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ON SOME CONNECTIONS BETWEEN RUNGE-KUTTA AND ADAMS METHODS

Abstract

The one- and multi-step methods whose classical representatives are Runge-Kutta and Adams methods are popular methods of numerical solution of the Cauchy problem for the ordinary differential equations. Each of these methods has its lacks and advantages. Here one scheme is given with the help of which it's possible to obtain the Adams method from Runge-Kutta method and vice versa.

Introduction. As is known, the first direct numerical method has been worked out by Euler (see [1]). The Euler method is developed in two directions as a result of that the Runge-Kutta and Adams methods appeared. Some technicians called the Adams method a developed form of the Runge-Kutta method (see, for example [1, p.293]). It's obvious that the Runge-Kutta and Adams methods cross at a point in which the Euler method is found. Thus we obtain that the Euler method is developed in two directions as a result of that the one- and multi-step methods appeared.

Taking into account that each of these directions has its advantage, some authors tried to construct the methods that combine the best properties of Runge-Kutta and Adams methods. As a result of such investigations the hybrid methods (see, for example [2]) appeared. Here we define some connections between Adams and Runge-Kutta methods.

1. Construction of k-step methods on the base of explicit Runge-Kutta form methods.

Consider the following Cauchy problem

$$y' = f(x, y), \ y(x_0) = y_0. \tag{1}$$

Assume that problem (1) has a unique solution $y(x)$ that is defined on the segment $[x_0, X]$.

It's known that by using the Runge-Kutta methods at every step we have repeatedly to compute the function $f(x, y)$. Note that by increase of accuracy of the method the amount of computations of the function $f(x, y)$ also increases. Moreover, the computed values of the function $f(x, y)$ at some fixed point aren't used at a neighboring point. For example, the following method

$$y_{n+1} = h(f(x_n, y_n) + f(x_n + h, y_n + hf(x_n, y_n)))/2 \tag{2}$$

is the second order Runge-Kutta method. As it follows from description of method (2), by increasing the values of variable n by 1 (unit), the values of the functions

$f(x_{n+1}, y_{n+1})$ and $f(x_{n+1}+h, hf(x_{n+1}, y_{n+1}))$ are computed again. These facts are the basic lacks of the Runge-Kutta methods.

In order on the base of method (2) to show the connection between Runge-Kutta and Adams methods, we construct the Adams method. To this end we consider the following scheme

$$\hat{y}_{n+1} = y_n + hf(x_n, y_n), \tag{3}$$

$$y_{n+1} = y_n + h(f(x_n, y_n) + f(x_{n+1}, \hat{y}_{n+1}))/2. \tag{4}$$

The indicated scheme is the predictor-correction method in which as a predictor the Euler method is used, and as a correction the method of trapezoid is used. Therefore, we can write method (2) subject to the Euler method in the following form

$$y_{n+1} = y_n + h(f(x_n, y_n) + f(x_{n+1}, \hat{y}_{n+1}))/2,$$

that enters to a class of Adams methods.

Consider the third order Runge-Kutta method having the following form

$$y_{n+1} = y_n + h(k_1 + 4k_2 + k_3)/6, \tag{5}$$

where

$$k_1 = f(x_n, y_n),\ k_2 = f\left(x_n + \frac{h}{2},\ y_n + \frac{h}{2}k_1\right), k_3 = f(x_n + h,\ y_n - hk_1 + 2hk_2).$$

In method (5) we substitute h for $2h$. Then we have

$$y_{n+2} = y_n + 2h(f(x_n, y_n) + 4f(x_n + h, y_n + hf(x_n, y_n)) +$$

$$+ f(x_n + 2h, y_n - 2hf(x_n, y_n) + 4hf(x_n + h, y_n + hf(x_n, y_n))))/6. \tag{6}$$

Using the Euler method in correlation (6) we can rewrite it in the following form

$$y_{n+2} = y_n + 2h(f_n + 4f_{n+1} + f(x_n + 2h, y_n - 2hf_n + 4hf_{n+1}))/6, \tag{7}$$

where $f_m = f(x_m, y_m) \quad (m = 0, 1, 2, ...)$.

Using the following scheme

$$y_{n+2} = y_n - 2hf_n + 4hf_{n+1} \tag{8}$$

in method (7) we obtain

$$y_{n+2} = y_n + h(f_n + 4f_{n+1} + f_{n+2})/3. \tag{9}$$

This method is the known Simpson method and enters to the class of Adams methods. Note that method (5) has the third order, and method (9) has the fourth order of accuracy. By obtaining method (9) from method (5) we used the Euler method and method (8) each of them has the second order of accuracy. Since we use these substitutions in the function $f(x, y)$, we can expect that as a result of above mentioned substitutions, the obtained method (9) will have the third order of accuracy. Really, if we denote by $y(x_n)$ an exact value of solution of problem (1) at the point x_n, then the error on step for the Euler method will have the following form

$$\varepsilon_{n+1} = y(x_{n+1}) - y_{n+1} = Ch^2 + O\left(h^3\right).$$

In method (6) we substitute the approximated values y_n of solution of problem (1) by exact ones. Then we have

$$y(x_{n+2}) = y(x_n) + 2hf(x_n, y(x_n)) + 4f(x_n + h, y(x_n) + hf(x_n, y(x_n)) +$$
$$+f(x_n + 2h, y(x_n) + 2hf(x_n, y(x_n)) +$$
$$+4f(x_n + h, y(x_n) + hf(x_n, y(x_n))))/6 + R_n, \qquad (10)$$

where R_n is an error of method (5). Since it is a representative of a class of the third order Runge-Kutta methods, we can write the error on step in the following form

$$R_n = C_1 h^4 + O\left(h^5\right). \qquad (11)$$

In order to estimate the error of method (9), we rewrite correlation (10) in the form

$$y(x_{n+2}) = y(x_n) + h(f(x_n, y(x_n)) + 4f(x_{n+1}, y(x_{n+1}) +$$
$$+O\left(h^2\right)) + f(x_{n+2}, y(x_{n+2}) + O\left(h^3\right)))/3 + R_n. \qquad (12)$$

Hence we can write

$$y(x_{n+2}) = y(x_n) + \frac{h}{3}(f(x_n, y(x_n)) + 4f(x_{n+1}, y(x_{n+1}) +$$
$$+f(x_{n+2}, y(x_{n+2}))) + R_n + O\left(h^3\right). \qquad (13)$$

If we subtract relation (9) from (13), we obtain that the error of the method has the following order

$$\varepsilon_{n+2} = Ch^3 + O\left(h^4\right). \qquad (14)$$

Here we assume that $\varepsilon_{n+1} = \varepsilon_n = 0$, since we compute error on step, i.e. local error of the method obtained from (5). Note that error of method (9) can be written in the form (see for example [3])

$$\varepsilon_{n+1} = Ch^4 + O\left(h^5\right). \qquad (15)$$

Here it's assumed that $\varepsilon_n = 0$. However, a local error of method (9) has the following form (see [4])

$$Ch^5 + O\left(h^6\right). \qquad (16)$$

As was noted method (5) has the third order of accuracy, but in it using superposition in the function $f(x, y)$ with respect to the second argument, the method of fourth order was constructed. In order to determine reason of increase the accuracy of the obtained method, we consider the following fourth order Runge-Kutta method

$$y_{n+1} = y_n + h\left(k_1 + 2k_2 + 2k_3 + k_4\right)/6, \tag{17}$$

where

$$k_1 = f\left(x_n, y_n\right), \quad k_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right),$$

$$k_3 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_2\right), \quad k_4 = f\left(x_n + h, y_n + hk_3\right).$$

For clearness of method (17) we rewrite it in the following form

$$y_{n+1} = y_n + h(f\left(x_n, y_n\right) + 2f(x_n + \frac{h}{2}, y_n + \frac{h}{2}f\left(x_n, y_n\right))+$$

$$+2f(x_n + \frac{h}{2}, y_n + \frac{h}{2}f(x_n + \frac{h}{2}, y_n + \frac{h}{2}f\left(x_n, y_n\right))+$$

$$+f(x_n + h, y_n + hf(x_n + \frac{h}{2}, y_n + \frac{h}{2}f(x_n + \frac{h}{2}, y_n + \frac{h}{2}f\left(x_n, y_n\right))))/6, \tag{18}$$

Using the explicit Euler method, we rewire (18) in the following form

$$y_{n+1} = y_n + h(f_n + 2f(x_n + \frac{h}{2}, y_{n+\frac{1}{2}}) + 2f(x_n + \frac{h}{2}, y_n + \frac{h}{2}f_{n+\frac{1}{2}})+$$

$$+f(x_n + h, y_n + hf(x_n + \frac{h}{2}, y_n + \frac{h}{2}f_{n+\frac{1}{2}})))/6,$$

where $f_m = f\left(x_m, y_m\right), \quad f_{m+\frac{1}{2}} = f\left(x_m + \frac{h}{2}, y_{m+\frac{1}{2}}\right), \quad (m = 0, 1, 2, ...).$

By using the implicit Euler method, we have

$$y_{n+1} = y_n + h(f_n + 2f_{n+\frac{1}{2}} + 2f_{n+\frac{1}{2}} + f(x_n + h, y_n + hf_{n+\frac{1}{2}}))/6. \tag{19}$$

In method (19) we substitute h for $2h$. Then we have

$$y_{n+2} = y_n + 2h\left(f_n + 4f_{n+1} + f\left(x_{n+2}, y_n + 2hf_{n+1}\right)\right)/6. \tag{20}$$

Using the midpoint method

$$y_{n+2} = y_n + 2hf_{n+1},$$

in method (20), we obtain

$$y_{n+2} = y_n + h(f_n + 4f_{n+1} + f_{n+2})/3. \tag{21}$$

Thus, the obtained method coincides with method (9). Note that method (21) is obtained from method (17) that is the fourth order Runge-Kutta method. With the help of the described scheme we constructed the two-step method of the fourth order accuracy on the base of the one-step methods of the third and fourth order accuracy. However, if we use the Runge-Kutta methods of the fifth and higher order for construction of k-steps methods ($k \geq 3$), then as a result we obtain the unstable methods. Therefore, the above-described scheme will be of useful at construction of one- and two-step methods. Now we consider the construction of the Runge-Kutta methods on base of the Adams method. Since the second order Runge-Kutta method is easily obtained from the method of trapezoids, we consider the Adams method of order higher than second order. It's obvious that such schemes will be from k-step methods, where $k \geq 2$. By the Dahlquist result we obtain that if the method

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h\sum_{i=0}^{k}\beta_i f_{n+i} \tag{22}$$

is stable, then $p \leq 2\,[k/2] + 2$ (see [5]). Here the coefficients $\alpha_i, \beta_i\,(i = 0, 1, ..., k)$ are some real numbers. Consequently, at $k = 2$ maximum order of accuracy for stable methods is equal to 4, i.e. $p_{\max} = 4$. Therefore for the construction of the Adams stable methods with accuracy order $p > 2$, the quantity k should satisfy the condition $k \geq 2$.

Let's consider the following method

$$y_{n+2} = y_{n+1} + h(5f_{n+2} + 8f_{n+1} - f_n)/12. \tag{23}$$

which is the Adams method and has the accuracy order $p = 3$.

It's easy to note that for obtaining the one-step method, we should assume $h = h/2$. Then method (23) takes the form of

$$y_{n+1} = y_{n+\frac{1}{2}} + h(5f_{n+1} + 8f_{n+\frac{1}{2}} - f_n)/24. \tag{24}$$

Using the Euler method for computation of $y_{n+\frac{1}{2}}$ from relation (24) we have

$$y_{n+1} = y_n + h(5f_{n+1} + 8f_{n+\frac{1}{2}} + 11f_n)/24.$$

The obtained method is the Runge-Kutta type method. Now from this method we construct semiexplicit Runge-Kutta type method. To this end we use the method of trapezoids at $h = h/2$. Then we have

$$y_{n+1} = y_n + h(11k_1 + 8k_2 + 5k_3)/24, \tag{25}$$

where $k_1 = f\,(x_n, y_n)\,, k_2 = f(x_n + \frac{h}{2}, y_n + h(k_1 + k_2)/4),\; k_3 = f\,(x_n + h, y_n + hk_2)\,.$

Batcher has suggested a such scheme earlier. However we can select k_2 and k_3 in the following form

$$k_2 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1), \quad k_3 = f\,(x_n + h, y_n + h\,(k_1 + k_3)\,/2)\,.$$

Thus, we showed that with the help of the above described scheme, we can construct implicit and semiexplicit Runge-Kutta methods. Now we consider the constructed explicit methods of Runge-Kutta type. To this end we consider method (23).

For obtaining the Runge-Kutta method form method (23) we substitute h for $h/2$. Then we have

$$y_{n+1} = y_{n+\frac{1}{2}} + h(5f(x_n + h, y_{n+1}) + 8f(x_n + \frac{h}{2}, y_{n+\frac{1}{2}}) - f\,(x_n, y_n))/24. \qquad (26)$$

The free term $y_{n+\frac{1}{2}}$ participating in the right hand side of (26) we compute with the help of the following method:

$$y_{n+\frac{1}{2}} = y_n + h(f(x_n, y_n) + f(x_n + \frac{h}{2}, y_{n+\frac{1}{2}}))/4. \qquad (27)$$

Subject to (27), we rewrite method (26) in the following form

$$y_{n+1} = y_n + h(5f(x_n + h, y_{n+1}) + 14f(x_n + \frac{h}{2}, y_{n+\frac{1}{2}}) + 5f\,(x_n, y_n))/24. \qquad (28)$$

In the obtained method for computation of the quantities y_{n+1} and $y_{n+\frac{1}{2}}$ participating in the right hand side of (28), we use the Euler explicit method. Then we can write the method obtained from (28) in the form of

$$y_{n+1} = y_n + h(5f\,(x_n, y_n) + 14f(x_n + \frac{h}{2}, y_n + \frac{h}{2}f\,(x_n, y_n)) +$$
$$+5f(x_n + h, y_n + hf\,(x_n, y_n))/24.$$

If denote by

$$k_1 = f\,(x_n, y_n)\,, \quad k_2 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1), \quad k_3 = f\,(x_n + h, y_n + hk_1)\,,$$

then we can write

$$y_{n+1} = y_n + h(5k_1 + 14k_2 + 5k_3)/24. \qquad (29)$$

This method remind the Runge-Kutta method, however, it doesn't enter to the class of the third order Runge-Kutta methods.

Now we consider method (9).

For construction of the Runge-Kutta method on the base of Simpson's method, we substitute h for $h/2$. Then we have

$$y_{n+1} = y_n + h(f_{n+1} + 4f_{n+\frac{1}{2}} + f_n)/6. \qquad (30)$$

Using the notation

$$k_1 = f\,(x_n, y_n)\,, \quad k_2 = f(x_n + \frac{h}{2}, y_n + \frac{1}{2}hk_1), \quad k_3 = f\,(x_n + h, y_n + 2hk_2 - hk_1)\,,$$

we can write method (30) in the following form

$$y_{n+1} = y_n + h(k_1 + 4k_2 + k_3)/6. \qquad (31)$$

At construction of method (31), it is used the Euler implicit method, next its modification based on the following relation

$$y'_{n+1} = 2y'_{n+\frac{1}{2}} - y'_n + O\left(h^3\right).$$

Really, after substitution $y_{n+1} = y_n + hy'_{n+1}$, $y'_{n+\frac{1}{2}} = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1)$ and $y'_n = f\left(x_n, y_n\right)$, we obtain method (31).

As is know method (31) is the third order Runge-Kutta method. Thus, we showed that we can construct a multi-step method with constant coefficients from each explicit Runge-Kutta method. However the inverse isn't always valid, i.e. we cannot construct an explicit Runge-Kutta method of appropriate orders from each Adams method.

Note that with the help of the predictor-correction method we can extend the domain of stability of the Adams method (see [6]). As is known, the boundary of the domain of stability for forward jumping method is equal to zero. However applying the predictor-correction method to these methods, we can extend their domain of stability (see [7]).

References

[1]. Krylov A.N. *Lectures on approximated computations.* M.: "Gostekhisdat", 1954, 400p. (Russian)

[2]. Ibrahimov V.R. *One nonlinear method of numerical solution of the Cauchy problem for ordinary differential equations.* Differential equations and its application. Proceedings of lectures Second Intern. Conf., June 29 – Jule 4, 1987, Russa, Bulgaria, Russ, 1982, pp.310-319. (Russian)

[3]. Mamedov Ya.J. *Approximated methods of solutions of ordinary differential equations.* Baku, Maarif, 1974, 175p. (Russian)

[4]. Berezin I.S., Zhidkov N.P. *Computing methods.* M., Fizmatgiz, 1962, 639p. (Russian)

[5]. Dahlquist G. *Convergence and stability in the numerical integration of ordinary differential equations.* Math.Scand. – 1956, No4, pp.33-53.

[6]. Lapidus L., Seinfeld J.H. *Numerical solution of ordinary differential equations.* New York, Londan, 1971.

[7]. Ibrahimov V.R., Mehdiyeva G.Yu., Nasirova I.I. *On the expansion of stability region for the forward jumping methods.* Transactions of NAS of Azerbaijan, 2005.

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