Afat N. JABRAYILOVA

ON DOUBLE COMPLETENESS OF PART OF ROOT VECTORS OF A CLASS OF POLYNOMIAL OPERATOR BUNDLES OF THE FOURTH ORDER

Abstract

The principles of double completeness of part of a system of eigen and adjoint vectors of fourth order operator bundles depending on parameter are established. For some values of parameters the characteristic polynomial of the principal part of an operator bundle has a multiple root. By fulfilling some algebraic conditions imposed on smallness of relative norms of coefficients we obtain a theorem on double completeness of part of a system of root vectors and establish their dependence on parameters.

On a separable Hilbert space H let's consider a polynomial operator bundle of fourth order depending on a spectral parameter $\lambda \in C$ and real parameter $\varepsilon \in [0,1]$:

$$L(\lambda;\varepsilon) = \lambda^4 E - 2\varepsilon\lambda^2 G^2 + G^4 + \sum_{j=0}^{3} \lambda^j F_{4-j}, \tag{1}$$

where for the coefficients of the bundle $L(\lambda;\varepsilon)$ the following conditions are assumed to be fulfilled:

1. G is a positive definite self-adjoint operator;
2. G^{-1} is a completely continuous operator in H, i.e. $G^{-1} \in \sigma_\infty$;
3. The operators $F_j G^{-j}$ $(j = 1,2,3)$ are bounded in H;
4. The operator $G^4 + F_4$ has a bounded inverse in H.

By fulfilling conditions 1)-4) the operator bundle $L(\lambda;\varepsilon)$ has a discrete spectrum [1], [2].

Obviously, at $\varepsilon \in [0,1)$ the principal part of operator bundle (1)

$$L_0(\lambda;\varepsilon;G) = \lambda^4 E - 2\varepsilon\lambda^2 G^2 + G^4$$

has a simple spectrum in the sense that the characteristic polynomial

$$L_0(\lambda;\varepsilon;\mu) = \lambda^4 - 2\varepsilon\lambda^2\mu^2 + \mu^4, \quad \mu \in \sigma(G)$$

has simple roots for $\varepsilon \in [0,1)$ and the operator bundle $L_0(\lambda;\varepsilon;G)$ has a multiple spectrum for $\varepsilon = 1$, i.e. the characteristic polynomial $L_0(\lambda;\varepsilon;\mu)$ has a multiple root.

In the paper we'll find conditions on the coefficients of operator bundle (1) that will provide double completeness of part of eigen and adjoint vectors of the bundle $L(\lambda;\varepsilon)$. (Note that the operator bundle (1) was studied in the paper [3] for $\varepsilon = 0$).

Definition 1. *Let there exist a vector $\varphi_0(\varepsilon) \neq 0$ that satisfies the equation*

$$L(\lambda_0;\varepsilon)\varphi_0(\varepsilon) = 0.$$

Then $\lambda_0(\varepsilon)$ is said to be the eigen-number and $\varphi_0(\varepsilon)$ the eigen vector of the bundle $L_0(\lambda;\varepsilon)$. The system $\varphi_1(\varepsilon), ..., \varphi_{m_\varepsilon}(\varepsilon)$ is said to be adjoint to the vector $\varphi_0(\varepsilon)$ if these vectors satisfy the equations

$$\sum_{k=0}^{4}\frac{1}{k!}\frac{\partial^k L(\lambda_0;\varepsilon)}{\partial\lambda^k}\varphi_{q-k}(\varepsilon)=0,\qquad q=0,1,...m(\varepsilon). \tag{2}$$

Let $\varphi_0(\varepsilon)$ be an eigen-vector, and $\varphi_1(\varepsilon),...,\varphi_{m(\varepsilon)}(\varepsilon)$ be adjoint vectors responding to the eigen-value $\lambda_0(\varepsilon)$. Then the vector-functions

$$u_p(\lambda_0(\varepsilon),t)=e^{\lambda_0(\varepsilon)t}\left(\frac{t^p}{p!}\varphi_p(\varepsilon)+\frac{t^{p-1}}{(p-1)!}\varphi_{p-1}(\varepsilon)+...+\varphi_0(\varepsilon)\right), \tag{3}$$
$$p=0,1,...,m(\varepsilon)$$

satisfy the equation

$$L\left(\frac{d}{dt};\varepsilon\right)u(t)=0 \tag{4}$$

and are said to be elementary solutions of homogeneous equation (4) (see [1]).

Assume $\lambda_0(\varepsilon)\in\Pi_-=\{\lambda:\operatorname{Re}\lambda<0\}$ and we form the following vectors

$$u_p^{(\nu)}(\lambda_0(\varepsilon);0)=\varphi_p^{(\nu)}(\varepsilon),\qquad \nu=0,2;\ \ p=0,1,...,m(\varepsilon) \tag{5}$$

Definition 2. *If the system* $\left\{\varphi_{p(\varepsilon)}^{(0)},\varphi_{p(\varepsilon)}^{(2)}\right\}\in H^2$ *constructed with respect to all possible eigen and adjoint vectors responding to eigen-values from the left half-plane* Π_- *is complete in the space* H^2 *to the sum of two copies of the space* H, *then we'll say that a part of a system of eigen and adjoint vectors of the bundle* $L(\lambda;\varepsilon)$ *is double complete in* H.

The following theorem establishes sufficient conditions providing the double completeness of part of a system of root vectors $L(\lambda;\varepsilon)$.

Theorem. *Let* $\varepsilon\in[0,1]$ *the operators* $G,F_j\quad(j=1,2,3)$ *satisfy equations 1)-3) and there be fulfilled the inequality*

$$\alpha(\varepsilon)=\sum_{j=0}^{3}d_j(\varepsilon)\left\|F_{4-j}G^{-(4-j)}\right\|<1, \tag{6}$$

where $d_0(\varepsilon)=1,\ d_2(\varepsilon)=\dfrac{1}{2(1+\varepsilon)},\ d_1(\varepsilon)=d_3(\varepsilon)=\dfrac{3\sqrt{3}\left(-\varepsilon+\sqrt{\varepsilon^2+3}\right)^{1/2}}{4\left(3+\varepsilon\sqrt{\varepsilon^2}+3-\varepsilon^2\right)}.$

And one of the conditions holds:

a) $G^{-1}\in\sigma_p\quad(0<p\le1)$

b) $G^{-1}\in\sigma_p\quad(0<p<\infty),\quad F_jG^{-j}\in\sigma_\infty\ \ (j=1,2,3,4).$

Then a part of a system of eigen and adjoint vectors of operator bundle (1) is double complete in H.

Proof. We show first that by fulfilling inequality (6) the operator bundle $L(\lambda;\varepsilon)$ is invertible on an imaginary axis. Since

$$L(\lambda;\varepsilon)=L_0(\lambda;\varepsilon;G)+L_1(\lambda;\varepsilon), \tag{7}$$

where $L_0(\lambda,\varepsilon;G)=\lambda^4E-2\varepsilon\lambda^2G^2+G^4,\ L_1(\lambda;\varepsilon)=\sum_{j=0}^{3}\lambda^jF_{4-j}$ and the operator bundle $L_0(\lambda;\varepsilon;G)$ is invertible on an imaginary axis, then

$$L(\lambda;\varepsilon) = \left(E + L_1(\lambda;\varepsilon) L_0^{-1}(\lambda;\varepsilon;G)\right) L_0(\lambda;\varepsilon;G) \tag{8}$$

For $\lambda = i\xi$, $\xi \in R = (-\infty;\infty)$, we have

$$\begin{aligned} \left\| L_1(\lambda;\varepsilon) L_0^{-1}(\lambda;\varepsilon;G)\right\| &\le \sum_{j=0}^{3} \left|\xi^j\right| \left\| F_{4-j}\left(\xi^4 E + 2\xi^2 G^2 + G^4\right)^{-1}\right\| \le \\ &\le \sum_{j=0}^{3} \left\| F_{4-j} G^{-(4-j)}\right\| \left\| \xi^j G^{4-j}\left(\xi^4 E + 2\xi^2 G^2 + G^4\right)^{-1}\right\|. \end{aligned} \tag{9}$$

Using the spectral expansion of the operator G we easily get that for $\xi \in R$ the following estimates hold:

$$\begin{aligned} &\left\| \xi^j G^{4-j}\left(\xi^4 E + 2\xi^2 G^2 + G^4\right)^{-1}\right\| = \\ &= \sup_{\mu\in\sigma(G)} \left| \xi^j \mu^{4-j}\left(\xi^4 + 2\xi^2\mu^2 + \mu^4\right)^{-1}\right| \le d_j(\varepsilon), \end{aligned} \tag{10}$$

where the numbers $d_j(\varepsilon)$ are defined from the conditions of the theorem. Then it follows from (6) and (8) that on the imaginary axis $L^{-1}(\lambda;\varepsilon)$ there exists and

$$\begin{aligned} &\left\| L^{-1}(\lambda;\varepsilon)\right\| \le \\ &\le \left\| L_0^{-1}(\lambda;\varepsilon;G)\right\| \cdot \left\| \left(E + L_1(\lambda;\varepsilon) L_0^{-1}(\lambda;\varepsilon;G)\right)\right\| \le const\,(1+|\lambda|)^{-4} \end{aligned} \tag{11}$$

Let's consider the Hilbert spaces

$$W_2^4(R_+;H) = \left\{ u/u^{(4)} \in L_2(R_+;H),\, G^4 u \in L_2(R_+;H)\right\}$$

$$\overset{0}{W}{}_2^4(R_+;H) = \left\{ u/u \in W_2^4(R_+;H),\, u(0) = u''(0) = 0\right\}$$

with norm

$$\|u\|_{W_2^4} = \left(\left\| u^{(4)}\right\|^2_{L_2(R_+;H)} + \left\| G^4 u\right\|^2_{L_2(R_+;H)}\right)^{1/2},$$

where

$$\|f\|_{L_2(R_+;H)} = \left(\int_0^\infty \|f(t)\|^2 dt\right)^{1/2}.$$

Now let's show that by fulfilling condition (6) the problem

$$L(d/dt;\varepsilon)\, u = 0 \tag{12}$$

$$u(0) = \psi_0, \quad u''(0) = \psi_2, \quad \psi_\nu \in H_{4-\nu-1/2}, \quad \nu = 0,2 \tag{13}$$

has a unique regular solution $u(t) \in W_2^4(R_+;H)$, i.e. $u(t)$ satisfies equation (12) almost everywhere and boundary conditions (13) in the sense that

$$\lim_{t\to+0} \left\| G^{4-\nu-1/2} u(t)\right\| = 0, \quad \nu = 0,2.$$

Here $H_\gamma = D\left(A^\gamma\right), \quad \gamma \geq 0.$

We can reduce problem (12)-(13) to the problem (see [3]):

$$L\left(d/dt;\varepsilon\right)u\left(t\right) = f\left(t\right) \tag{14}$$

$$u\left(o\right) = 0, \quad u''\left(0\right) = 0, \tag{15}$$

where $f\left(t\right) \in L_2\left(R_+;H\right)$. And we can write problem (14)-(15) in the form of the operator equation

$$L\left(d/dt;\varepsilon\right)u = f\left(t\right), \tag{16}$$

where $f\left(t\right) \in L_2\left(R_+;H\right)$ and $u \in \overset{0}{W}{}_2^4\left(R_+;H\right)$. First we show that the equation

$$L_0\left(d/dt;\varepsilon;G\right)u = f \tag{17}$$

has a unique solution $u \in \overset{0}{W}{}_2^4\left(R_+;H\right)$ for any $f \in L_2\left(R_+;H\right)$. Really, using a sine-transformation we can easily see that the vector-function

$$u_0\left(t\right) = \frac{4}{\pi}\int\limits_0^\infty \left(\xi^4 + 2\xi^2G^2 + G^4\right)^{-1}\left(\int\limits_0^\infty f\left(s\right)\sin\xi s ds\right)\sin\xi t$$

is a unique solution of equation (17) from $\overset{0}{W}{}_2^4\left(R_+;H\right)$. Then by Banach theorem on the inverse operator $L_0^{-1}\left(d/dt;\varepsilon;G\right)$ is a continuous operator from $L_2\left(R_+;H\right)$ on $\overset{0}{W}{}_2^4\left(R_+;H\right)$.

We write equation (16) as

$$L_0\left(d/dt;\varepsilon;G\right)u + L_1\left(d/dt\right)u = f, \tag{18}$$

where

$$L_1\left(d/dt\right)u = \sum_{j=0}^{3} F_{4-j}u^{(j)}.$$

Denoting $L_0u = \nu$ we get from (18)

$$\nu + L_1L_0^{-1}\nu = f.$$

Let's show that by the norm the operator $L_1L_0^{-1}$ is lesser than unit

$$\begin{aligned}\left\|L_1L_0^{-1}\nu\right\| = \left\|L_1u\right\|_{L_2} \leq \sum_{j=0}^{3}\left\|F_{4-j}u^{(j)}\right\|_{L_2} \leq \\ \leq \sum_{j=0}^{3}\left\|F_{4-j}G^{-(4-j)}\right\|\left\|G^{4-j}u^{(j)}\right\|_{L_2}.\end{aligned} \tag{19}$$

But by Plancharel theorem

$$\left\|G^{4-j}u^{(j)}\right\|_{L_2} = \left\|G^{4-j}\left(L^{-1}\nu\right)^{(j)}\right\|_{L_2} =$$

$$= \left\| G^{4-j}\xi^j \left(\xi^4 + 2\xi^2 G^2 + G^4\right)^{-1} \hat{\nu}(\xi) \right\|_{L_2} \leq$$

$$\leq \sup_{\xi} \left\| G^{4-j}\xi^j \left(\xi^4 + 2\xi^2 G^2 + G^4\right)^{-1} \right\|_{L_2} \cdot \|\nu\|_{L_2} \leq d_j(\varepsilon) \|\nu\|_{L_2}.$$

Then it follows from (19) that

$$\left\| L_1 L_0^{-1} \nu \right\|_{L_2} \leq \alpha(\varepsilon) \|\nu\|_{L_2}.$$

Since $\alpha(\varepsilon) < 1$, then $E + L_1 L_0^{-1}$ is invertible and $\nu = \left(E + L_1 L_0^{-1}\right)^{-1} f$. Therefore

$$u = L_0^{-1} \left(E + L_1 L_0^{-1}\right)^{-1} f.$$

Thus, problem (14)-(15) and the more problem (12)-(13) has a unique regular solution $u(t)$.

Assume that the theorem is not true. Then there exist the vectors φ_0 and φ_2 from H such that (φ_0, φ_1) are orthogonal to all $\left\{\varphi_p^{(0)}(\varepsilon), \varphi_p^{(2)}(\varepsilon)\right\}$. Then the vector-function ([1], [2])

$$R(\lambda) = \left(L^{*-1}\left(\bar{\lambda}; \varepsilon\right), \varphi_0 + \lambda^2 \varphi_2\right)$$

is analytic on the half-plane $\Pi_- = \{\lambda : \operatorname{Re}\lambda < 0\}$.

Let's consider for $t > 0$ the function

$$K(t) = \sum_{j=0}^{1} \left(u^{(2j)}(t), \varphi_{2j}\right),$$

where $u(t)$ is the solution of problem (12)-(13). Then considering that

$$u(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} L^{-1}(\xi) Q(\xi) e^{\xi t} d\xi,$$

where $Q(\xi)$ is a polynomial of power of at most three, and condition (11) we'll have

$$K(t) = \frac{1}{2\pi i} \int_{\Gamma} \left(Q(\xi), L^{-1}(\xi) \left(\varphi_0 + \bar{\xi}^2 \varphi_2\right)\right) e^{\xi t} d\xi.$$

Considering condition a) or b) of the theorem we get that we can apply Fragmen-Lindelof theorem to the integrand expression before $e^{\xi t}$ ([1], [2]) and get that this expression is a polynomial and therefore $K(t) = 0, \quad t > 0$. Then passing to limit as $t \to +0$ in the expression for $K(t)$ we get

$$\sum_{j=0}^{1} \left(\psi_2, \varphi_{2j}\right) = 0.$$

Since $\psi_{2j} \in D\left(G^{4-j-1/2}\right)$ $(j = 0, 1)$ is any vector, then $\varphi_{2j} = 0$, $j = 0, 1$. This contradiction proves the theorem.

From the theorem proved above we can write the Corollary.

Corollary. *Let conditions 1)-3) be fulfilled and the inequality*

$$\alpha(0)=\sum_{j=0}^{3} d_j(0)\left\|F_{4-j}G^{-(4-j)}\right\|<1$$

hold and one of the conditions a) or b) of the theorem be fulfilled. Then part of eigen and adjoint vectors of the bundle $L(\lambda;\varepsilon)$ *is double complete in* H *for all* $\varepsilon\in[0,1]$. *Here*

$$d_j(0)=\begin{cases} 1, & j=0 \\ \left(\frac{j}{4}\right)^{\frac{j}{4}}\left(\frac{4-j}{4}\right)^{\frac{4-j}{4}}, & j=1,2,3. \end{cases}$$

References

[1]. Keldysh M.V. *On eigen-values and eigen-functions of some classes not self-adjoint equations.* DAN SSSR, v.77, No1, 1951, pp.11-14. (Russian)

[2]. Gasymov M.G. *On multiple completeness of part of eigen and adjoint vectors of polynomial operator bundles.* Izv. Arm.SSR., v.VI, No2-3, 1971, pp.131-147. (Russian)

[3]. Mirzoyev S.S. *Problems of the theory of solvability of operator-differential equations in Hilbert space and related spectral problems. Author's thesis for Doctor's degree.* Baku, 1993, 32 p. (Russian)

[4]. Hochberg I.T., Krein M.G. *Introduction to the theory of not self-adjoint operators.* M.: "Nauka", 1965. (Russian)

Lions J.S., Majenes E. *Non-homogeneous boundary problems and their applications.* M.: "Mir", 1971, 374 p. (Russian)

Afat N. Jabrayilova
Institute of Mathematics and Mechanics of NAS of Azerbaijan
9, F.Agayev str., AZ1141, Baku, Azerbaijan
Tel.: (99412) 439 47 20 (off.)

Received February 02, 2005; Revised May 05, 2005.
Translated by Nazirova S.H.