_{D_{\Sigma'}} |\nabla u|^2\, dx + C_2\varepsilon \le d_0\alpha\int\limits_{D_{\Sigma'}} |\nabla u|^2\, dx + \frac{d_0 \cdot n \cdot M^2 mes_n D}{\alpha} + k_1 + C_2\varepsilon. \quad (19)$$

Choosing $\alpha = \frac{\gamma - 1}{d_0}$ from (19) we have

$$\int\limits_{D_{\Sigma'}} |\nabla u|^2\, dx \le C_3, \quad (20)$$

where C_3 depends on $M, d_0, \gamma, mes_n D,\ n,\ k_1$.

Without loss of generality we'll assume that $\varepsilon \leq 1$. Hence we have

$$\int\limits_D |\nabla u|^2 \, dx \leq C_4,$$

where C_4 depends on $C_3,\ E,\ D$.

Further, acting analogously to the linear case we'll obtain

$$\int\limits_{D_{\Sigma'}} \sum_{i,j=1}^{n} a_{ij}(x)\, u_{x_i x_j} dx \leq C_5 \varepsilon.$$

From the last inequality by virtue of (4) it follows that $u \equiv 0$. Thus, the following theorem is proved.

Theorem 2. *Let D be a bounded domain in R^n, $n \geq 2$, $E \subset \overline{D}$ be some compact. For the coefficients of equation (16) conditions (4)-(6) be fulfilled. Then for removability of the compact E for problem (18), it suffices that*

$$m_H^{n-2+\lambda}(E) = 0.$$

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