Nizameddin Sh. ISKENDEROV, Kamilla A. JABBAROVA

THE INVERSE SCATTERING PROBLEM FOR THE SYSTEM OF FIVE FIRST ORDER HYPERBOLIC EQUATIONS ON SEMI-AXIS

Abstract

The factorization properties of the scattering operator are established and solution of the inverse scattering problem for the hyperbolic system of five first order equations on semi-axis with two given incident waves in the absence of the first or second interacting waves is given.

Direct and inverse scattering problems for a system of two equations on semi-axis and also for a system of $n \geq 2$ equations on the whole axis have been studied in the papers [1, 2, 3]. In the papers [4, 5] the case of the $n-1$ given incident waves and the case of the one given incident wave for a system of $n \geq 3$ equations have been considered. Direct and inverse scattering problems for a system of four equations with two incident waves on semi-axis have been solved in [6]. Direct scattering problem for a system of five hyperbolic equations on semi-axis in case of two given incident waves have been considered and some properties of scattering operator for this system have been established in the papers [7] and [8], respectively.

1. Problem statement. On a semi-axis $x \geq 0$ we consider the following hyperbolic system of the first order equations:

$$\xi_i \frac{\partial u_i(x,t)}{\partial t} - \frac{\partial u_i(x,t)}{\partial x} = \sum_{j=1}^{5} C_{ij}(x,t)\, u_j(x,t)$$

$$(i = 1, ..., 5;\ -\infty < t < +\infty), \tag{1}$$

where coefficients $C_{ij}(x,t)$ are complex-valued functions measurable by x and t and satisfying estimations:

$$|C_{ij}(x,t)| \leq c\left[(1+|x|)(1+|t|)\right]^{-1-\varepsilon},\ c > 0,\ \varepsilon > 0 \tag{2}$$

and condition $C_{ii}(x,t) = 0,\ i = 1, ..., 5;\ u_i(x,t),\ i = 1, ..., 5$ are unknown functions, ξ_i are constants, where $\xi_1 > \xi_2 > 0 > \xi_3 > \xi_4 > \xi_5$.

For system (1) we consider three problems.

In the first problem we have to find solution of system (1) which satisfies the boundary conditions:

$$\begin{cases} u_5(0,t) = u_1(0,t), \\ u_4(0,t) = u_2(0,t), \\ u_3(0,t) = 0 \end{cases} \tag{3}$$

by the given incident waves $a_1(t), a_2(t) \in L_\infty(R)$ which determine as $x \to +\infty$ asymptotics of the solutions $u_1(x,t), u_2(x,t)$ of the form:

$$\begin{cases} u_1(x,t) = a_1(t+\xi_1 x) + o(1), \\ u_2(x,t) = a_2(t+\xi_2 x) + o(1). \end{cases}$$

In the second problem it is necessary to find solutions of system (1) by such given incident waves $a_1(t), a_2(t) \in L_\infty(R)$, that satisfy the boundary conditions:

$$\begin{cases} u_5(0,t) = u_2(0,t), \\ u_3(0,t) = u_1(0,t), \\ u_4(0,t) = 0. \end{cases} \tag{4}$$

In the third problem the solution of system (1) by the given incident waves a_1, a_2 has to satisfy the boundary conditions

$$\begin{cases} u_4(0,t) = u_1(0,t), \\ u_3(0,t) = u_2(0,t), \\ u_5(0,t) = 0. \end{cases} \tag{5}$$

Joint consideration of these three problems is called the scattering problem for system (1).

In the paper [7] existence and uniqueness of the scattering problem solution for system (1) have been proved; scattering operator for this system $S = (S_1, S_2, S_3)$ has been determined, where operators

$$S_k = \begin{pmatrix} S_{11}^k & S_{12}^k \\ S_{21}^k & S_{22}^k \\ S_{31}^k & S_{32}^k \end{pmatrix}$$

transfer incident waves $a_1(t), a_2(t)$ into scattered waves $b_3^k(t), b_4^k(t), b_5^k(t) \in L_\infty(R)$, $k = 1,2,3$.

2. Scattering operator properties. Integral representations of solutions of the system of equations (1), which are represented by their asymptotics and by value of these solutions at zero have been studied in the paper [8].

At solving the inverse problem we need some properties of scattering operator, which we get from the following integral representation ([8]):

$$u_i(x,t) = h_i^6(t+\xi_i x) + \int_{t+\xi_1 x}^{+\infty} A_{i1}^6(x,t,s)\, h_1^6 ds + \int_{-\infty}^{+\infty} \sum_{j=2}^{4} A_{ij}^6(x,t,s)\, h_j^6(s)\, ds +$$

$$+ \int_{-\infty}^{t+\xi_5 x} A_{i5}^6(x,t,s)\, h_5^6(s)\, ds; \quad i = \overline{1,5}, \tag{6}$$

where $h^6(t) = \{a_1(t), a_2(t), b_3(t), b_4(t), b_5(t)\}$; transformation kernels

$A^6_{ij}(x,t,s)$, $i,j = \overline{1,5}$ at fixed x are Hilbert-Shmidt kernels, and are uniquely determined by the coefficients of a system of equations (1).

From representation (6) we have:

$$\begin{cases} u_1 = \left(I + A^6_{11-}\right) a_1 + A^6_{12} a_2 + A^6_{13} b_3 + A^6_{14} b_4 + A^6_{15+} b_5, \\ u_2 = A^6_{21-} a_1 + \left(I + A^6_{22}\right) a_2 + A^6_{23} b_3 + A^6_{24} b_4 + A^6_{25+} b_5, \\ u_3 = A^6_{31-} a_1 + A^6_{32} a_2 + \left(I + A^6_{33}\right) b_3 + A^6_{34} b_4 + A^6_{35+} b_5, \\ u_4 = A^6_{41-} a_1 + A^6_{42} a_2 + A^6_{43} b_3 + \left(I + A^6_{44}\right) b_4 + A^6_{45+} b_5, \\ u_5 = A^6_{51-} a_1 + A^6_{52} a_2 + A^6_{53} b_3 + A^6_{54} b_4 + \left(I + A^6_{55+}\right) b_5, \end{cases} \tag{7}$$

where

$$A^6_{i1-} f(x) = \int_x^{+\infty} A^6_{i1}(x,t,s) f(s)\, ds,$$

$$A^6_{i5+} f(x) = \int_{-\infty}^{x} A^6_{i5}(x,t,s) f(s)\, ds,$$

$$A^6_{ik} f(x) = \int_{-\infty}^{+\infty} A^6_{ik}(x,t,s) f(s)\, ds, \quad i = \overline{1,5}; \quad k = \overline{2,4}.$$

It is easy to see, that for all three scattering problems for the boundary conditions (3), (4), (5) the equality

$$u_1^k(0,t) + u_2^k(0,t) = u_3^k(0,t) + u_4^k(0,t) + u_5^k(0,t), \quad k = 1,2,3$$

or

$$u_1^k + u_2^k - u_3^k - u_4^k - u_5^k = 0.$$

holds.

Let's take it into account in the system (7):

$$\left(I + A^6_{11-} + A^6_{21-} - A^6_{31-} - A^6_{41-} - A^6_{51-}\right) a_1 +$$

$$+ \left(I + A^6_{12} + A^6_{22} - A^6_{32} - A^6_{42} - A^6_{52}\right) a_2 =$$

$$= \left(I + A^6_{53} + A^6_{43} + A^6_{33} - A^6_{23} - A^6_{13}\right) b_3^k +$$

$$+ \left(I + A^6_{54} + A^6_{44} + A^6_{34} - A^6_{24} - A^6_{14}\right) b_4^k +$$

$$+\left(I+A^6_{55+}+A^6_{45+}+A^6_{35+}-A^6_{25+}-A^6_{15+}\right)b^k_5,\quad k=1,2,3.$$

Multiplying the both sides of the last equality from the left by

$$\left(I+A^6_{55+}+A^6_{45+}+A^6_{35+}-A^6_{25+}-A^6_{15+}\right)^{-1}$$

and introducing some designation (see [8]), we'll get:

$$(I+A_+)^{-1}(I+A_-)\,a_1+(I+A_+)^{-1}(I+B)\,a_2=(I+M)\,b^k_3+(I+N)\,b^k_4+b^k_5,$$

where

$$A_+=A^6_{55+}+A^6_{45+}+A^6_{35+}-A^6_{25+}-A^6_{15+},$$

$$A_-=A^6_{11-}+A^6_{21-}-A^6_{31-}-A^6_{41-}-A^6_{51-},$$

$$B=A^6_{12}+A^6_{22}-A^6_{32}-A^6_{42}-A^6_{52},$$

$$M=(I+A_+)^{-1}\left(I+A^6_{53}+A^6_{43}+A^6_{33}-A^6_{23}-A^6_{13}\right)-I,$$

$$N=(I+A_+)^{-1}\left(I+A^6_{54}+A^6_{44}+A^6_{34}-A^6_{24}-A^6_{14}\right)-I.$$

Since b^k_3,b^k_4,b^k_5 are of the form:

$$b^k_3=S^k_{11}a_1+S^k_{12}a_2,\ b^k_4=S^k_{21}a_1+S^k_{22}a_2,\ b^k_5=S^k_{31}a_1+S^k_{32}a_2,\quad k=1,2,3,$$

therefore, for arbitrary $a_1,a_2\in L_\infty(R)$ the following factorization holds:

$$(I+M)\,S^k_{11}+(I+N)\,S^k_{21}+S^k_{31}=(I+A_+)^{-1}(I+A_-)\,, \tag{8}$$

$$(I+M)\,S^k_{12}+(I+N)\,S^k_{22}+S^k_{32}=(I+A_+)^{-1}(I+B)\,,\ k=1,2,3 \tag{9}$$

By Gohberg-Crain theory from these factorization properties one can find the expressions

$$\left\{\begin{array}{r} A^6_{11-}+A^6_{21-}-A^6_{31-}-A^6_{41-}-A^6_{51-}=D_1\\ A^6_{12}+A^6_{22}-A^6_{32}-A^6_{42}-A^6_{52}=D_2,\\ A^6_{53}+A^6_{43}+A^6_{33}-A^6_{23}-A^6_{13}=D_3\\ A^6_{54}+A^6_{44}+A^6_{34}-A^6_{24}-A^6_{14}=D_4,\\ A^6_{55+}+A^6_{45+}+A^6_{35+}-A^6_{25+}-A^6_{15+}=D_5,\end{array}\right. \tag{D}$$

Let's prove, that operators M and N are expressed by the scattering operator elements in the following way

$$(I+M, \quad I+N)=\left(S_{31}^{1}-S_{31}^{2},\ S_{32}^{1}-S_{32}^{2}\right)\widetilde{S}_{1}^{-1}, \tag{10}$$

where

$$\widetilde{S}_{1}=\begin{pmatrix} S_{11}^{2} & -S_{11}^{1} & S_{12}^{2} & -S_{12}^{1} \\ S_{21}^{2} & -S_{21}^{1} & S_{22}^{2} & -S_{22}^{1} \end{pmatrix}$$

By determination of the scattering operator the functions $b_5^1(t)$, $b_5^2(t)$ have the form:

$$b_5^1=S_{31}^{1}a_1+S_{32}^{1}a_2,\ b_5^2=S_{31}^{2}a_1+S_{32}^{2}a_2.$$

So, $b_5^1-b_5^2=\left(S_{31}^{1}-S_{31}^{2}\right)a_1+\left(S_{32}^{1}-S_{32}^{2}\right)a_2$. We can rewrite this equality in a vector form:

$$b_5^1-b_5^2=\left(S_{31}^{1}-S_{31}^{2},\ S_{32}^{1}-S_{32}^{2}\right)\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{$*$}$$

On the other hand, in the paper [8] the following property has been proved:

$$b_5^1-b_5^2=(I+M)\left(b_3^2-b_3^1\right)+(I+N)\left(b_4^2-b_4^1\right), \tag{$**$}$$

which we can also rewrite in the following form:

$$b_5^1-b_5^2=(I+M,\ I+N)\begin{pmatrix} b_3^2 & -b_3^1 \\ b_4^2 & -b_4^1 \end{pmatrix}$$

Operator $S_1=\begin{pmatrix} S_{11}^{2} & -S_{11}^{1} & S_{12}^{2} & -S_{12}^{1} \\ S_{21}^{2} & -S_{21}^{1} & S_{22}^{2} & -S_{22}^{1} \end{pmatrix}$ transfers $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ in $\begin{pmatrix} b_3^2 & -b_3^1 \\ b_4^2 & -b_4^1 \end{pmatrix}$, therefore we have:

$$b_5^1-b_5^2=(I+M,\ I+N)\,\widetilde{S}_1\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Comparing the last equality with $(*)$ and taking arbitrariness of a_1, a_2, into account, we get:

$$(I+M,\ I+N)\,\widetilde{S}_1=\left(S_{31}^{1}-S_{31}^{2},\ S_{32}^{1}-S_{32}^{2}\right).$$

Since the operator $\widetilde{S}_1$ is reversible (see [8]), then from the last correlation it follows:

$$(I+M,\ I+N)=\left(S_{31}^{1}-S_{31}^{2},\ S_{32}^{1}-S_{32}^{2}\right)\widetilde{S}_{1}^{-1}.$$

Thus, property (10) is proved.

Lemma. Operators $A^6_{ij},\ A^6_{i1-},\ A^6_{i5+} i=\overline{1,5},\ j=\overline{2,4}$; which take part in representation (6) satisfy the following systems of equations:

$$\begin{cases} I+A^6_{11-}-A^6_{51-}=\left(A^6_{53}-A^6_{13}\right)S^1_{11}+\left(A^6_{54}-A^6_{14}\right)S^1_{21}+ \\ \quad +\left(I+A^6_{55+}-A^6_{15+}\right)S^1_{31}, \\ A^6_{21-}-A^6_{41-}=\left(A^6_{43}-A^6_{23}\right)S^1_{11}+\left(I+A^6_{44}-A^6_{24}\right)S^1_{21}+ \\ \quad +\left(A^6_{45+}-A^6_{25+}\right)S^1_{31}, \\ -A^6_{31-}=\left(I+A^6_{33}\right)S^1_{11}+A^6_{34}S^1_{21}+A^6_{35+}S^1_{31}, \end{cases} \tag{11}$$

$$\begin{cases} A^6_{12}-A^6_{52}=\left(A^6_{53}-A^6_{13}\right)S^1_{12}+\left(A^6_{54}-A^6_{14}\right)S^1_{22}+ \\ \quad +\left(I+A^6_{55+}-A^6_{15+}\right)S^1_{32}, \\ I+A^6_{22}-A^6_{42}=\left(A^6_{43}-A^6_{23}\right)S^1_{12}+\left(I+A^6_{44}-A^6_{24}\right)S^1_{22}+ \\ \quad +\left(A^6_{45+}-A^6_{25+}\right)S^1_{32}, \\ -A^6_{32}=\left(I+A^6_{33}\right)S^1_{12}+A^6_{34}S^1_{22}+A^6_{35+}S^1_{32}, \end{cases} \tag{12}$$

$$\begin{cases} A^6_{21-}-A^6_{51-}=\left(A^6_{53}-A^6_{23}\right)S^2_{11}+\left(A^6_{54}-A^6_{24}\right)S^2_{21}+ \\ \quad +\left(I+A^6_{55+}-A^6_{25+}\right)S^2_{31}, \\ I+A^6_{11-}-A^6_{31-}=\left(I+A^6_{33}-A^6_{13}\right)S^2_{11}+\left(A^6_{34}-A^6_{14}\right)S^2_{21}+ \\ \quad +\left(A^6_{35+}-A^6_{15+}\right)S^2_{31}, \\ -A^6_{41-}=A^6_{43}S^2_{11}+\left(I+A^6_{44}\right)S^2_{21}+A^6_{45+}S^2_{31}, \end{cases} \tag{13}$$

$$\begin{cases} I+A^6_{22}-A^6_{52}=\left(A^6_{53}-A^6_{23}\right)S^2_{12}+\left(A^6_{54}-A^6_{24}\right)S^2_{22}+ \\ \quad +\left(I+A^6_{55+}-A^6_{25+}\right)S^2_{32}, \\ A^6_{12}-A^6_{32}=\left(I+A^6_{33}-A^6_{13}\right)S^2_{12}+\left(A^6_{34}-A^6_{14}\right)S^2_{22}+ \\ \quad +\left(A^6_{35+}-A^6_{15+}\right)S^2_{32}, \\ -A^6_{42}=A^6_{43}S^2_{12}+\left(I+A^6_{44}\right)S^2_{22}+A^6_{45+}S^2_{32}, \end{cases} \tag{14}$$

$$\begin{cases} I+A^6_{11-}-A^6_{41-}=\left(A^6_{43}-A^6_{13}\right)S^3_{11}+\left(I+A^6_{44}-A^6_{14}\right)S^3_{21}+ \\ \quad +\left(A^6_{45+}-A^6_{15+}\right)S^3_{31}, \\ A^6_{21-}-A^6_{31-}=\left(I+A^6_{33}-A^6_{23}\right)S^3_{11}+\left(A^6_{34}-A^6_{24}\right)S^3_{21}+ \\ \quad +\left(A^6_{35+}-A^6_{25+}\right)S^3_{31}, \\ -A^6_{51-}=A^6_{53}S^3_{11}+A^6_{54}S^3_{21}+\left(I+A^6_{55+}\right)S^3_{31}, \end{cases} \tag{15}$$

$$\begin{cases} A^6_{12}-A^6_{42}=\left(A^6_{43}-A^6_{13}\right)S^3_{12}+\left(I+A^6_{44}-A^6_{14}\right)S^3_{22}+ \\ \quad +\left(A^6_{45+}-A^6_{15+}\right)S^3_{32}, \\ I+A^6_{22}-A^6_{32}=\left(I+A^6_{33}-A^6_{23}\right)S^3_{12}+\left(A^6_{34}-A^6_{24}\right)S^3_{22}+ \\ \quad +\left(A^6_{35+}-A^6_{25+}\right)S^3_{32}, \\ -A^6_{52}=A^6_{53}S^3_{12}+A^6_{54}S^3_{22}+\left(I+A^6_{55+}\right)S^3_{32}, \end{cases} \tag{16}$$

Proof. Let's consider system (7). If we substitute the boundary conditions (3) $(u_5(0,t)=u_1(0,t))$ in this system, we'll get:

$$\left(I+A^6_{11-}-A^6_{51-}\right)a_1+\left(A^6_{12}-A^6_{52}\right)a_2=\left(A^6_{53}-A^6_{13}\right)b^1_3+\left(A^6_{54}-A^6_{14}\right)b^1_4+$$

$$+\left(I+A^6_{55+}-A^6_{15+}\right)b^1_5,$$

Allowing here the form of functions b_3^1, b_4^1, b_5^1, at arbitrary $a_1, a_2 \in L_\infty(R)$ we'll have:

$$I + A_{11-}^6 - A_{51-}^6 = \left(A_{53}^6 - A_{13}^6\right) S_{11}^1 + \left(A_{54}^6 - A_{14}^6\right) S_{21}^1 + \left(I + A_{55+}^6 - A_{15+}^6\right) S_{31}^1,$$

$$A_{12}^6 - A_{52}^6 = \left(A_{53}^6 - A_{13}^6\right) S_{12}^1 + \left(A_{54}^6 - A_{14}^6\right) S_{22}^1 + \left(I + A_{55+}^6 - A_{15+}^6\right) S_{32}^1.$$

Thus, the first equalities of systems (11) and (12) are proved.

Further, by condition (3) $u_2(0,t) = u_4(0,t)$, and $u_3(0,t) = 0$. Therefore

$$\left(A_{21-}^6 - A_{41-}^6\right) a_1 + \left(I + A_{22}^6 - A_{42}^6\right) a_2 = \left(A_{43}^6 - A_{23}^6\right) b_3^1 +$$

$$+ \left(I + A_{44}^6 - A_{24}^6\right) b_4^1 + \left(A_{45+}^6 - A_{25+}^6\right) b_5^1;$$

$$-A_{31-}^6 a_1 - A_{32}^6 a_2 = \left(I + A_{33}^6\right) b_3^1 + A_{34}^6 b_4^1 + A_{35+}^6 b_5^1.$$

Hence

$$\left(A_{21-}^6 - A_{41-}^6\right) = \left(A_{43}^6 - A_{23}^6\right) S_{11}^1 + \left(I + A_{44}^6 - A_{24}^6\right) S_{21}^1 + \left(A_{45+}^6 - A_{25+}^6\right) S_{31}^1$$

$$-A_{31-}^6 = \left(I + A_{33}^6\right) S_{11}^1 + A_{34}^6 S_{21}^1 + A_{35+}^6 S_{31}^1;$$

$$I + A_{22}^6 - A_{42}^6 = \left(A_{43}^6 - A_{23}^6\right) S_{12}^1 + \left(I + A_{44}^6 - A_{24}^6\right) S_{22}^1 + \left(A_{45+}^6 - A_{25+}^6\right) S_{32}^1$$

$$-A_{32}^6 = \left(I + A_{33}^6\right) S_{12}^1 + A_{34}^6 S_{22}^1 + A_{35+}^6 S_{32}^1.$$

So, the correctness of systems (11) and (12) has been established. Analogously, one can prove systems (13), (14) and (15), (16), taking the boundary conditions (4) and (5) into account in system (7), respectively.

3. The inverse scattering problem. At this item, in the absence of the first or second interacting waves, we solve the inverse scattering problem for system (1) on semi-axis, i.e. coefficients of system (1) are uniquely defined by the known scattering operator on semi-axis. First of all, let's consider the case when $C_{12} = ... = C_{15} = 0$. In this case $u_1(x,t) = a_1(t + \xi_1 x)$; thereafter, the boundary conditions will have the form:

$$\text{I.} \begin{cases} u_5^1(0,t) = u_1^1(0,t) = a_1(t) \\ u_4^1(0,t) = u_2^1(0,t) \\ u_3^1(0,t) = 0. \end{cases}$$

$$\text{II.}\ \begin{cases} u_5^2(0,t) = u_2^2(0,t) \\ u_3^2(0,t) = u_1^2(0,t) = a_1(t) \\ u_4^2(0,t) = 0. \end{cases}$$

$$\text{III.}\ \begin{cases} u_4^3(0,t) = u_1^3(0,t) = a_1(t) \\ u_3^3(0,t) = u_2^3(0,t) \\ u_5^3(0,t) = 0. \end{cases}$$

Allowing for the boundary conditions, one can write system (7) in the form:

$$\begin{cases} u_2^1 - u_2^2 = A_{23}^6\left(b_3^1 - b_3^2\right) + A_{24}^6\left(b_4^1 - b_4^2\right) + A_{25+}^6\left(b_5^1 - b_5^2\right), \\ -u_1^2 = \left(I + A_{33}^6\right)\left(b_3^1 - b_3^2\right) + A_{34}^6\left(b_4^1 - b_4^2\right) + A_{35+}^6\left(b_5^1 - b_5^2\right), \\ u_2^1 = A_{43}^6\left(b_3^1 - b_3^2\right) + \left(I + A_{44}^6\right)\left(b_4^1 - b_4^2\right) + A_{45+}^6\left(b_5^1 - b_5^2\right), \\ u_1^1 - u_2^2 = A_{53}^6\left(b_3^1 - b_3^2\right) + A_{54}^6\left(b_4^1 - b_4^2\right) + \left(I + A_{55+}^6\right)\left(b_5^1 - b_4^2\right). \end{cases}$$

From the second equality, allowing for $(**)$, we get:

$$-a_1 = \left[I + A_{33}^6 - A_{35+}^6(I+M)\right]\left(b_3^1 - b_3^2\right) + \left[A_{34}^6 - A_{35+}^6(I+N)\right]\left(b_4^1 - b_4^2\right).$$

It is known, that the operator

$$S_1 = \begin{pmatrix} S_{11}^2 & -S_{11}^1 & S_{12}^2 & -S_{12}^1 \\ S_{21}^2 & -S_{21}^1 & S_{22}^2 & -S_{22}^1 \end{pmatrix} : \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \to \begin{pmatrix} b_3^2 & -b_3^1 \\ b_4^2 & -b_4^1 \end{pmatrix}$$

has the inverse $\tilde{\gamma}^1 = \left\|\tilde{\gamma}_{ij}^1\right\|_{i,j=1}^2$ (see [8]), i.e.

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \tilde{\gamma}_{11}^1 & \tilde{\gamma}_{12}^1 \\ \tilde{\gamma}_{21}^1 & \tilde{\gamma}_{22}^1 \end{pmatrix}\begin{pmatrix} b_3^2 & -b_3^1 \\ b_4^2 & -b_4^1 \end{pmatrix}$$

Hence, we see that $a_1 = \tilde{\gamma}_{11}^1\left(b_3^2 - b_3^1\right) + \tilde{\gamma}_{12}^1\left(b_4^2 - b_4^1\right)$.
Therefore

$$I + A_{33}^6 - A_{35+}^6(I+M) = \tilde{\gamma}_{11}^1, \tag{17}$$

$$A_{34}^6 - A_{35+}^6(I+N) = \tilde{\gamma}_{12}^1 \tag{18}$$

Let's consider the third equality of system (11):

$$-A_{31-}^6 = \left(I + A_{33}^6\right)S_{11}^1 + A_{34}^6 S_{21}^1 + A_{35+}^6 S_{31}^1.$$

Let's put here expressions for $I + A_{33}^6$ and A_{34}^6 according to (17) and (18):

$$-A_{31-}^6 = \tilde{\gamma}_{11}^1 S_{11}^1 + A_{35+}^6(I+M)S_{11}^1 + \tilde{\gamma}_{12}^1 S_{21}^1 + A_{35+}^6(I+N)S_{21}^1 + A_{35+}^6 S_{31}^1$$

or

$$A^6_{31-} = -A^6_{35+}\left[(I+M)\,S^1_{11} + (I+N)\,S^1_{21} + S^1_{31}\right] - \tilde{\gamma}^1_{11}S^1_{11} - \tilde{\gamma}^1_{12}S^1_{21}.$$

Allowing for factorization (8), we get:

$$A^6_{31-} = -A^6_{35+}\,(I+A_+)^{-1}\,(I+A_-) - \tilde{\gamma}^1_{11}S^1_{11} - \tilde{\gamma}^1_{12}S^1_{21}.$$

We multiply the both sides by $(I+A_-)^{-1}$ from the right:

$$A^6_{31-}\,(I+A_-)^{-1} + A^6_{35+}\,(I+A_+)^{-1} = -\tilde{\gamma}^1_{11}S^1_{11}\,(I+A_-)^{-1} - \tilde{\gamma}^1_{12}S^1_{21}\,(I+A_-)^{-1}.$$

Hence we can find:

$$A^6_{31-}\,(I+A_-)^{-1} = \left[\left(-\tilde{\gamma}^1_{11}S^1_{11} - \tilde{\gamma}^1_{12}S^1_{21}\right)(I+A_-)^{-1}\right]_-;$$

$$A^6_{35+}\,(I+A_+)^{-1} = \left[\left(-\tilde{\gamma}^1_{11}S^1_{11} - \tilde{\gamma}^1_{12}S^1_{21}\right)(I+A_-)^{-1}\right]_+;$$

i.e.

$$A^6_{31-} = \left[\left(-\tilde{\gamma}^1_{11}S^1_{11} - \tilde{\gamma}^1_{12}S^1_{21}\right)(I+A_-)^{-1}\right]_-(I+A_-); \tag{19}$$

$$A^6_{35+} = \left[\left(-\tilde{\gamma}^1_{11}S^1_{11} - \tilde{\gamma}^1_{12}S^1_{21}\right)(I+A_-)^{-1}\right]_+(I+A_+) \tag{20}$$

Now, putting (20) into (17) and (18), we find A^6_{33} and A^6_{34}, respectively.

$$A^6_{33} = \tilde{\gamma}^1_{11} + \left[\left(-\tilde{\gamma}^1_{11}S^1_{11} - \tilde{\gamma}^1_{12}S^1_{21}\right)(I+A_-)^{-1}\right]_+(I+A_+)\,(I+M) - I; \tag{21}$$

$$A^6_{34} = \tilde{\gamma}^1_{12} + \left[\left(-\tilde{\gamma}^1_{11}S^1_{11} - \tilde{\gamma}^1_{12}S^1_{21}\right)(I+A_-)^{-1}\right]_+(I+A_+)\,(I+N) \tag{22}$$

For finding A^6_{32}, let's take into account (20), (21) and (22) in the third equality of system (12):

$$A^6_{32} - \left(I + A^6_{33}\right)S^1_{12} - A^6_{34}S^1_{22} - A^6_{35+}S^1_{32}$$

$$A^6_{32} = -\tilde{\gamma}^1_{11}S^1_{12} + \left[\left(\tilde{\gamma}^1_{11}S^1_{11} + \tilde{\gamma}^1_{12}S^1_{21}\right)(I+A_-)^{-1}\right]_+(I+A_+)\,(I+M)\,S^1_{12}-$$

$$-\tilde{\gamma}_{12}^{1}S_{22}^{1}+\left[\left(\tilde{\gamma}_{11}^{1}S_{11}^{1}+\tilde{\gamma}_{12}^{1}S_{21}^{1}\right)(I+A_{-})^{-1}\right]_{+}(I+A_{+})(I+N)\,S_{22}^{1}+$$

$$+\left[\left(\tilde{\gamma}_{11}^{1}S_{11}^{1}+\tilde{\gamma}_{12}^{1}S_{21}^{1}\right)(I+A_{-})^{-1}\right]_{+}(I+A_{+})\,S_{32}^{1}=-\tilde{\gamma}_{11}^{1}S_{12}^{1}-\tilde{\gamma}_{12}^{1}S_{22}^{1}+$$

$$+\left[\left(\tilde{\gamma}_{11}^{1}S_{11}^{1}+\tilde{\gamma}_{12}^{1}S_{21}^{1}\right)(I+A_{-})^{-1}\right]_{+}(I+A_{+})\left[(I+M)\,S_{12}^{1}+(I+N)\,S_{22}^{1}+S_{32}^{1}\right]$$

Because of the property (9) the last expression takes the form:

$$A_{32}^{6}=\left[\left(\tilde{\gamma}_{11}^{1}S_{11}^{1}+\tilde{\gamma}_{12}^{1}S_{21}^{1}\right)(I+A_{-})^{-1}\right]_{+}(I+B)-\tilde{\gamma}_{11}^{1}S_{12}^{1}-\tilde{\gamma}_{12}^{1}S_{22}^{1} \qquad (23)$$

Thus, all elements $A_{3j}^{6},\ j=1,...,5$ are found.

For finding $A_{4j}^{6},\ j=1,...,5$ let's consider the following system, which can be obtained from system (7) allowing for the boundary conditions (4), (5):

$$\begin{cases} u_2^2-u_2^3=A_{23}^{6}\left(b_3^2-b_3^3\right)+A_{24}^{6}\left(b_4^2-b_4^3\right)+A_{25+}^{6}\left(b_5^2-b_5^3\right), \\ u_1^2-u_2^3=\left(I+A_{33}^{6}\right)\left(b_3^2-b_3^3\right)+A_{34}^{6}\left(b_4^2-b_4^3\right)+A_{35+}^{6}\left(b_5^2-b_5^3\right), \\ -u_1^3=A_{43}^{6}\left(b_3^2-b_3^3\right)+\left(I+A_{44}^{6}\right)\left(b_4^2-b_4^3\right)+A_{45+}^{6}\left(b_5^2-b_5^3\right), \\ u_2^2=A_{53}^{6}\left(b_3^2-b_3^3\right)+A_{54}^{6}\left(b_4^2-b_4^3\right)+\left(I+A_{55+}^{6}\right)\left(b_5^2-b_5^3\right). \end{cases}$$

Since $u_1^3(0,t)=a_1(t)$, then from the third equality of the system, allowing for $(**)$, we get:

$$a_1=\left[A_{43}^{6}-A_{45+}^{6}(I+M)\right]\left(b_3^3-b_3^2\right)+\left[I+A_{44}^{6}-A_{45+}^{6}(I+N)\right]\left(b_4^3-b_4^2\right)$$

In [8] it has been proved the reversibility of the operator $\widetilde{S}_3$, which has the form:

$$\widetilde{S}_3:\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\rightarrow\begin{pmatrix} b_4^3 & -b_4^2 \\ b_3^3 & -b_3^2 \end{pmatrix}.$$

Its reverse:

$$\tilde{\gamma}^{3}=\begin{pmatrix} \tilde{\gamma}_{11}^{3} & \tilde{\gamma}_{12}^{3} \\ \tilde{\gamma}_{21}^{3} & \tilde{\gamma}_{22}^{3} \end{pmatrix}\begin{pmatrix} b_4^3 & -b_4^2 \\ b_3^3 & -b_3^2 \end{pmatrix}=\begin{pmatrix} a_1 \\ a_2 \end{pmatrix};$$

i.e. $a_1=\tilde{\gamma}_{11}^{3}\left(b_4^3-b_4^2\right)+\tilde{\gamma}_{12}^{3}\left(b_3^3-b_3^2\right)$.

Therefore,

$$A_{43}^{6}-A_{45+}^{6}(I+M)=\tilde{\gamma}_{12}^{3}, \qquad (24)$$

$$I+A_{44}^{6}-A_{45+}^{6}\left(I+N\right)=\tilde{\gamma}_{11}^{3} \tag{25}$$

Then, from the third equality of system (13) we have:

$$-A_{41-}^{6}=A_{45+}^{6}\left[\left(I+M\right)S_{11}^{2}+\left(I+N\right)S_{21}^{2}+S_{31}^{2}\right]+\tilde{\gamma}_{12}^{3}S_{11}^{2}+\tilde{\gamma}_{11}^{3}S_{21}^{2}.$$

Allowing for factorization (8), we get:

$$-A_{41-}^{6}=A_{45+}^{6}\left(I+A_{+}\right)^{-1}\left(I+A_{-}\right)+\tilde{\gamma}_{12}^{3}S_{11}^{2}+\tilde{\gamma}_{11}^{3}S_{21}^{2}$$

or

$$A_{41-}^{6}\left(I+A_{-}\right)^{-1}+A_{45+}^{6}\left(I+A_{+}\right)^{-1}=-\tilde{\gamma}_{12}^{3}S_{11}^{2}\left(I+A_{-}\right)^{-1}-\tilde{\gamma}_{11}^{3}S_{21}^{2}\left(I+A_{-}\right)^{-1}.$$

From here we find

$$A_{41-}^{6}=\left[\left(-\tilde{\gamma}_{12}^{3}S_{11}^{2}-\tilde{\gamma}_{11}^{3}S_{21}^{2}\right)\left(I+A_{-}\right)^{-1}\right]_{-}\left(I+A_{-}\right) \tag{26}$$

$$A_{45+}^{6}=\left[\left(-\tilde{\gamma}_{12}^{3}S_{11}^{2}-\tilde{\gamma}_{11}^{3}S_{21}^{2}\right)\left(I+A_{-}\right)^{-1}\right]_{+}\left(I+A_{+}\right) \tag{27}$$

Putting (27) in (24) and (25), we find A_{43}^{6} and A_{44}^{6}, respectively:

$$A_{43}^{6}=\left[\left(-\tilde{\gamma}_{12}^{3}S_{11}^{2}-\tilde{\gamma}_{11}^{3}S_{21}^{2}\right)\left(I+A_{-}\right)^{-1}\right]_{+}\left(I+A_{+}\right)\left(I+M\right)+\tilde{\gamma}_{12}^{3}, \tag{28}$$

$$A_{44}^{6}=\left[\left(-\tilde{\gamma}_{12}^{3}S_{11}^{2}-\tilde{\gamma}_{11}^{3}S_{21}^{2}\right)\left(I+A_{-}\right)^{-1}\right]_{+}\left(I+A_{+}\right)\left(I+N\right)+\tilde{\gamma}_{11}^{3}-I. \tag{29}$$

Finally, A_{42}^{6} is found for the third equality of system (14):

$$A_{42}^{6}=-A_{43}^{6}S_{12}^{2}-\left(I+A_{44}^{6}\right)S_{22}^{2}-A_{45+}^{6}S_{32}^{2}$$

After transformations we get:

$$A_{42}^{6}=-\tilde{\gamma}_{12}^{3}S_{12}^{2}-\tilde{\gamma}_{11}^{3}S_{22}^{2}+\left[\left(\tilde{\gamma}_{12}^{3}S_{11}^{2}+\tilde{\gamma}_{11}^{3}S_{21}^{2}\right)\left(I+A_{-}\right)^{-1}\right]_{+}\left(I+B\right). \tag{30}$$

From system (7), allowing for the boundary conditions (3), (5), we get the following system:

$$\left\{\begin{aligned}
u_{2}^{1}-u_{2}^{3}&=A_{23}^{6}\left(b_{3}^{1}-b_{3}^{3}\right)+A_{24}^{6}\left(b_{4}^{1}-b_{4}^{3}\right)+A_{25+}^{6}\left(b_{5}^{1}-b_{5}^{3}\right),\\
-u_{2}^{3}&=\left(I+A_{33}^{6}\right)\left(b_{3}^{1}-b_{3}^{3}\right)+A_{34}^{6}\left(b_{4}^{1}-b_{4}^{3}\right)+A_{35+}^{6}\left(b_{5}^{1}-b_{5}^{3}\right),\\
u_{2}^{1}-u_{1}^{3}&=A_{43}^{6}\left(b_{3}^{1}-b_{3}^{3}\right)+\left(I+A_{44}^{6}\right)\left(b_{4}^{1}-b_{4}^{3}\right)+A_{45+}^{6}\left(b_{5}^{1}-b_{5}^{3}\right),\\
u_{1}^{1}&=A_{53}^{6}\left(b_{3}^{1}-b_{3}^{3}\right)+A_{54}^{6}\left(b_{4}^{1}-b_{4}^{3}\right)+\left(I+A_{55+}^{6}\right)\left(b_{5}^{1}-b_{5}^{3}\right).
\end{aligned}\right.$$

As in the previous cases, using reversibility of the operator $\widetilde{S}_2$:

$$\widetilde{S}_2=\begin{pmatrix} S^3_{21} & -S^1_{21} & S^3_{22} & -S^1_{22}\\ S^3_{11} & -S^1_{11} & S^3_{12} & -S^1_{12}\end{pmatrix}:\begin{pmatrix} a_1\\ a_2\end{pmatrix}\to\begin{pmatrix} b^3_4 & -b^1_4\\ b^3_3 & -b^1_3\end{pmatrix};$$

factorization (8) and property (∗∗), from the last system, and also from system (15) and (16) we find coefficients $A^6_{5j},\ j=\overline{1,5}$:

$$\left\{\begin{array}{c} A^6_{51-}=\left[\left(\tilde{\gamma}^2_{12}S^3_{11}+\tilde{\gamma}^2_{11}S^3_{21}\right)(I+A_-)^{-1}\right]_-(I+A_-),\\ A^6_{55+}=\left[\left(\tilde{\gamma}^2_{12}S^3_{11}+\tilde{\gamma}^2_{11}S^3_{21}\right)(I+A_-)^{-1}\right]_+(I+A_+)-I,\\ A^6_{53}=\left[\left(\tilde{\gamma}^2_{12}S^3_{11}+\tilde{\gamma}^2_{11}S^3_{21}\right)(I+A_-)^{-1}\right]_+(I+A_+)(I+M)-\tilde{\gamma}^2_{12},\\ A^6_{54}=\left[\left(\tilde{\gamma}^2_{12}S^3_{11}+\tilde{\gamma}^2_{11}S^3_{21}\right)(I+A_-)^{-1}\right]_+(I+A_+)(I+N)-\tilde{\gamma}^2_{11},\\ A^6_{52}=\tilde{\gamma}^2_{12}S^3_{12}+\tilde{\gamma}^2_{11}S^3_{22}-\left[\left(\tilde{\gamma}^2_{12}S^3_{11}+\tilde{\gamma}^2_{11}S^3_{21}\right)(I+A_-)^{-1}\right]_+(I+B), \end{array}\right. \tag{31}$$

where $\tilde{\gamma}^2_{11},\tilde{\gamma}^2_{12}$ are the elements of the operator $\tilde{\gamma}^2$, reverse to the operator $\widetilde{S}_2$. In the considered case the system (D) takes the from:

$$A^6_{21-}-A^6_{31-}-A^6_{41-}-A^6_{51-}=D_1,$$

$$A^6_{22}-A^6_{32}-A^6_{42}-A^6_{52}=D_2,$$

$$A^6_{53}+A^6_{43}+A^6_{33}-A^6_{23}=D_3,$$

$$A^6_{54}+A^6_{44}+A^6_{34}-A^6_{24}=D_4,$$

$$A^6_{55+}+A^6_{45+}+A^6_{34+}-A^6_{25+}=D_5.$$

Here allowing for the values of the known operators, we find A^6_{21-}, A^6_{22}, A^6_{23}, A^6_{24}, A^6_{25+}.

Let's now consider case when $C_{21}=C_{23}=C_{24}=C_{25}=0$. Then, $u_2(x,t)=a_2(t+\xi_2 x)$. In this case the boundary conditions have the form:

$$\text{I.}\ \left\{\begin{array}{c} u^1_5(0,t)=u^1_1(0,t),\\ u^1_4(0,t)=u^1_2(0,t)=a_2(t),\\ u^1_3(0,t)=0\end{array}\right.$$

$$\text{II.}\ \left\{\begin{array}{c} u^2_5(0,t)=u^2_2(0,t)=a_2(t),\\ u^2_3(0,t)=u^2_1(0,t),\\ u^2_4(0,t)=0\end{array}\right.$$

$$\text{III.}\ \left\{\begin{array}{c} u^3_4(0,t)=u^3_1(0,t),\\ u^3_3(0,t)=u^3_2(0,t)=a_2(t)\\ u^3_5(0,t)=0\end{array}\right.$$

Analogously to the first case we find coefficients $A_{ij}^6,\ i,j=\overline{1,5}$ from system (7) and systems (11)-(16), using reversibility of operators $\widetilde{S}_1,\ \widetilde{S}_2,\ \widetilde{S}_3$, property $(**)$ and factorization properties (8), (9).

From systems (11), (12):

$$\begin{cases} A_{41-}^6=\left[\left(\tilde\gamma_{21}^1S_{11}^1+\tilde\gamma_{22}^1S_{21}^1\right)(I+A_-)^{-1}\right]_-(I+A_-)\,,\\ A_{45+}^6=\left[\left(\tilde\gamma_{21}^1S_{11}^1+\tilde\gamma_{22}^1S_{21}^1\right)(I+A_-)^{-1}\right]_+(I+A_+)\,,\\ A_{43}^6=\left[\left(\tilde\gamma_{21}^1S_{11}^1+\tilde\gamma_{22}^1S_{21}^1\right)(I+A_-)^{-1}\right]_+(I+A_+)(I+M)-\tilde\gamma_{21}^1,\\ A_{44}^6=\left[\left(\tilde\gamma_{21}^1S_{11}^1+\tilde\gamma_{22}^1S_{21}^1\right)(I+A_-)^{-1}\right]_+(I+A_+)(I+N)-\tilde\gamma_{22}^1-I,\\ A_{42}^6=I+\tilde\gamma_{21}^1S_{12}^1+\tilde\gamma_{22}^1S_{22}^1-\left[\left(\tilde\gamma_{21}^1S_{11}^1+\tilde\gamma_{22}^1S_{21}^1\right)(I+A_-)^{-1}\right]_+(I+B)\,; \end{cases} \tag{32}$$

from systems (15), (16):

$$\begin{cases} A_{31-}^6=\left[\left(\tilde\gamma_{22}^2S_{11}^3+\tilde\gamma_{21}^2S_{21}^3\right)(I+A_-)^{-1}\right]_-(I+A_-)\,,\\ A_{35+}^6=\left[\left(\tilde\gamma_{22}^2S_{11}^3+\tilde\gamma_{21}^2S_{21}^3\right)(I+A_-)^{-1}\right]_+(I+A_+)\,,\\ A_{33}^6=\left[\left(\tilde\gamma_{22}^2S_{11}^3+\tilde\gamma_{21}^2S_{21}^3\right)(I+A_-)^{-1}\right]_+(I+A_+)(I+M)-\tilde\gamma_{22}^2-I,\\ A_{34}^6=\left[\left(\tilde\gamma_{22}^2S_{11}^3+\tilde\gamma_{21}^2S_{21}^3\right)(I+A_-)^{-1}\right]_+(I+A_+)(I+N)-\tilde\gamma_{21}^2,\\ A_{32}^6=\tilde\gamma_{22}^2S_{12}^3+\tilde\gamma_{21}^2S_{22}^3-\left[\left(\tilde\gamma_{22}^2S_{11}^3+\tilde\gamma_{21}^2S_{21}^3\right)(I+A_-)^{-1}\right]_+(I+B)\,; \end{cases} \tag{33}$$

from systems (13), (14):

$$\begin{cases} A_{51-}^6=\left[\left(\tilde\gamma_{22}^3S_{11}^2+\tilde\gamma_{21}^3S_{21}^2\right)(I+A_-)^{-1}\right]_-(I+A_-)\,,\\ A_{55+}^6=\left[\left(\tilde\gamma_{22}^3S_{11}^2+\tilde\gamma_{21}^3S_{21}^2\right)(I+A_-)^{-1}\right]_+(I+A_+)-I,\\ A_{53}^6=\left[\left(\tilde\gamma_{22}^3S_{11}^2+\tilde\gamma_{21}^3S_{21}^2\right)(I+A_-)^{-1}\right]_+(I+A_+)(I+M)-\tilde\gamma_{22}^3,\\ A_{54}^6=\left[\left(\tilde\gamma_{22}^3S_{11}^2+\tilde\gamma_{21}^3S_{21}^2\right)(I+A_-)^{-1}\right]_+(I+A_+)(I+N)-\tilde\gamma_{21}^3,\\ A_{52}^6=I+\tilde\gamma_{22}^3S_{12}^2+\tilde\gamma_{21}^3S_{22}^2-\left[\left(\tilde\gamma_{22}^3S_{11}^2+\tilde\gamma_{21}^3S_{21}^2\right)(I+A_-)^{-1}\right]_+(I+B)\,. \end{cases} \tag{34}$$

We find remaining coefficients, i.e. $A_{11-}^6,A_{12}^6,A_{13}^6,A_{14}^6,A_{15+}^6$, from the system (D).

Thus, in the considered cases we proved that by scattering operator on semi-axis it is uniquely reestablished the passage operator Π, which ties together asymptotics $\{a_1(t),a_2(t),b_3(t),b_4(t),b_5(t)\}$ with the boundary values of solutions at zero $\{u_1(0,t),u_2(0,t),u_3(0,t),u_4(0,t),u_5(0,t)\}$.

In the paper [3] it was established, that by the known passage operator Π one can uniquely determine coefficients of system (1). So, we prove the theorem:

Theorem. *If coefficients of system (1) satisfy conditions (2) and condition* $C_{12}=\ldots=C_{15}=0$ $(C_{21}=\ldots=C_{25}=0)$, *then all remaining coefficients of this system are uniquely determined by the scattering operator* S.

References

[1]. Nizhnik L.P. *Inverse nonstationary scattering problem.* Kiev: Nauk. Dumka, 1973, 182 p. (Russian)

[2]. Nizhnik L.P. *Inverse scattering problems for the hyperbolic equations.* Kiev: Nauk. Dumka, 1991, 232 p. (Russian)

[3]. Nizhnik L.P., Tarasov V.G. *Inverse nonstationary scattering problem for the hyperbolic system of equations* // DAN SSSR, 1977, v.233, No3, pp.300-303. (Russian)

[4]. Iskenderov N.Sh. *Inverse scattering problem for the hyperbolic system of n equations of the first order on semi-axis* // Ukr. math. journ., 1991, v.43, No12, pp.1638-1646. (Russian)

[5]. Iskenderov N.Sh. *Nonstationary inverse scattering problems for a system of hyperbolic equations of the first order on semi-axis.* Baku, Elm, 2000, 185 p. (Russian)

[6]. Iskenderov N.Sh., Ismailov M.I. *Inverse nonstationary scattering problem for the hyperbolic system of four first order equations on semi-axis.* Proceedings of IMM of NASA, 1996, IV (XII), pp.161-168. (Russian)

[7]. Iskenderov N.Sh., Jabbarova K.A. *The scattering problem for a hyperbolic system of five first order equations on semi-axis with two given incident waves.* Transactions of NAS Azerbaijan, 2001, v.XXI, No4, pp.77-82.

[8]. Jabbarova K.A. *Properties of the scattering operator for a system of five first order hyperbolic equations with two given incident waves.* Proceedings of IMM, 2004, v.XX (XXVIII), pp.41-56.

Nizameddin Sh. Iskenderov, Kamilla A. Jabbarova
Institute of Mathematics and Mechanics of NAS of Azerbaijan
9, F.Agayev str., AZ1141, Baku, Azerbaijan
Tel.: (99412) 439 47 20 (off.)

Received September 02, 2005; Revised October 20, 2005.
Translated by Mamedova V.A.