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ON THE ERGODICITY OF A MICROSCOPIC TRAFFIC MODEL FOR A ROAD NETWORK

Abstract

Vehicular traffic is analyzed in computer simulations. Often, microscopic traffic models, i.e. models in which every vehicle is modeled on its own, are used. The high-dimensional microscopic states are transformed into macroscopic values by taking the mean of some interesting values, like the traffic flow, the traffic density, or the fundamental diagram. The question is, whether such an approach delivers reliable results. Necessary for this is the existence of a Law of Large Numbers.

We will prove an ergodic theorem for a prototypical traffic model with general state space, based on a model introduced by Poppinga. From this ergodic theorem some LLNs will be derived. Finally we will give possible extensions of the model and discuss how the proof might be extended to such models.

1.Introduction

Vehicular traffic plays an important role in our society and much scientific work is done in this area. One great challenge is to simulate macroscopic effects like traffic densities or traffic flow in microscopic traffic models. The microscopic models are usually stochastic, since deterministic models are unable to reproduce observed facts like the occurrence from traffic jams "out of nothing" (c.f. [4], p. 109). These traffic models are therefore stochastic processes X_t with $t \in \mathbb{N}_0$ or $t \in \mathbb{R}_+$. The macroscopic values are functionals defined on the values of X_t. In fact, the practically important functionals are the mean traffic density (measure in vehicles per meter), the mean traffic flow (measured in vehicles per second), and the fundamental diagram. The last describes the (mean) traffic flow as a function of the traffic density.

Observing these mean values one might ask, whether one can prove limit theorems about these random variables. Of course, the answer depends on the chosen model. In this paper, we analyse a model for vehicles driving on a network of roads. We will prove, that the stochastic process is ergodic and hence the distribution converges towards a stationary distribution. Furthermore, the mean values converge towards the respective limit values, which can be expressed with the stationary distribution.

The notion of microscopic traffic models is quite old. It appeared already in [5]. The idea is always to derive traffic properties by modeling the behavior the single vehicles. For an overview of the proposed models we refer to [4, 2]. The evaluation of such a model is done as follows: Let the model run on a computer and compare the results with what should occur. With this paper, we can show prototypically for one model, that such a physical reasoning can be justified mathematically. For all starting distributions, the distribution over the state space converges towards a stationary distribution, and we get a law of large numbers for the mean values. Questions of this kind are e.g. analyzed for traffic at a bottleneck in [3] or for traffic on a circle in [7]. We will use a dynamic, inspired by the one introduced in [7], for the traffic on a network of roads. The proof technique should extend to various other models and especially to those with a discrete state space, e.g. the various cellular automaton models, which were developed during the last years, starting with [8] and culminating in [9].

We will describe and formalize our model and the macroscopic values in section 2. The theorems about the limit behavior of the model will be given in section . It will be proved that the model is uniformly ergodic and hence that we have the desired convergency of the distribution. Moreover we will get some Laws of Large Numbers for the macroscopic values. The corresponding proofs are in section . With the theoretical background of section 3 the derived values from this simulation are known to come from consistent estimators for the corresponding theoretical values, which justifies the use of these estimators in practical traffic observation or in simulations of traffic. A discussion about possible extensions of the model and some open questions will be given in section 5.

2. The Model

2.1. Overview

We will model the following traffic situation: Let there be $n \in \mathbb{N} = \{1, 2, \dots\}$ vehicles which stay in an enclosed area, e.g. a city. Some vehicles are passive, i.e. they are parking. Other vehicle are moving and we assume that they have a destination and a route which connects their current position and the destination. As it was said in the introduction, the model will be a Markov chain. The state space should therefore contain for each vehicle the information where it is, where its destination is, and how it will get there. The formal definition of the state space $(H, \mathcal{H})$ will be given in subsection 2.2.

The motion of the vehicles during the next time step will depend stochastically on the positions of the other vehicles. We will model this as a (time discrete) Markov chain and the dynamic will be described by a stochastic kernel $P : H \times \mathcal{H} \to [0, 1]$. Note however that we make no assumptions on the length (e.g. in seconds) of a time step. The details are in subsection 2.3.

The state space $(H, \mathcal{H})$ and the kernel P define a Markov chain $(X_t)_{t \in \mathbb{N}_0}$. The macroscopic traffic values, i.e. the traffic density, the traffic flow, and the fundamental diagram, will be defined as functions of X_t. This is done in subsection 2.4.

2.2.The State Space

The road network is modeled as a directed graph (V, E) where each edge e has a positive length $l_0(e)$. A two way road is simply modeled by two edges in the graph. We assume, that the graph (V, E) is strongly connected, i.e. for every $(x, y) \in V \times V$, there exists a directed path from x to y. We will call the tripel (V, E, l_0) a **road network**.

A position of a vehicle in the graph will be described by a tuple (e, r), where e is an edge and r is the distance from the beginning of the edge, i.e. $0 \le r < l_0(e)$. Every edge is isomorphic to an interval $[0, l[$ on the real line which is equipped with the Borel-σ-algebra $\mathcal{B} \cap [0, l[$. All edges together form the position-space. Formally, the **position space** is the measure space $(Q, \mathcal{Q}, \lambda)$, where Q is the set of all positions, $\mathcal{Q}$ is the respective Borel-σ-algebra, and λ is the respective Lebesgue measure.

Let W be the set of all directed paths in the graph. This set is countable and thus $(W, \mathcal{P}(W))$ is a measurable space. We call it the **path space**. In the following, we will write a path $w \in W$ usually in the form $w = e_1 \dots e_k$, with the e_i being the edges which form the path.

A route should consist of a starting point, a path, and a destination. Formally, we define the **route space** to be

$$(H_0, \mathcal{H}_0) := (Q \times W \times Q, \mathcal{Q} \otimes \mathcal{W} \otimes \mathcal{Q}).$$

Before we give the definition of the state space, as the product of route spaces we have to consider some aspects of routes. An element $(a_0, w_0, b_0) \in H_0$ might have no real meaning since a_0 might not lie on the first edge of w_0 or b_0 might not lie on the last edge of w_0. Nevertheless, H_0 contains these routes and hence H_0 keeps its product space structure. However, the stochastic process will be concentrated on the meaningful routes, which we will call valid. In the following we will use the terminology "a point lies on an edge" or "a point is an inner point of the route". The meaning of this should be clear without formal definition.

Definition 2.1 *Let $(H_0, \mathcal{H}_0)$ be a route space of a graph G. A route $(a_0, w_0, b_0) \in H_0$ is called* **valid**, *if a_0 lies on the first edge of w_0, b_0 lies on the last edge of w_0 and no inner point of the route will be passed twice. w_0 is called* **passive** *if w_0 consists of one edge and $a_0 = b_0$. w_0 is* **active** *if w_0 is valid and not passive.*

Note, that passive routes are of course also valid. The above defined sets are all measurable, which is shown in the next lemma.

Lemma 2.2 *Let G be a road network and $(H_0, \mathcal{H}_0)$ the route space of G. The set of all valid routes is measurable (i.e. an element of $\mathcal{H}_0$), the set of all passive routes is measurable, and the set of all active routes is measurable.*

Proof For the set of all valid routes consider a fixed path w_0. The set of all valid routes, which use w_0 is trivially measurable and there are only countably many paths in W. The same idea works for passive and for active routes.

The definition of the state space can now be given as the product of route spaces.

Definition 2.3 *Let (V, E, l_0) be a road network, let be $n \in \mathbb{N}$ and for all $i = 1, \dots, n$ let $(H_i, \mathcal{H}_i)$ be a copy of the respective route space $(H_0, \mathcal{H}_0)$. Then*

$$(H, \mathcal{H}) := \bigotimes_{i=1}^{n} (H_i, \mathcal{H}_i)$$

is the **state space** *for G and n. The set of all* **valid states** *is*

$$H_v := \{(x_1, \dots, x_n) \in H \,|\, x_i \text{ is valid}, i = 1, \dots, n\}.$$

The set of all **passive states** *is*

$$H_p := \{(x_1, \dots, x_n) \in H \,|\, x_i \text{ is passive}, i = 1, \dots, n\}.$$

H_v and H_p are of course measurable (c.f. Lemma 2.2). In the following we will write a state $x \in H$ also in the form $(x_1, \dots, x_n) \in H$ or $(a, w, b) = = ((a_1, w_1, b_1), \dots, (a_n, w_n, b_n)) \in H$.

2.3.The Dynamic

The dynamic of the Markov chain is motivated by the following assumptions: A vehicle is either passive or active. If the i-th vehicle is passive then in each time step there is a probability p_i that the vehicle gets activated again. If vehicle i is activated then a destination (i.e. an edge and a position on that edge) is chosen with respect to a measure μ_i on the point space $(Q, \mathcal{Q})$. A valid route is chosen with respect to a kernel J_i.

If the i-th vehicle is active, then in every time step it travels an amount which depends on the free space that this vehicle has in front of it. We will make almost no assumptions on this amount. The only requirements are: The amount can be described by a family of Lebesgue densities and this amount should be a sensible part of the free space (c.f. 3.2).

The rest of this subsection is used to describe this dynamic formally as a stochastic kernel $P : H \times \mathcal{H} \to \mathbb{R}_+$. For this, we use $n \in \mathbb{N}$, (V, E, l_0), $(Q, \mathcal{Q})$, $(W, \mathcal{W})$, $(H_0, \mathcal{H}_0)$ and $(H, \mathcal{H})$ as defined above.

Assume, that $x_0 = (a_0, w_0, b_0) \in H_0$ is a valid route with $w_0 = e_1 \dots e_k$, $k \geq 1$, $a_0 = (e_1, r_1)$ and $b_0 = (e_k, r_k)$. Then the length of x_0 is given by

$$l(x_0) := \sum_{i=1}^{k-1} l_0(e_i) - r_1 + r_k.$$

The mapping $l : H_0 \to \mathbb{R}_+$ is measurable (by a similar argument as in the proof of Lemma 2.2) and $l(a_0, w_0, b_0) = 0$ iff (a_0, w_0, b_0) is passive.

Let $x_0 \in H_0$ be a valid route and let $q \in Q$ be a point which lies on the route x_0. Then, without formal definition, we define $d_{x_0}(q)$ to be the distance from the start of x_0 to q. If $q' \in Q$ is a point on the graph which does not lie on x_0 then define $d_{x_0}(q') := +\infty$. With this we define the free space $\alpha_i(x)$ for the i-th vehicle in a state x:

Definition 2.4 *For all* $i = 1, \dots, n$ *define the function* $\alpha_i : H \to \overline{\mathbb{R}}_+$ *as*

$$\alpha_i(x) := \inf\{d_{x_i}(a_j) \,|\, j \in \{1, \dots, n\}, x_j \text{ is active}, d_{x_i}(a_j) > 0\},$$

with the notation

$$x = (x_1, \dots, x_n) = ((a_1, w_1, b_1), \dots, (a_n, w_n, b_n)).$$

The mapping α_i is measurable. $\alpha_i(x) = +\infty$ means that at the moment the i-th vehicle has free space in front of it up to its destination.

A new route for the i-th vehicle will be chosen in two steps. First a new destination is chosen via a destination measure. Afterwards a path in the graph is selected.

The former is described via a probability measure μ_i on $\mathcal{Q}$. The later is given by a stochastic kernel $J_i : (H_0 \times H_0) \times \mathcal{W} \to \mathbb{R}$. Together with the start and the end, the path should give a valid and active route. To be precise we formalize this in the following definition. Note that the set U_{a_0,b_0}, which is used in Definition 2.5, is measurable since $\mathcal{W}$ contains all subsets of W. Furthermore U_{a_0,b_0} is not empty, since the graph is strongly connected and we allow active routes where the starting point and destination are the same (c.f. Definition 2.1).

Definition 2.5 *For all* $a_0, b_0 \in Q$ *define*

$$U_{a_0,b_0} := \{w_0 \in W \mid (a_0, w_0, b_0) \text{ is active}\}.$$

A **route-choice-kernel** *is a stochastic kernel* $J_0 : (H_G \times H_G) \times \mathcal{W}_G \to [0, 1]$ *such that* $J_0((a_0, b_0), U_{a_0,b_0}) = 1$ *for all* $a_0, b_0 \in Q$.

Now we define how far a vehicle will move during the next time step. In our model this space depends stochastically on the amount of free space in front of the vehicle but it does not depend on the earlier positions of the vehicle. We will discuss possible extensions of the model and the implications on the ergodic theorem in section 5.

Let the free space be α. Then the used part of the free space is realized stochastically. The respective probability measure is given by a Lebesgue-density f_α.

Definition 2.6 *Let* $\alpha \in \mathbb{R}$, $\alpha > 0$. *A mapping* $f(\alpha, .) : \mathbb{R} \to \mathbb{R}$ *is a* **driving-density**, *if* $f(\alpha, .)$ *is a Lebesgue-density with* $\operatorname{supp} f(\alpha, .) \subset [0, \alpha]$.

The condition $\operatorname{supp} f(\alpha, .) \subset [0, \alpha]$ ensures that no accident occurs, i.e. that no two vehicles are at the same position. We will now define the stochastic kernel P for the dynamic. Without formal definition we will use the following notation. If x_0 is a valid route and $a, b \in Q$ are points which lie on x_0 and a lies before b or equals b then $x_0|_a^b$ is the **shortened route**, starting in a using parts of the path of x_0 and ending in b.

The dynamic is given by a stochastic kernel. It formalizes the above described intuition about the behavior of passive and active cars. A passive car with number i stays passive with probability $(1 - p_i)$ and gets activated with probability p_i. If it is activated, then it gets a new destination and a route. This is the meaning of equation (1) in Definition 2.7.

If the i-th vehicle is active, then it travels an amount which is given via the Lebesgue-Density $f_i(\alpha_i(x), .)$. If the vehicle reaches its destination before driving the whole (stochastically realized) distance then it stops earlier. In any case, the destination does not change and the path will become a shortened path as defined above. In the next definition we use again the notation $d_{x_i}(a')$ for the distance from the start of the route x_i to a'.

Definition 2.7 *For all* $i = 1, \dots, n$ *let be* $p_i \in (0, 1)$, *let* μ_i *be a probability measure on* $(Q, \mathcal{Q})$, *let* J_i *be a route-choice-kernel and let* f_i *be a driving-density. Then we define the stochastic kernel* $P_i : H \times \mathcal{H}_i \to \mathbb{R}_+$ *as follows: If* x_i *is passive then let be*

$$P_i(x, .) := (1 - p_i)\delta_{x_i} + p_i \cdot (\delta_{a_i} \otimes \mu_i \otimes J_i). \tag{1}$$

If $x_i = (a_i, w_i, b_i)$ is active then let be

$$R_i := \{x_i|_{a'_i}^{b_i} \mid a'_i \text{ lies on } x_i\},$$

(i.e. the set of all reachable states). Define

$$P_i(x, \{(b_i, w_i, b_i)\}) := \int\limits_{[d_{x_i}(b_i), +\infty[} f_i(\alpha_i(x), .)\, \mathrm{d}\lambda. \tag{2}$$

For each measurable set $S_i \subset R_i \backslash \{(b_i, w_i, b_i)\}$ define

$$P_i(x, S_i) := \int\limits_{\left\{d_{x_i}(a'_i) \,\middle|\, x_i|_{a'_i}^{b_i} \in S_i\right\}} f_i(\alpha_i(x), .)\, \mathrm{d}\lambda. \tag{3}$$

We have $P_i(x, R_i) = \int_{[0,+\infty[} f_i(\alpha_i(x), .)\, \mathrm{d}\lambda = 1$ for all valid x and hence the probability measure $P_i(x, .)$ is uniquely determined on R_i. Finally we define $P_i(x, C_i) := P_i(x, C_i \cap R_i)$ for all measurable sets $C_i \in \mathcal{H}_i$.

If x_i is not valid, define $P_i(x, .)$ as an arbitrary (but fixed, i.e. equal for all invalid x) measure which is concentrated on the valid states. The kernel P is then defined as

$$P(x, .) := \bigotimes_{i=1}^{n} P_i(x, .). \tag{4}$$

P is a stochastic kernel for the Markov chain which describes the dynamic of our model. From Definition 2.5 and Definition 2.7 follows for all $x \in H$

$$P(x, H_v) = 1.$$

Hence the paths of the Markov chain will stay in H_v with probability one (even when it starts in an invalid state).

2.4. Macroscopic values

Suppose now that $(X_t)_{t\in\mathbb{N}}$ is a Markov chain with state space $(H, \mathcal{H})$ and transition kernel P as defined above. Such a Markov chain always exists (c.f. [6], Theorem 3.4.1.). We call such a Markov chain a **traffic model** for the respective parameters.

Easy to prove is that $(X_t, X_{t+1})_{t\in\mathbb{N}}$ is a Markov chain with state space $(H \times H, \mathcal{H} \otimes \mathcal{H})$ and transition Kernel P' given by

$$P'((x, x'), C \times C') = \delta_{x'}(C) \cdot P(x', C'). \tag{5}$$

A state of this Markov chain combines two consecutive states of the Markov chain $(X_t)_{t\in\mathbb{N}}$.

First we define the traffic flow as a function of the Markov chain. Informally, the traffic flow in a point $q \in Q$ at time $t \in \mathbb{N}$ is the number of vehicles that pass the point from time t to time $t + 1$.

Definition 2.8 *Let* $(X_t, X_{t+1})_{t\in\mathbb{N}}$ *be as defined above. Let further be* $q \in Q$ *and* $t \in \mathbb{N}$. *Then the* **traffic flow in** q **at time** t *is*

$$F_{q,t} := h_q \circ (X_t, X_{t+1})$$

with

$$h_q(x, x') := |\{i \in \{1, \dots, n\} \,|\, q \text{ lies on } x \text{ and } q \text{ does not lie on } x'\}|$$

for all $x, x' \in H$. *The* **mean traffic flow in** q **at time** t *is*

$$\overline{F}_{q,t} := \frac{1}{t} \sum_{k=1}^{t} F_{q,k}.$$

The traffic density in a measurable part C of the roads is the number of vehicles in C divided by the length of C.

Definition 2.9 *Let* $(X_t)_{t\in\mathbb{N}}$ *be as defined above. Let further be* $C \in \mathcal{Q}$ *with* $\lambda(C) > 0$ *and* $t \in \mathbb{N}$. *Then the* **traffic density in** C **at time** t *is*

$$D_{C,t} := h_C \circ X_t$$

with

$$h_C(x) := \frac{|\{i \in \{1, \dots, n\} \,|\, x_i = (a_i, w_i, b_i) \text{ is active}, a_i \in C\}|}{\lambda(C)}$$

for all $x \in H$. *The* **mean traffic density in** C **at time** t *is*

$$\overline{D}_{C,t} := \frac{1}{t} \sum_{k=1}^{t} D_{C,k}.$$

The fundamental diagram maps every possible traffic density r to the mean traffic flow over all time steps where the traffic density was r.

Definition 2.10 *Let* $(X_t, X_{t+1})_{t\in\mathbb{N}}$ *be as defined above, let be* $q \in Q$, $C \in \mathcal{Q}$ *with* $\lambda(C) > 0$, $r \in \{0, \frac{1}{\lambda(C)}, \frac{2}{\lambda(C)}, \dots, \frac{n}{\lambda(C)}\}$, *and let be* $t \in \mathbb{N}$. *Then define*

$$\overline{\Phi}_{q,C,t}(r) := \frac{\sum_{k=1}^{t} F_{q,k} 1_{\{D_{C,k}=r\}}}{\sum_{k=1}^{t} 1_{\{D_{C,k}=r\}}}, \tag{6}$$

provided the density r *is realized at least once (and hence we do not divide by zero). Otherwise let the value be zero.*

3. The Ergodic Theorem

Given a Markov chain $(X_t)_{t\in\mathbb{N}}$ with state space $(H, \mathcal{H})$ and transition kernel P, we can look for the sequence of kernels $(P^m)_{m\in\mathbb{N}}$. It describes the evolution of the Markov chain after m time steps. These kernels are usually defined inductively as $P^0(x, A) := 1_A(x)$ and

$$P^{m+1}(x, A) := \int P^m(y, A)\, P(x, \mathrm{d}y)$$

for all $m \geq 0$.

The two main questions are:

1. Is there a limit distribution $\lim_{m\to\infty} P^m(x,.)$?
2. Do the mean values (as they are defined in subsection 2.4) converge, i.e. do we have a LLN (Law of Large Numbers)?

Before answering these questions we have to define the type of convergence. We will use the terminology as it is used in [6]. For the first question the distance of the distributions is measured in terms of the total variation norm which is defined for signed measures μ on $\mathcal{H}$ as

$$\| \mu \| := \sup_{A\in\mathcal{H}} \mu(A) - \inf_{A\in\mathcal{H}} \mu(A).$$

The following Definition 3.1 is taken from [6], equation (16.6). Recall, that an invariant measure π for the Markov chain is one which fulfills

$$\pi(A) = \int P(x,A)\,\pi(\mathrm{d}x)$$

for all $A \in \mathcal{H}$.

Definition 3.1 *A Markov chain $(X_t)_{t\in\mathbb{N}}$ with state space $(H,\mathcal{H})$ and transition kernel P is* **uniformly ergodic** *if there exists an invariant measure π and furthermore*

$$\lim_{m\to\infty} \sup_{x\in H} \| P^m(x,.) - \pi \| = 0. \tag{7}$$

If π fulfills equation (7) then π is in fact a probability measure (since $P^m(x,H) = 1$ for all $m \in \mathbb{N}$). We remark that an aperiodic, irreducible Markov chain is uniformly ergodic iff it fulfills Doeblin's Condition (c.f. [6], Theorem 16.2.3). We will not make use of this fact and hence we will not go into the details.

Our model is uniformly ergodic provided the driving-densities fulfill the condition of the following theorem which is a generalization of *Satz* 2.1.3 in [7].

Theorem 3.2 [Ergodic Theorem] *Let $(f_i(\alpha,.))_{\alpha>0}$, $i = 1,\dots,n$ be the driving-densities for a traffic model $(X_t)_{t\in\mathbb{N}}$. Assume that there are constants $\alpha_0, c, \beta_1 > 0$ such that for all $i = 1,\dots,n$ we have*

$$\int_{[c\alpha;\alpha]} f_i(\alpha,.)\,\mathrm{d}\lambda \geq \beta_1 \quad \textit{for all } 0 < \alpha \leq \alpha_0 \tag{8}$$

and

$$\int_{[c\alpha_0;\alpha]} f_i(\alpha,.)\,\mathrm{d}\lambda \geq \beta_1 \quad \textit{for all } \alpha > \alpha_0. \tag{9}$$

Then the traffic model (resp. the Markov chain $(X_t)_{t\in\mathbb{N}}$) is uniformly ergodic.

We remark that these conditions are not very strong. Stated informally we assume that each vehicle uses a sensible amount of the free space with probability at least β_1.

For little free space ($\alpha \leq \alpha_0$) the sensible amount is $c\alpha$ and for much free space ($\alpha > \alpha_0$) the amount is $c\alpha_0$. The theorem will be proved in section 4.

Corollary 3.3 *If the conditions of Theorem 3.2 are fulfilled for the traffic model $(X_t)_{t\in\mathbb{N}}$ then the Markov chain $(X_t, X_{t+1})_{t\in\mathbb{N}}$ is uniformally ergodic.*

The corollary will also be proved in section 4. Having these theorems, we can answer the second question. Herein we use the notation P_*-a.s. as in [6], p. 69: A statement holds P_*-a.s. iff it holds P_x-a.s. for all $x \in H$. As usual P_x denotes the distribution of the Markov chain if it starts in the state x.

Corollary 3.4 *Let X_t be a traffic-model which fulfills the conditions of Theorem 3.2 with state space $(H, \mathcal{H})$ and transition kernel P. Define P' as in equation (5) and let π and π' be the invariant measures for $(X_t)_{t\in\mathbb{N}}$ resp. $(X_t, X_{t+1})_{t\in\mathbb{N}}$. Then we have*

$$\lim_{t\to\infty} \overline{F}_{q,t} = \mathbb{E}_{\pi'}(h_q) \quad P_*\text{-}a.s.,$$

$$\lim_{t\to\infty} \overline{D}_{C,t} = \mathbb{E}_{\pi}(h_C) \quad P_*\text{-}a.s.,$$

and for r as in Definition 2.10 we have: If $\pi(h_C^{-1}(\{r\})) > 0$, then

$$\lim_{t\to\infty} \overline{\Phi}_{q,C,t}(r) = \frac{1}{\pi(h_C^{-1}(\{r\}))} \int_{h_C^{-1}(\{r\})} h_q \mathrm{d}\pi' \quad P_*\text{-}a.s..$$

In the proof of Corollary 3.4 we will use Theorem 17.0.1 from [6]. It states the following: Suppose that X_t is a positive Harris chain with state space $(H, \mathcal{H})$ and with invariant probability measure π. Assume further that $g : H \to \mathbb{R}$ is an integrable mapping. Then we have the following LLN:

$$\lim_{t\to\infty} \frac{1}{t} \sum_{k=1}^{t} g(X_k) = \int g \,\mathrm{d}\pi \quad P_*\text{-a.s.}.$$

Proof of Corollary 3.4 We have

$$\overline{F}_{q,t} = \frac{1}{t} \sum_{k=1}^{t} h_q(X_k, X_{k+1})$$

and

$$\overline{D}_{C,t} = \frac{1}{t} \sum_{k=1}^{t} h_C(X_k).$$

The chains (X_t) resp. (X_t, X_{t+1}) are uniformly ergodic and hence they are positive Harris chains with invariant probability measures π resp. π'. Furthermore the mappings h_q and h_C are bounded and thus integrable. Using the above quoted Theorem 17.0.1 from [6] we conclude that we have a Law of Large Number for h_q and h_C. This proves the first two equations. Further by equation (6) we have

$$\overline{\Phi}_{q,C,t}(r) = \frac{\frac{1}{t}\sum_{k=1}^{t} g_1(X_k, X_{k+1})}{\frac{1}{t}\sum_{k=1}^{t} g_2(X_k)},$$

with $g_1(x,x') = h_q(x,x') \cdot 1_{h_C^{-1}(\{r\})}(x)$ and $g_2(x) = 1_{h_C^{-1}(\{r\})}(x)$. The mappings g_1 and g_2 are again bounded and we therefore have a LLN for the enumerator and the denominator, which proves the third equation.

Corollary 3.4 can be formulated in the following way: The mean values are consistent estimators for parameters of the traffic model. These parameters can thus be estimated by computer simulations, which is the usual approach in applied sciences. Our contribution is to prove that this approach is indeed justified from a mathematical point of view.

4. The Proof

4.1. Overview

In this section we will prove Theorem 3.2 and Corollary 3.3. As in [7] we will do this by showing that there are constants $t_0 \in \mathbb{N}$ and $\beta > 0$ and a probability measure ν on $\mathcal{H}$ such that for all $x \in H$, $C \in \mathcal{H}$

$$P^{t_0}(x, C) \geq \beta\nu(C). \tag{10}$$

Equation (10) means that the whole space H is ν-small. Theorem 16.0.2 in [6] states, that H is ν-small iff the Markov chain is uniformly ergodic. Thus equation (10) would prove Theorem 3.2. Having this we can prove Corollary 3.3 by showing a simar equation for the kernel in equation (5).

In the proof we will use the Chapman-Kolmogorov equation: Let be $t_1, t_2 \in \mathbb{N}$ with $t_1 + t_2 = t$. Then we have for all $x \in H$, $C \in \mathcal{H}$

$$P^t(x, C) = \int P^{t_2}(y, C)\, \mathrm{d}P^{t_1}(x, dy).$$

Given a set $D \in \mathcal{H}$, which might depend on x and C, we have as a corollary

$$P^t(x, C) \geq \int_D P^{t_2}(y, C)\, \mathrm{d}P^{t_1}(x, dy). \tag{11}$$

4.2. The main parts of the proof

Showing equation (10) requires a lot of details. In this section the key steps of the proof are given. They are formulated as four lemmata which are proved in subsection 4.3. These lemmata can be explained with the intuition about traffic. The outline of the proof was motivated by the corresponding proofs in [7].

For the rest of this section we use the notation of Theorem 3.2 and Corollary 3.3. The first lemma states the following: There exists a constant $\gamma > 0$ such that with probability at least β_1 the following happens: If there is at least one active vehicle then one vehicle will move at least $c\gamma$ or one vehicle will reach its destination.

Lemma 4.1 *Let L' be the length of a shortest circle in the graph G, let be $\gamma = \min\{L'/n, \alpha_0\}$ and let be $x \in H_v \backslash H_p$. Then there is an $i \in \{1, \dots, n\}$ such that x_i is active and*

$$P_i\left(x, \left\{x_i' \in H_i \,|\, l(x_i') \leq (l(x_i) - c\gamma)_+\right\}\right) \geq \beta_1.$$

Applying Lemma 4.1 often enough, we get the probability that all vehicles reach there destinations and stay there passive. A detailed calculation (given in the next subsection) shows that this probability can be bounded uniformly for all states x. For the formulation we need some further notation. Let $q \in Q$ be a point on the graph. Then w_q denotes the path which consists of one edge, namely the edge on which q lies. Thus (q, w_q, q) is the state of a passive vehicle being in the position q.

Lemma 4.2 *There are constants $t_1 \in \mathbb{N}$ and $\beta_2 \in]0,1[$ such that for all $x = (a_i, w_i, b_i)_{i=1,\ldots,n} \in H_v$ and all rectangular sets $C \in \mathcal{H}$ of the form*

$$C = ((B_1 \times U_1 \times B_1) \times \cdots \times (B_n \times U_n \times B_n)) \tag{12}$$

with $b_i \in B_i$, $i = 1, \ldots, n$ and $w_q \in U_i$ for all $q \in B_i$, $i = 1, \ldots, n$ we have

$$P^{t_1}(x, C \cap H_p) \geq \beta_2.$$

The measure ν in equation (10) should be concentrated on a set which is visited by the Markov process independently of the starting distribution. The previous lemma indicates that we might use the set of all passive states for this. Following this idea we define

$$T : Q^n \to H$$

by

$$T(q_1, \ldots, q_n) := ((q_1, w_{q_1}, q_1), \ldots, (q_n, w_{q_n}, q_n)).$$

$T(q_1, \ldots, q_n)$ is the state where each vehicle is passive and the i-th vehicle stays at the position q_i. It is easy to prove that T is measurable. Further, we define the probability measure ν over $\mathcal{H}$ as

$$\nu := \left(\bigotimes_{i=1}^{n} \mu_i \right)_T .$$

Suppose x is a state in H_v. The dynamic might evolve in the following way: After t_1 time steps (c.f. Lemma 4.2) all vehicles are passive. Then they get a new destination with resp. to the destination measures μ_i, $i = 1, \ldots, n$. After another t_1 time steps they reach again there destination. The state of the Markov chain is then in the set on which ν is concentrated. This is first proved for rectangular sets.

Lemma 4.3 *There are constants $t_2 \in \mathbb{N}$ and $\beta \in]0,1[$ such that for all $x \in H_v$ and all rectangular sets $C \in \mathcal{H}$ of the form*

$$C = ((A_1 \times U_1 \times B_1) \times \cdots \times (A_n \times U_n \times B_n)) \tag{13}$$

we have

$$P^{t_2}(x, C) \geq \beta\nu(C).$$

With a standard method the result is afterwards extended to general measurable sets C and to all (i.e. also invalid) states $x \in H$.

Lemma 4.4 *There are constants $t_0 \in \mathbb{N}$ and $\beta \in]0,1[$ such that for all $x \in H$ and all $C \in \mathcal{H}$ we have*

$$P^{t_0}(x, C) \geq \beta\nu(C).$$

Suppose now we had proven Lemma 4.4. Then equation (10) would follow and hence, as explained above, Theorem 3.2 would hold. Proving Corollary 3.3 is now quite easy and it is similar to the proof of *Satz* 4.2.9 in [7].

Proof of Corollary 3.3 Let P be the transition kernel for the Markov chain $(X_t)_{t\in\mathbb{N}}$. P fulfills equation (10). Let t_0, β and ν be chosen accordingly. The transition kernel P' for the Markov chain $(X_t, X_{t+1})_{t\in\mathbb{N}}$ is given by equation (5). We will show that P' fulfills an equation similar to (10) which proves the uniform ergodicity of $(X_t, X_{t+1})_{t\in\mathbb{N}}$. Let be $t_0' := t_0 + 1$ and define the measure ν' on $\mathcal{H} \otimes \mathcal{H}$ by extending

$$\nu'(C \times C') := \int_C P(x, C')\, \nu(\mathrm{d}x)$$

to the σ-algebra.

Let be $(x, x') \in H \times H$ and $C \times C' \in \mathcal{H} \times \mathcal{H}$. By induction over t_0' one can prove

$$P'^{t_0'}((x, x'), C \times C') = \int_C P(y, C')\, P^{t_0}(x', \mathrm{d}y).$$

Using Lemma 4.4 we get

$$P'^{t_0'}((x, x'), C \times C') \geq \beta \int_C P(y, C')\, \nu(\mathrm{d}y) = \beta\nu'(C \times C').$$

The inequality extends to the whole σ-algebra, which was to be shown.

It remains to show the lemmata 4.1 to 4.4. These rather technical proofs are given in subsection 4.3.

4.3. The details of the proof

For the rest of this section let L be the length of all edges in the graph and let L' be defined as above, i.e. L' is the smallest length of all circles in the graph.

Lemma 4.5 *For all $x = (x_1, \dots, x_n) \in H_v \backslash H_p$ there exists an $i \in \{1, \dots, n\}$ such that x_i is active and $\alpha_i(x) \geq \frac{L'}{n}$.*

Proof Assume, that the claim were not true. Since $x \notin H_p$ there has to be an $i \in \{1, \dots, n\}$ such that x_i is active and hence $\alpha_i(x) > 0$. If there is even an $j \in \{1, \dots, n\}$ with $\alpha_j(x) = +\infty$ then the statement of the lemma would be true (in contrast to the assumption). Therefore, each active vehicle has a predecessor with a distance which lies proper between 0 and $\frac{L'}{n}$. If we follow this link of predecessors then eventually we will reach a vehicle which has already been visited before. The edges on which the positions of the vehicles lie induce a circle in the graph with a length greater than 0 and less then L'. This contradicts the construction of L'.

Proof of Lemma 4.1 Chose the i from Lemma 4.5. From the measurability of l follows the measurability of the set

$$\left\{x_i' \in H_i \,|\, l(x_i') \leq (l(x_i) - c\gamma)_+\right\}.$$

A straight calculation which uses equation (2) and equation (3) shows

$$P_i(x, \left\{x_i' \in H_i \,|\, l(x_i') \leq (l(x_i) - c\gamma)_+\right\}) = \int\limits_{[\min\{c\gamma, l(x_i)\}, +\infty[} f_i(\alpha_i(x), .)\, \mathrm{d}\lambda$$

and thus

$$P_i(x, \left\{x_i' \in H_i \,|\, l(x_i') \leq (l(x_i) - c\gamma)_+\right\}) \geq \int\limits_{[c\gamma, +\infty[} f_i(\alpha_i(x), .)\, \mathrm{d}\lambda.$$

Case 1: $\gamma = L'/n$ and $\alpha_i(x) \leq \alpha_0$. Then $\alpha_i(x) \geq \gamma$ (by the choice of i) and therefore

$$\int\limits_{[c\gamma, +\infty[} f_i(\alpha_i(x), .)\, \mathrm{d}\lambda \geq \int\limits_{[c\cdot\alpha_i(x), \alpha_i(x)]} f_i(\alpha_i(x), .)\, \mathrm{d}\lambda \underset{(8)}{\geq} \beta_1.$$

Case 2: $\gamma = L'/n$ and $\alpha_i(x) > \alpha_0$. By definition of γ we have $\gamma \leq \alpha_0$ and hence

$$\int\limits_{[c\gamma, +\infty[} f_i(\alpha_i(x), .)\, \mathrm{d}\lambda \geq \int\limits_{[c\alpha_0, \alpha_i(x)]} f_i(\alpha_i(x), .)\, \mathrm{d}\lambda \underset{(9)}{\geq} \beta_1.$$

Case 3: $\gamma = \alpha_0$. By the choice of i we have $\alpha_i(x) \geq \frac{L'}{n} \geq \gamma = \alpha_0$. We can assume that $\alpha_i(x) > \alpha_0$ since otherwise this case is covered by case 1. Then we have

$$\int\limits_{[c\gamma, +\infty[} f_i(\alpha_i(x), .)\, \mathrm{d}\lambda \geq \int\limits_{[c\alpha_0, \alpha_i(x)]} f_i(\alpha_i(x), .)\, \mathrm{d}\lambda \underset{(9)}{\geq} \beta_1.$$

Proof of Lemma 4.2 Let be $I_1(x) := \{i \in \{1, \ldots, n\} \,|\, x_i \text{ is active}\}$ and let be $I_2(x) := \{i \in \{1, \ldots, n\} \,|\, x_i \text{ is passive}\}$. For each i in $I_1(x)$ let R_i be the set of all reachable states, as defined in Definition 2.7 and let

$$R_i' := \left\{x_i' \in H_i \,|\, l(x_i') \leq (l(x_i) - c\gamma)_+\right\}.$$

Define further

$$D_1(x) := \bigcup_{i \in I_1(x)} \left(R_i' \times \mathop{\times}_{j \in I_1(x) \setminus \{i\}} R_j \right),$$

$$D_2(x) := \mathop{\times}_{i \in I_2(x)} \{x_i\},$$

and

$$D(x) := D_1(x) \times D_2(x).$$

The set $D(x)$ contains all states which are reached in such a way that all passive vehicles stay passive and at least one active vehicle moves according to Lemma 4.1.

Now fix the $i \in I_1(x)$ which exists due to Lemma 4.1. The set $D_1(x)$ includes the set $R'_i \times \times_{j\in I_1(x)\setminus\{i\}} R_j$. Using equation (4) we have therefore

$$P(x, D(x)) \geq P_i(x, R'_i) \cdot \prod_{j\in I_1(x)\setminus\{i\}} P_j(x, R_j) \cdot \prod_{j\in I_2(x)} P_j(x, \{x_j\}).$$

We have $P_i(x, R'_i) \geq \beta_1$ by the choice of i. For all $j \in I_1(x)\setminus\{i\}$ we have $P_j(x, R_j) = 1$ (c.f. Definition 2.7). Finally for each $j \in I_2(x)$ we have $P_j(x, \{x_j\}) = 1 - p_j$ (c.f. equation (1)). Thus we have

$$P(x, D(x)) \geq \beta_1(1 - p_i)^{|I_2(x)|}.$$

Define $p_{\max} := \max\{p_i \,|\, i = 1, \ldots, n\}$. Then

$$P(x, D(x)) \geq \beta_1(1 - p_{\max})^n. \tag{14}$$

We define the sequence of sets $(E_t(x))_{t\in\{1,\ldots,n\}}$ by $E_0(x) := \{x\}$ and

$$E_{t+1}(x) := \bigcup_{y\in E_t(x)} D(y).$$

$E_t(x)$ is the set of all states that are reachable in t time steps such that in every time step all passive vehicles stay passive and for at least one active vehicle we know that it moves a distance of $c\gamma$ (as defined in Lemma 4.1) or reaches its destination. In the state x the total length of all active routes is at most Ln and there are at most n active vehicles. Hence for $t_1 \geq \frac{nL}{c\gamma} + n$ the set E_{t_1} contains only passive states, i.e. $E_{t_1} \subset H_p$. Since the destinations of the vehicles never change, we have in fact

$$E_{t_1} = \{((b_1, w_{b_1}, b_1), \ldots, (b_n, w_{b_n}, b_n))\},$$

and hence $E_{t_1} \subset C$. From equation (11) follows by using equation (14) t_1-times

$$P^{t_1}(x, E_{t_1}) \geq (\beta_1(1 - p_{\max})^n)^{t_1}.$$

The lemma is therefore true for $\beta_2 := (\beta_1(1 - p_{\max})^n)^{t_1}$.

In analogy to the definition of w_q for $q \in Q$ we define now q_w. Let be $w \in W$ (i.e. w is a path in the graph) and let $|w|$ be the number of edges in w. Then

$$q_w := \begin{cases} \emptyset & \text{if } |w| \neq 1 \\ \{q \in Q \,|\, q \text{ lies on } w\} & \text{if } |w| = 1 \end{cases}$$

Accordingly we define for a set of paths $U \subset W$

$$q_U := \bigcup_{w\in U} q_w.$$

With this we have for all $q \in Q$, $U \subset W$

$$w_q \in U \iff q \in q_U. \tag{15}$$

Lemma 4.6 *Let be $x = (a, w, b) \in H_p$ and let $C \in \mathcal{H}$ be a rectangular set of the form*

$$C = (A_1 \times U_1 \times B_1) \times \cdots \times (A_n \times U_n \times B_n).$$

Let furthermore be

$$C' := \left\{(a', w', b') \in H_v \,|\, (a'_i, w'_i, b'_i) \text{ is active}, b'_i \in A_i \cap q_{U_i} \cap B_i, i = 1, \dots, n\right\}$$

and $p_{min} := \min\{p_i \,|\, i = 1, \dots, n\}$. Then we have

$$P\left(x, C'\right) \geq p^n_{min} \cdot \nu(C).$$

Proof All routes x_i are passive, $i = 1, \dots, n$. The set C' contains no passive states. Using equation (1) we get

$$P\left(x, C'\right) = \prod_{i=1}^{n} p_i \cdot \mu_i(A_i \cap q_{U_i} \cap B_i) \geq p^n_{\min} \prod_{i=1}^{n} \mu_i(A_i \cap q_{U_i} \cap B_i).$$

For all $q = (q_1, \dots, q_n) \in Q^n$ we get with equation (15)

$$\begin{aligned} T(q) \in C \quad &\Longleftrightarrow \quad (q_i, w_{q_i}, q_i) \in A_i \times U_i \times B_i \quad (i \in \{1, \dots, n\}) \\ &\Longleftrightarrow \quad q_i \in A_i \cap q_{U_i} \cap B_i \quad (i \in \{1, \dots, n\}) \end{aligned}$$

and hence

$$\prod_{i=1}^{n} \mu_i(A_i \cap q_{U_i} \cap B_i) = \nu(C).$$

Proof of Lemma 4.3 We define $t_2 := 2t_1 + 1$ and $\beta := \beta_2^2 p^n_{\min}$, where t_1 and β_2 are chosen according to Lemma 4.2. Furthermore let be

$$\begin{aligned} D \;\; := \;\; &(((A_1 \cap q_{U_1} \cap B_1) \times U_1 \times (A_1 \cap q_{U_1} \cap B_1)) \times \dots \\ &\times((A_n \cap q_{U_n} \cap B_n) \times U_n \times (A_n \cap q_{U_n} \cap B_n))). \end{aligned}$$

Obviously, $D \subset C$. Define

$$C' := \{(a', w', b') \in H_v \,|\, (a'_i, w'_i, b'_i) \text{ is active}, b'_i \in A_i \cap q_{U_i} \cap B_i, i = 1, \dots, n\}.$$

Let $x' = (a', w', b') \in C'$ be arbitrary. Then $x' \in H_v$ and D is of the form of the C in equation (12). Furthermore we have $b'_i \in A_i \cap q_{U_i} \cap B_i$ by the definition of C'. Finally for all $q \in A_i \cap q_{U_i} \cap B_i$ we have $w_q \in U_i$ (c.f. equation (15)). Hence x' and D fulfill the conditions of Lemma 4.2 and we get

$$P^{t_1}(x', C) \geq P^{t_1}(x', D \cap H_p) \geq \beta_2.$$

Also from Lemma 4.2 we get

$$P^{t_1}(x, H_p) \geq \beta_2.$$

Let $x'' \in H_p$ be arbitrary. Then x'' (instead of x), C and C' fulfill the conditions of Lemma 4.6 and hence we get

$$P(x'', C') \geq p^n_{\min}\nu(C).$$

From equation (11) follows

$$P^{t_2}(x,C) \geq \int_{H_p} \int_{C'} P^{t_1}(x',C)\, P(x'',\mathrm{d}x')\, P^{t_1}(x,\mathrm{d}x'').$$

The above stated inequalities thus yield

$$P^{t_2}(x,C) \geq \beta_2^2 p_{\min}^n \nu(C).$$

Proof of Lemma 4.4 Let be t_2 and β as in Lemma 4.3.

For every set-system $\mathcal{E}' \subset \mathcal{H}$ we denote by $\sigma(\mathcal{E}')$ the σ-algebra generated by $\mathcal{E}'$ and we denote by $\delta(\mathcal{E}')$ the generated Dynkin System, i.e. the intersection of all $\mathcal{E}'$ including Dynkin Systems (c.f. [1], chapter 2). Define

$$\mathcal{D} := \{C \in \mathcal{H} \,|\, P^{t_2}(x,C) \geq \beta\nu(C), P^{t_2}(x,H\backslash C) \geq \beta\nu(H\backslash C) \text{ for all } x \in H_v\}$$

and

$$\mathcal{E} := \{C \in \mathcal{H} \,|\, C \text{ is of the form of the } C \text{ in equation (13)}\}.$$

The set $\mathcal{E}$ is intersection stable, i.e. for all $A, B \in \mathcal{E}$ we have $A \cap B \in \mathcal{E}$. Hence $\delta(\mathcal{E}) = \sigma(\mathcal{E})$ ([1], Theorem 2.4).

Let be $x \in H_v$ and $C \in \mathcal{E}$. From Lemma 4.3 follows $P^{t_2}(x,C) \geq \beta\nu(C)$. The set $H\backslash C$ is the complement of a rectangular set. Therefore, it and can be written as the disjoint union of $3n$ rectangular sets C_i, $i = 1,\ldots,3n$. Thus, we have

$$P^{t_2}(x,H\backslash C) = \sum_{i=1}^{3n} P^{t_2}(x,C_i) \underset{\text{Lemma 4.3}}{\geq} \sum_{i=1}^{3n} \beta\nu(C_i) = \beta\nu(H\backslash C).$$

Hence $C \in \mathcal{D}$. Since x and C where arbitrary we have $\mathcal{E} \subset \mathcal{D}$.

From $\beta < 1$ follows $H \in \mathcal{D}$. For all $D \in \mathcal{D}$ we have by definition $H\backslash D \in \mathcal{D}$. Let $(D_n)_{n\in\mathbb{N}}$ be a sequence of pairwise disjoint sets from $\mathcal{D}$. The measures $P^{t_2}(x,.)$ and ν are σ-additive and thus $\cup_{n\in\mathbb{N}} D_n \in \mathcal{D}$. Therefore, $\mathcal{D}$ is a Dynkin-System and hence $\delta(\mathcal{E}) \subset \mathcal{D}$.

By definition of $\mathcal{H}$ we have $\sigma(\mathcal{E}) = \mathcal{H}$. Together this implies

$$\mathcal{H} = \sigma(\mathcal{E}) = \delta(\mathcal{E}) \subset \mathcal{D}$$

and thus we have for all $x \in H_v$ and $C \in \mathcal{H}$

$$P^{t_2}(x,C) \geq \beta\nu(C).$$

Define $t_0 := t_2 + 1$. Then for all $x \in H$ (especially for $x \notin H_v$) and all $C \in \mathcal{H}$ we conclude from equation (11)

$$P^{t_0}(x,C) \geq \int_{H_v} P^{t_2}(x',C)\, P(x,\mathrm{d}x') \geq \beta\nu(C).$$

5. Conclusion and Open Questions

In this paper, a stochastic process which models the traffic on a network of roads is presented. It is proved that this process is uniformly ergodic and hence the mean random variables converge towards limits. Furthermore these limits can be seen as parameters of the invariant distribution and the mean random variables are consistent estimators for these parameters. This delivers the mathematical justification for the definition of the mean values which are observed in practice.

Using these consistent estimators we can determine the density, the flow and the fundamental diagram for given graphs and driving-densities. This might lead to a deeper understanding of the invariant measure and how it depends on the graph structure.

Of course, our model can be improved in various ways. One might think of using the speed or even the acceleration. This can be modeled via a Markov chain which simulates a finite memory, i.e. there is a fixed $k \in \mathbb{N}$ such that at time t a state in the model contains the positions of the vehicles at time points $t-k, t-k+1, \ldots, t$. This might be quite useful in practical applications. We think however that it should be possible to generalize the theorems. If one can proof a version of Lemma 4.1 and the destinations of the vehicles are chosen in such a way that the system mixes somehow, then the ergodic proof should work and thus the LLNs should hold.

Similarly, it should be possible to prove similar ergodic theorems for traffic models with discrete state space, as e.g. the cellular automaton models in [8, 9].

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