Niyazi A. ILYASOV

ESTIMATIONS OF THE SMOOTHNESS MODULES OF CONVOLUTION OF FUNCTIONS BY MEANS OF THEIR BEST APPROXIMATIONS IN $L_p(\mathbb{T})$

Abstract

In the paper the upper estimations of smoothess modules $\omega_k(h;\delta)_r$ of the convolution $h = f * g$ of two 2π periodic functions $f \in L_p(\mathbb{T})$ and $g \in L_q(\mathbb{T})$ are obtained by means of exspession containing the product $E_{n-1}(f)_p\, E_{n-1}(g)_q$ of the best approximations of these functions in the metrics of $L_p(\mathbb{T})$ and $L_q(\mathbb{T})$ respectively, where $p, q \in [1,\infty]$, $1/r = 1/p + 1/q - 1 \geq 0$, $k \in \mathbb{N}$. It is proved in the case $p, q \in (1,\infty)$ that the obtained estimations are exact in the terms of order on the scale of power majorants of sequences of the best approximations of functions forming the convolution.

In what follows we use the following notation.

- $\mathbb{T}$ is the interval $(-\pi,\pi]$ in $\mathbb{R}$.
- $L_p(\mathbb{T})$, $1 \leq p < \infty$, is the space of all measurable 2π periodic functions $f : \mathbb{R} \to \mathbb{C}$ with finite L_p-norm $\|f\|_p = \left((2\pi)^{-1}\int_{\mathbb{T}} |f(x)|^p\, dx\right)^{1/p} < \infty$.
- $C(\mathbb{T}) \equiv L_\infty(\mathbb{T})$ is the space of all continuous 2π periodic functions with uniform norm $\|f\|_\infty \equiv \max\{|f(x)| : x \in \mathbb{T}\}$.
- $E_n(f)_p$ is the best approximation of a function f in the metric of $L_p(\mathbb{T})$ by the trigonometric polynomials of order $\leq n \in \mathbb{Z}_+$.
- $T_{n,p}(f)$ is the polynomial of the best approximation of a function f in the metric $L_p(\mathbb{T}) : \|f - T_{n,p}(f)\|_p = E_n(f)_p$, $n \in \mathbb{Z}_+$.
- $S_n(f;\cdot)$ is the partial sum of order $n \in \mathbb{Z}_+$ of the Fourier-Lebesgue series of a function $f \in L_1(\mathbb{T}) : S_n(f;x) = \sum_{|\nu|=0}^{n} c_\nu(f)\, e^{i\nu x}$, $x \in \mathbb{T}$.
- $\omega_k(f;\delta)_p$ is the smoothness module of k-th order of a function $f \in L_p(\mathbb{T})$:

  $\omega_k(f;\delta)_p = \sup\left\{\left\|\Delta_t^k f\right\|_p : t \in \mathbb{R},\ |t| \leq \delta\right\}$, $k \in \mathbb{N}$, $\delta \geq 0$, where $\Delta_t^k f(x) = \sum_{\nu=0}^{k} (-1)^{k-\nu} \binom{k}{\nu} f\ (x + \nu t)$, $x \in \mathbb{R}$.
- M_0 is the class of all sequences $\varepsilon = \{\varepsilon_n\}_{n=1}^{\infty}$ such that $0 < \varepsilon_n \downarrow 0\,(n \uparrow \infty)$.
- $E_p[\varepsilon] = \{f \in L_p(\mathbb{T}) : E_{n-1}(f)_p \leq \varepsilon_n,\ n \in \mathbb{N}\}$ for $p \in [1,\infty]$ and $\varepsilon \in M_0$.

The convolution $h = f * g$ of $f \in L_1(\mathbb{T})$ and $g \in L_1(\mathbb{T})$ is defined by the formula: $h(x) = (f * g)(x) = (1/2\pi) \int_{\mathbb{T}} f(x-y)\, g(y)\, dy$; it is known (see f.e. [1], v.1, § 2.1, pp.64-65, [2], v.1, § 3.1, pp.65-66) that the function h is defined almost everywhere, 2π periodic, measurable and $\|h\|_1 \le \|f\|_1 \|g\|_1$ (whence it follows in particular that $h = f * g \in L_1(\mathbb{T})$). The last statement is a particular case of the following result known as the W.Young's inequality (see, f.e. [1], v.1, Theorem (1.15), pp.67-68; [2], v.2, Theorem 13.6.1, pp.176-177; [2], v.1, Theorem 3.1.4, p.70, Theorem 3.1.6, p.72). Given $p \in [1, \infty]$, let $p' = p/(p-1)$ be the exponent conjugate to p. As usual, we assume that $p' = 1$ for $p = \infty$ and $p' = \infty$ for $p = 1$.

Theorem A. *Let* $p, q \in [1, \infty]$, $f \in L_p(\mathbb{T})$ and $g \in L_q(\mathbb{T})$, $h = f * g$, $1/r = 1/p + 1/q - 1$. *Then*

- *If* $1/r > 0$ *then* h *belongs to* $L_r(\mathbb{T})$ *and* $\|h\|_r \le \|f\|_p \|g\|_q$.
- *If* $1/r = 0$ *then* h *belongs to* $C(\mathbb{T}) \equiv L_\infty(\mathbb{T})$ *and* $\|h\|_\infty \le \|f\|_p \cdot \|g\|_{p'}$.

Recall that the Fourier coefficients $c_n(h)$ of $h = f * g$ of two arbitrary functions $f \in L_1(\mathbb{T})$ and $g \in L_1(\mathbb{T})$ are calculated by the formula (see [1], v.1, Theorem (1.5), p.64; [2], v.1, p.66, formula (3.1.5)) $c_n(h) = c_n(f * g) = c_n(f) \cdot c_n(g)$ for every $n \in \mathbb{Z}$.

Between the smoothness modulus and the best approximation of a function $f \in L_p(\mathbb{T})$ there exists the known connection expressed by the following inverse theorem of the approximation theory.

Theorem B. *Let* $f \in L_p(\mathbb{T})$, $1 \le p \le \infty$, $k \in \mathbb{N}$, $\theta = \theta(p) = \min\{2, p\}$for $p \in [1, \infty)$ and $\theta = \theta(p) = 1$ for $p = \infty$. *Then*

$$\omega_k(f; \pi/n)_p \le C_1(k,p)\, n^{-k} \left(\sum_{\nu=1}^{n} \nu^{\theta k - 1} E_{\nu-1}^{\theta}(f)_p \right)^{1/\theta} \quad \text{for every } n \in \mathbb{N}, \qquad (1)$$

where $C_1(k,p)$ *is a positive constant depending only on parameters* k and p.

Estimation (1) in an equvalent form for the first time was anonced by R Salem [3; inequality (1) on p. 703] for $p = \infty$ and $k = 1$.The same estimation in the present form was proved by S. B. Stechkin in [4; Theorem 8, inequality (5.14), p.234] for $p = \infty, k \in N$, [5; Lemma 1, p. 502] for $p = 2, k = 1$, by A. F. Timan (see [6; Section 6.1.1, inequality (1), p. 344]) for $p = 1, p = \infty, k \in N$, by M.F.Timan [7; Theorem 1, inequalities (7), p. 126] for $1 < p < \infty$, $k \in N$ (see also [6; Section 6.1.5], [8; Section 7.3, Theorem 3.4, inequality (3.9), p. 210]).

A. F. Timan and M. F. Timan proved in [9; Theorem 6 for $p = q = 2$, p. 20] that if $f \in L_2(\mathbb{T})$ and $E_{n-1}(f)_2 = O(n^{-1})$, $n \in \mathbb{N}$, then $\omega_1(f;\delta)_2 = O(\delta(\ln(\pi e/\delta))^{1/2})$, $\delta \in (0, \pi]$ (see also [6; Section 3.3.41, p. 121-122]). Note that this statement follows from the proof of the implication (3) $\Rightarrow$ (2) of Theorem 1 [10; Section 82, p. 185-187] for $\alpha = 1$. Earlier E. S. Quade [11; p. 536] (see also [10; Section 93, p. 230]) gave an example of function $f_0 \in L_2(\mathbb{T})$ with $E_{n-1}(f_0)_2 = O(n^{-1})$, $n \in \mathbb{N}$, such that $\delta(\ln(\pi e/\delta))^{1/2} = O(\omega_1(f_0;\delta)_2)$, $\delta \in (0,\pi]$.

The exponent $\theta = \theta(p) = \min\{2,p\}$, $p \in [1,\infty)$, which take into account the influence of L_p-metrics to structural properties of functions with given sequence of the best in $L_p(\mathbb{T})$ approximations appeared for the first time in A. Zigmund's [12] in which the following implication was proved: $f \in L_p(\mathbb{T})$, $\omega_2(f;\delta)_p = O(\delta) \Rightarrow \omega_1(f;\delta)_p = O(\delta(\ln(\pi e/\delta))^{1/\theta})$, $\delta \in (0,\pi]$. It follows from the result of [13; Theorems 8 and 8$'$, p. 52], namely if $f \in L_p(\mathbb{T})$ for $p \in [1,\infty]$ then $E_{n-1}(f)_p = O(n^{-1})$, $n \in \mathbb{N}$, $\Leftrightarrow \omega_2(f;\delta)_p = O(\delta)$, $\delta \in (0,\pi]$, that the following statement holds: if $f \in L_p(\mathbb{T})$ for $p \in [1,\infty)$ and $E_{n-1}(f)_p = O(n^{-1})$, $n \in \mathbb{N}$, then $\omega_1(f;\delta)_p = O(\delta(\ln(\pi e/\delta))^{1/\theta})$, $\delta \in (0,\pi]$.

Besides, A. Zigmund gave in [12; points 3-6] examples of functions $f_0(\cdot,p) \in L_p(\mathbb{T})$ for every $p \in [1,\infty)$ with $\omega_2(f_0;\delta)_p = O(\delta)$, $\delta \in (0,\pi]$ ($\Leftrightarrow E_{n-1}(f_0)_p = O(n^{-1})$, $n \in \mathbb{N}$) such that $\delta(\ln(\pi e/\delta))^{1/\theta} = O(\omega_1(f_0;\delta)_p)$, $\delta \in (0,\pi]$. This says that the exponent θ is exact.

Inequality (1) is exact in the sense of order on the class $E_p[\varepsilon]$ for all $p \in [1,\infty]$, namely

$$\sup\left\{\omega_k\left(f;\pi/n\right)_p : f \in E_p[\varepsilon]\right\} \asymp n^{-k}\left(\sum_{\nu=1}^{n}\nu^{\theta k-1}\varepsilon_\nu^\theta\right)^{1/\theta}, n \in \mathbb{N}. \tag{2}$$

The upper estimation in (2) immediately follows from (1). The lower estimation in (2) is realized by means of individual functions in $E_p[\varepsilon]$; more precisely, for every $p \in [1,\infty]$ and $\varepsilon \in M_0$ there exists a function $f_0(\cdot;p;\varepsilon) \in L_p(\mathbb{T})$ with $E_{n-1}(f_0)_p \le \varepsilon_n$, $n \in \mathbb{N}$, such that

$$\omega_k\left(f_0;\pi/n\right)_p \ge C_2(k,p)n^{-k}\left(\sum_{\nu=1}^{n}\nu^{\theta k-1}\varepsilon_\nu^\theta\right)^{1/\theta}, n \in \mathbb{N}. \tag{3}$$

The assertion (2) for $p = \infty$ and $k \in \mathbb{N}$ was obtained by M. F. Timan [14; Theorem 1, p. 19]. Examples of functions for (3) were given by V. E. Geit [15; points 1) and 2) of Lemma 4, p. 73] for $p = \infty$ and $k \in \mathbb{N}$. The assertion (2) for $p = 1$ and $k \in \mathbb{N}$ was proved by V. E. Geit [16; point 1) of Theorem 3, p. 572]. The proofs of (3) were presented in [17; points 1) and 2) of Theorem 13, p. 26-27] for $p = 1$, $k \in \mathbb{N}$ and [18; points 1) and 3) of Lemma 2, p. 175-176] for $p = 1$, $p = \infty$ and $k \in \mathbb{N}$. The proofs of (2) and (3) for $p \in [1,\infty]$ and $k \in \mathbb{N}$ were given by the author [19; Lemma 2, p. 1302-1303], [20; Lemma 3.7, p. 75], [21; Lemma 2, p. 46], [22; Lemma 3, p. 46]. As the author learned later, the statement (in the terms of equivalence of O-relations between $\omega_k(f;\delta)_p$ and $E_{n-1}(f)_p$ for $p \in [1,\infty)$) closed to

(2) was formulated by P.Osvald [23; Proposition 3, p. 210], who also noted that (2) for $p \in (1,\infty)$ is contained indirectly in M. F. Timan [24; Theorem 4, p. 777].

In this paper considered are similar problems for the convolution $h = f * g$ of arbitrary functions $f \in L_p(\mathbb{T})$ and $g \in L_q(\mathbb{T})$.

Theorem 1. *Let* $p, q \in [1,\infty]$, $1/r = 1/p + 1/q - 1 \geq 0$, $f \in L_p(\mathbb{T})$, $g \in L_q(\mathbb{T})$, $h = f * g$, $k \in \mathbb{N}$. *Then*

(i) *If* $1/r > 0$ *then* $h \in L_r(\mathbb{T})$ *and, for* $\theta = \theta(r) = \min\{2, r\}$,

$$\omega_k\left(h;\pi/n\right)_r \leq C_1\left(k,r\right) n^{-k} \left(\sum_{\nu=1}^{n} \nu^{\theta k-1} E_{\nu-1}^{\theta}\left(f\right)_p E_{\nu-1}^{\theta}\left(g\right)_q\right)^{1/\theta}, \ n \in \mathbb{N}.$$

(ii) *If* $1/r = 0$ *then* $h \in C(\mathbb{T}) \equiv L_\infty(\mathbb{T})$, $q = p'$ *and*

$$\omega_k\left(h;\pi/n\right)_\infty \leq C_1\left(k,r\right) n^{-k} \sum_{\nu=1}^{n} \nu^{k-1} E_{\nu-1}\left(f\right)_p E_{\nu-1}\left(g\right)_q, \ n \in \mathbb{N},$$

where $C_1(k,r)$ *is a constant in Theorem* B.

Proof. Note that $r = pq/(p+q-pq) \in [1,\infty)$ for $1/r > 0$ and $r = \infty$ for $1/r = 0$. We have that $h \in L_r(\mathbb{T})$ for $1/r > 0$ and $h \in C(\mathbb{T})$ for $1/r = 0$ by Theorem A. We need the following estimation (see [25; the inequality (2) in the proof of Theorem 1]):

$$E_{n-1}\left(h\right)_r \leq E_{n-1}\left(f\right)_p E_{n-1}\left(g\right)_q, n \in \mathbb{N}, r \in [1,\infty]. \tag{4}$$

Applying the inequalities (1) for h and (4), we obtain (i) and (ii). Theorem 1 is proved. □

Given $p, q \in [1,\infty]$ and $\alpha, \beta \in (0,\infty)$, let $E_{p,\alpha} = E_p[\{n^{-\alpha}\}_{n=1}^{\infty}]$ and $E_{p,\alpha} * E_{q,\beta} = \{h = f * g : f \in E_{p,\alpha},\ g \in E_{q,\beta}\}$. The estimations (i) and (ii) of Theorem 1 are exact in the sense of order on $E_{p,\alpha} * E_{q,\beta}$ for $p, q \in (1,\infty)$ and $\alpha, \beta \in (0,\infty)$.

Theorem 2. *Let* $p, q \in (1,\infty)$, $1/r = 1/p + 1/q - 1 \geq 0$, $\alpha, \beta \in (0,\infty)$, $k \in \mathbb{N}$, $\theta = \theta(r) = \min\{2, r\}$ *for* $r \in (1,\infty)$ *and* $\theta = \theta(r) = 1$ *for* $r = \infty$. *Then, for* $\delta \in (0,\pi]$,

(i) $$\sup\{\omega_k(h;\delta)_r : h \in E_{p,\alpha} * E_{q,\beta}\} \asymp \begin{cases} \delta^{\alpha+\beta} & (\alpha+\beta < k), \\ \delta^k(\ln(\pi e/\delta))^{1/\theta} & (\alpha+\beta = k), \\ \delta^k & (\alpha+\beta > k). \end{cases}$$

(ii) $\sup\{\omega_{k+1}(h;\delta)_r : h \in E_{p,\alpha} * E_{q,\beta}\} \asymp \delta^k$ for $\alpha + \beta = k$.

For the proof of the theorem we need preliminary lemmas.

Lemma 1. *Let* $f \in L_p(\mathbb{T})$, $p \in [1,\infty]$, $k \in \mathbb{N}$. *Then*

(i) $\max\{E_n(\operatorname{Re} f)_p, E_n(\operatorname{Im} f)_p\} \le E_n(f)_p \le E_n(\operatorname{Re} f)_p + E_n(\operatorname{Im} f)_p$.
(ii) $\max\{\omega_k(\operatorname{Re} f;\delta)_p, \omega_k(\operatorname{Im} f;\delta)_p\} \le \omega_k(f;\delta)_p$
$\le \omega_k(\operatorname{Re} f;\delta)_p + \omega_k(\operatorname{Im} f;\delta)_p$.

Proof. (i) For $f = \operatorname{Re} f + i \operatorname{Im} f$, we have that $(n \in \mathbb{Z}_+)$

$$\begin{aligned}
& \max\{E_n(\operatorname{Re} f)_p, E_n(\operatorname{Im} f)_p\} \\
= \ & \max\{\|\operatorname{Re} f - T_{n,p}(\operatorname{Re} f)\|_p, \|\operatorname{Im} f - T_{n,p}(\operatorname{Im} f)\|_p\} \\
\le \ & \max\{\|\operatorname{Re} f - \operatorname{Re}(T_{n,p}(f))\|_p, \|\operatorname{Im} f - \operatorname{Im}(T_{n,p}(f))\|_p\} \\
= \ & \max\{\|\operatorname{Re}(f - T_{n,p}(f))\|_p, \|\operatorname{Im}(f - T_{n,p}(f))\|_p\} \\
\le \ & \|f - T_{n,p}(f)\|_p = E_n(f)_p = E_n(\operatorname{Re} f + i \operatorname{Im} f)_p \\
\le \ & E_n(\operatorname{Re} f)_p + E_n(\operatorname{Im} f)_p.
\end{aligned}$$

(ii) We have that $\Delta_t^k f = \Delta_t^k \operatorname{Re} f + i\Delta_t^k \operatorname{Im} f$, whence

$$\left|\Delta_t^k f(x)\right| \ge \max\{\left|\Delta_t^k \operatorname{Re} f(x)\right|, \left|\Delta_t^k \operatorname{Im} f(x)\right|\}$$

and therefore $\max\{\left\|\Delta_t^k \operatorname{Re} f\right\|_p, \left\|\Delta_t^k \operatorname{Im} f\right\|_p\} \le \left\|\Delta_t^k f\right\|_p \le \omega_k(f;\delta)_p$ for every $t \in \mathbb{R}$, $|t| \le \delta \in [0,\infty)$. It follows from this estimation that the left inequality in (ii) holds.

On the other hand, $\omega_k(f;\delta)_p = \omega_k(\operatorname{Re} f + i \operatorname{Im} f;\delta)_p \le \omega_k(\operatorname{Re} f;\delta)_p + \omega_k(\operatorname{Im} f;\delta)_p$. Lemma 1 is proved. $\square$

Lemma 2. *Let $f \in L_p(T)$, $p \in (1,2]$, $k \in \mathbb{N}$. Then*

$$n^{-k}\left(\sum_{\nu=1}^{n} \nu^{pk+p-2} |c_\nu(f)|^p\right)^{1/p} \le C_3(k,p)\omega_k(f;\pi/n)_p, \quad n \in \mathbb{N}.$$

Proof. Since $\left\|\Delta_t^k f\right\|_p \le 2^{k+1}\|f\|_p$, $\Delta_t^k f \in L_p(\mathbb{T})$ for every $t \in \mathbb{R}$. Fix n and let $t \in \mathbb{R}$, $|t| \le \pi/n$. Since $\Delta_t^k(e^{i\nu x}) = (e^{i\nu t} - 1)^k e^{i\nu x}$, we obtain that

$$\Delta_t^k f(x) \sim \sum_{\nu \in \mathbb{Z}\setminus\{0\}} c_\nu(f)(e^{i\nu t} - 1)^k e^{i\nu x}, x \in \mathbb{T},$$

whence, by Hardy-Littlewood Theorem (see [1; Volume 2, Theorem 12.3.19, p. 165], [2; Volume 2, Theorem 13.11.1, p. 215]),

$$\begin{aligned}
\omega_k(f;\pi/n)_p \ & \ge \ \omega_k(f;|t|)_p \ge \left\|\Delta_{|t|}^k f\right\|_p \\
& \ge \ C_4(p)\left\{\sum_{|\nu|=1}^{\infty} |c_\nu(f)|^p \left|(e^{i\nu|t|} - 1)^k\right|^p |\nu|^{p-2}\right\}^{1/p} \\
& \ge \ C_4(p)\left\{\sum_{\nu=1}^{\infty} |c_\nu(f)|^p \left|e^{i\nu|t|} - 1\right|^{kp} \nu^{p-2}\right\}^{1/p}.
\end{aligned}$$

Since $\left|e^{i\nu|t|}-1\right| = |2\sin(\nu|t|/2)|$ and $\sin z \geq (2/\pi)z$ for $z \in [0,\pi/2]$, we have that $\left|e^{i\nu|t|}-1\right| = 2\sin(\nu|t|/2) \geq (2/\pi)\nu|t|$ for $\nu = 1,2,\ldots,n$. Therefore

$$\begin{aligned}\omega_k(f;\pi/n)_p &\geq C_4(p)\left\{\sum_{\nu=1}^{n}|c_\nu(f)|^p\left|e^{i\nu|t|}-1\right|^{kp}\nu^{p-2}\right\}^{1/p} \\ &\geq C_4(p)(2/\pi)^k|t|^k\left\{\sum_{\nu=1}^{n}|c_\nu(f)|^p\nu^{pk+p-2}\right\}^{1/p},\end{aligned}$$

whence

$$|t|^k\left\{\sum_{\nu=1}^{n}\nu^{pk+p-2}|c_\nu(f)|^p\right\}^{1/p} \leq (\pi/2)^k(C_4(p))^{-1}\omega_k(f;\pi/n)_p.$$

Since the right part of the last inequality does not depend from t, we have the required estimation with the constant $C_3(k,p) = 2^{-k}(C_4(p))^{-1}$. Lemma 2 is proved. □

Lemma 3. *Let $f \in L_2(T)$ have the Fourier series $f(x) \sim \sum_{n=0}^{\infty} c_n(f)e^{inx}$ and $k \in \mathbb{N}$. Then*

$$n^{-k}\left(\sum_{\nu=1}^{n}\nu^{2k-1}E_{\nu-1}^2(f)_2\right)^{1/2} \leq C_5(k)\omega_k(f;\pi/n)_2, n \in \mathbb{N}.$$

Proof. We have $E_{n-1}^2(f)_2 = \|f-S_{n-1}(f)\|_2^2 = \sum\limits_{\nu=n}^{\infty}|c_\nu(f)|^2$ by the Parseval ineguality, whence

$$\begin{aligned}\sum_{\nu=1}^{n}\nu^{2k-1}E_{\nu-1}^2(f)_2 &= \sum_{\nu=1}^{n}\nu^{2k-1}(\sum_{\mu=\nu}^{n}|c_\mu(f)|^2+\sum_{\mu=n+1}^{\infty}|c_\mu(f)|^2) \\ &= \sum_{\mu=1}^{n}|c_\mu(f)|^2\sum_{\nu=1}^{\mu}\nu^{2k-1}+\sum_{\nu=1}^{n}\nu^{2k-1}\sum_{\mu=n+1}^{\infty}|c_\mu(f)|^2 \\ &\leq \sum_{\mu=1}^{n}\mu^{2k}|c_\mu(f)|^2+n^{2k}E_n^2(f)_2.\end{aligned}$$

For every $t \in \mathbb{R}$, $|t| \leq \pi/n$, we have (see the proof of Lemma 2) that

$$\begin{aligned}\omega_k^2(f;\pi/n)_2 &\geq \omega_k^2(f;|t|)_2 \geq \left\|\Delta_{|t|}^k f\right\|_2^2 = \sum_{\nu=1}^{\infty}|c_\nu(f)|^2\left|e^{i\nu|t|}-1\right|^{2k} \\ &= \sum_{\nu=1}^{\infty}|c_\nu(f)|^2(2|\sin(\nu|t|/2)|)^{2k} \geq (2/\pi)^{2k}|t|^{2k}\sum_{\nu=1}^{n}\nu^{2k}|c_\nu(f)|^2,\end{aligned}$$

whence $n^{-2k}\sum\limits_{\nu=1}^{n}\nu^{2k}\left|c_\nu(f)\right|^2 \le 2^{-2k}\omega_k^2(f;\pi/n)_2$. Applying L_2-analog of Jackson-Stechkin inequality (see [6; Section 5.11, inequality (1), p. 338]), we obtain that

$$\begin{aligned} n^{-k}\left(\sum_{\nu=1}^{n}\nu^{2k-1}E_{\nu-1}^2(f)_2\right)^{1/2} &\le n^{-k}\left(\sum_{\nu=1}^{n}\nu^{2k}\left|c_\nu(f)\right|^2\right)^{1/2}+E_n(f)_2 \\ &\le 2^{-k}\omega_k(f;\pi/n)_2+E_n(f)_2 \\ &\le (2^{-k}+C_6(k))\omega_k(f;\pi/n)_2. \end{aligned}$$

Lemma 3 is proved. □

Lemma 4. *Let $f\in C(\mathbb{T})$ have the Fourier series $f(x)\sim\sum_{n=1}^{\infty}c_n(f)e^{inx}$ with $c_n(f)\ge 0$ for every $n\in\mathbb{N}$, and let $k\in N$. Then*

(i) $\omega_k(f;\pi/n)_\infty \ge \omega_k(\operatorname{Re} f;\pi/n)_\infty \ge C_7(k)n^{-\varkappa}\sum_{\nu=1}^{n}\nu^{\varkappa}c_\nu(f)$, $n\in\mathbb{N}$, *where* $\varkappa=k+(1-(-1)^k)/2=\{k$ *for even* k; $k+1$ *for odd* $k\}$.

(ii) $\omega_k(f;\pi/n)_\infty \ge \omega_k(\operatorname{Im} f;\pi/n)_\infty \ge C_8(k)n^{-\varkappa}\sum_{\nu=1}^{n}\nu^{\varkappa}c_\nu(f)$, $n\in\mathbb{N}$, *where* $\varkappa=k+(1+(-1)^k)/2=\{k+1$ *for even* k; k *for odd* $k\}$.

Proof. Note that lower estimations of smoothness modules of $f\in C(\mathbb{T})$ by means of the combinations of its Fourier coefficients in the real representation of the Fourier series were given in [15; Lemma 3, inequalities (23) and (24), p. 72]. The present proof differs from arguments of [15; p. 73].

It is clear that if f belongs to $C(\mathbb{T})$ then so do $\operatorname{Re} f$ and $\operatorname{Im} f$. Hence, since $c_n(f)\ge 0$ for every $n\in\mathbb{N}$, Fourier series of $\operatorname{Re} f$ and $\operatorname{Im} f$ (and f,respestively) uniformly converge everywhere on $\mathbb{T}$ by Paley's Theorem (see [26; Section 4.2, p. 277]). So

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty}c_n(f)e^{inx}=\sum_{n=1}^{\infty}c_n(f)\cos nx+i\sum_{n=1}^{\infty}c_n(f)\sin nx \\ &= \operatorname{Re} f(x)+i\operatorname{Im} f(x). \end{aligned}$$

Since, for every $t\in\mathbb{R}$,

$$\left\|\Delta_t^k f-S_n(\Delta_t^k f)\right\|_\infty=\left\|\Delta_t^k(f-S_n(f))\right\|_\infty\le 2^{k+1}\left\|f-S_n(f)\right\|_\infty=o(1)$$

as $n\to\infty$, we obtain taking into account $e^{i\nu t}-1=2i\sin(\nu t/2)e^{i(\nu t/2)}$ that

$$\begin{aligned} \Delta_t^k f(x) &= \sum_{\nu=1}^{\infty}c_\nu(f)(e^{i\nu t}-1)^k e^{i\nu x}=\sum_{\nu=1}^{\infty}c_\nu(f)\left(2i\sin(\nu t/2)\right)^k e^{i\nu(x+kt/2)} \\ &= 2^k i^k\sum_{\nu=1}^{\infty}c_\nu(f)\left(\sin(\nu t/2)\right)^k\left(\cos\nu(x+kt/2)+i\sin\nu(x+kt/2)\right), \end{aligned}$$

whence for even k

$$\Delta_t^k f(x-kt/2) = 2^k(-1)^{k/2}\sum_{\nu=1}^{\infty} c_\nu(f)\,(\sin(\nu t/2))^k\,(\cos\nu x + i\sin\nu x),$$

and for odd k

$$\Delta_t^k f(x-kt/2) = 2^k(-1)^{(k+1)/2}\sum_{\nu=1}^{\infty} c_\nu(f)\,(\sin(\nu t/2))^k\,(\sin\nu x - i\cos\nu x).$$

Let us prove (i). We have, for even k and $t\in\mathbb{R}$, $|t|\le\pi/n$, that

$$\begin{aligned}\omega_k\left(\operatorname{Re} f;\frac{\pi}{n}\right)_\infty &\ge \omega_k(\operatorname{Re} f;|t|)_\infty \ge \left\|\Delta_{|t|}^k \operatorname{Re} f\right\|_\infty \ge \left|\Delta_{|t|}^k \operatorname{Re} f\left(\frac{-k\,|t|}{2}\right)\right| \\ &= 2^k\left|\sum_{\nu=1}^{\infty} c_\nu(f)\left(\sin\frac{\nu\,|t|}{2}\right)^k\right| \ge 2^k\sum_{\nu=1}^{n} c_\nu(f)\left(\sin\frac{\nu\,|t|}{2}\right)^k,\end{aligned}$$

and, for odd k, that

$$\begin{aligned}\omega_k\left(\operatorname{Re} f;\frac{\pi}{n}\right)_\infty &\ge \omega_k(\operatorname{Re} f;|t|)_\infty \ge \left\|\Delta_{|t|}^k \operatorname{Re} f\right\|_\infty \\ &\ge \left|\Delta_{|t|}^k \operatorname{Re} f\left(\frac{-(k-1)\,|t|}{2}\right)\right| = 2^k\left|\sum_{\nu=1}^{\infty} c_\nu(f)\left(\sin\frac{\nu\,|t|}{2}\right)^{k+1}\right| \\ &\ge 2^k\sum_{\nu=1}^{n} c_\nu(f)\left(\sin\frac{\nu\,|t|}{2}\right)^{k+1}.\end{aligned}$$

Hence, taking into account that $\sin z\ge(2/\pi)z$, $z\in[0,\pi/2]$, we obtain that

$$\omega_k(\operatorname{Re} f;\pi/n)_\infty \ge 2^k\sum_{\nu=1}^{n} c_\nu(f)\,(\sin(\nu\,|t|\,/2))^{\varkappa} \ge 2^k\pi^{-\varkappa}\,|t|^{\varkappa}\sum_{\nu=1}^{n}\nu^{\varkappa}c_\nu(f),$$

where $\varkappa$ is equal to k for even k and $k+1$ for odd k, and therefore

$$n^{-\varkappa}\sum_{\nu=1}^{n}\nu^{\varkappa}c_\nu(f) \le 2^{-k}\omega_k(\operatorname{Re} f;\pi/n)_\infty \le 2^{-k}\omega_k(f;\pi/n)_\infty.$$

Let us prove (ii). Let F be a primitive function of f. Then

$$F(x)=\sum_{n=1}^{\infty}(in)^{-1}c_n(f)e^{inx} = \sum_{n=1}^{\infty} n^{-1}c_n(f)\sin nx - i\sum_{n=1}^{\infty} n^{-1}c_n(f)\cos nx$$

and we have, for even k and $t\in\mathbb{R}$, $|t|\le\pi/n$, that

$$\Delta_{|t|}^{k+1}\operatorname{Im} F(x) = (-1)^{\frac{k+2}{2}+1}2^{k+1}\sum_{\nu=1}^{\infty}\frac{c_\nu(f)}{\nu}\left(\sin\frac{\nu\,|t|}{2}\right)^{k+1}\sin\nu\frac{2x+(k+1)\,|t|}{2},$$

whence we obtain taking into account $\sin z \geq (2/\pi)z$, $z \in [0, \pi/2]$, that

$$\begin{aligned}\left\|\Delta_{|t|}^{k+1} \operatorname{Im} F\right\|_{\infty} &\geq \left|\Delta_{|t|}^{k+1} \operatorname{Im} F\left(\frac{-k\,|t|}{2}\right)\right| = 2^{k+1}\left|\sum_{\nu=1}^{\infty} \frac{c_\nu(f)}{\nu}\left(\sin\frac{\nu\,|t|}{2}\right)^{k+2}\right| \\ &\geq 2^{k+1}\sum_{\nu=1}^{n} \frac{c_\nu(f)}{\nu}\left(\sin\frac{\nu\,|t|}{2}\right)^{k+2} \geq \frac{2^{k+1}\,|t|^{k+2}}{\pi^{k+2}}\sum_{\nu=1}^{n} \nu^{k+1} c_\nu(f).\end{aligned}$$

Also, we have, for odd k and $t \in \mathbb{R}$, $|t| \leq \pi/n$, that

$$\Delta_{|t|}^{k+1} \operatorname{Im} F(x) = (-1)^{\frac{k+1}{2}} 2^{k+1}\sum_{\nu=1}^{\infty} \frac{c_\nu(f)}{\nu}\left(\sin\frac{\nu\,|t|}{2}\right)^{k+1} \cos\nu\frac{2x+(k+1)\,|t|}{2},$$

whence we obtain taking into account $\sin z \geq (2/\pi)z$, $z \in [0, \pi/2]$, that

$$\begin{aligned}\left\|\Delta_{|t|}^{k+1} \operatorname{Im} F\right\|_{\infty} &\geq \left|\Delta_{|t|}^{k+1} \operatorname{Im} F\left(\frac{-(k+1)\,|t|}{2}\right)\right| \\ &= 2^{k+1}\left|\sum_{\nu=1}^{\infty} \frac{c_\nu(f)}{\nu}\left(\sin\frac{\nu\,|t|}{2}\right)^{k+1}\right| \\ &\geq 2^{k+1}\sum_{\nu=1}^{n} \frac{c_\nu(f)}{\nu}\left(\sin\frac{\nu\,|t|}{2}\right)^{k+1} \\ &\geq 2^{k+1}\pi^{-(k+1)}\,|t|^{k+1}\sum_{\nu=1}^{n} \nu^{k} c_\nu(f).\end{aligned}$$

Taking the estimations obtained together, we have, for every $t \in \mathbb{R}$, $|t| \leq \pi/n$, that

$$\begin{aligned}\omega_{k+1}(\operatorname{Im} F; \pi/n)_\infty &\geq \omega_{k+1}(\operatorname{Im} F; |t|)_\infty \geq \left\|\Delta_{|t|}^{k+1} \operatorname{Im} F\right\|_{\infty} \\ &\geq 2^{k+1}\pi^{-(\varkappa+1)}\,|t|^{\varkappa+1}\sum_{\nu=1}^{n} \nu^{\varkappa} c_\nu(f),\end{aligned}$$

where $\varkappa$ is equal to $k+1$ for even k and k for odd k, whence we obtain using $\omega_{k+1}(\psi;\delta)_\infty \leq \delta\omega_k(\psi';\delta)_\infty$ for every $\psi \in C^1(\mathbb{T})$ that

$$\begin{aligned}n^{-\varkappa}\sum_{\nu=1}^{n} \nu^{\varkappa} c_\nu(f) &\leq 2^{-(k+1)} n\omega_{k+1}\left(\operatorname{Im} F; \frac{\pi}{n}\right)_\infty \leq 2^{-(k+1)}\pi\omega_k\left((\operatorname{Im} F)'; \frac{\pi}{n}\right)_\infty \\ &= 2^{-(k+1)}\pi\omega_k\left(\operatorname{Im} f; \frac{\pi}{n}\right)_\infty \leq 2^{-(k+1)}\pi\omega_k\left(f; \frac{\pi}{n}\right)_\infty.\end{aligned}$$

Lemma 4 is proved. $\square$

Lemma 5. *Let* $k \in \mathbb{N}$, $\alpha, \beta \in (0, \infty)$, $\gamma = \alpha + \beta$, $p, q \in (1, \infty)$, $1/r = 1/p + 1/q - 1 \geq 0$, $\theta = \theta(r) = \min\{2, r\}$ *for* $r \in (1, \infty)$ *and* $\theta = \theta(r) = 1$ *for* $r = \infty$. *Then there exist functions* $f_0(\cdot; p; \alpha) \in L_p(\mathbb{T})$ *and* $g_0(\cdot; q; \beta) \in L_q(\mathbb{T})$ *such that*

(i) $E_{n-1}(f_0)_p \leq C_9(p, \alpha)n^{-\alpha}$, $E_{n-1}(g_0)_q \leq C_9(q, \beta)n^{-\beta}$, $n \in \mathbb{N}$.

(ii) $\omega_k(f_0 * g_0; \pi/n)_r \geq C_{10}(k,\theta,\gamma) \begin{cases} n^{-\gamma} & (\gamma < k), \\ n^{-k}(\ln(n+1))^{1/\theta} & (\gamma = k), \\ n^{-k} & (\gamma > k). \end{cases}$

(iii) $\omega_{k+1}(f_0 * g_0; \pi/n)_r \geq C_{11}(k,\theta) n^{-k}$ *for* $\gamma = k$.

Proof. First we consider the case $1 < r \leq 2$. For $p, q \in (1, \infty)$ $(p' = p/(p-1)$, $q' = q/(q-1))$, let

$$f_0(x;p;\alpha) = \sum_{n=1}^{\infty} n^{-(\alpha+1/p')} e^{inx}, \quad g_0(x;q;\beta) = \sum_{n=1}^{\infty} n^{-(\beta+1/q')} e^{inx}, \quad x \in \mathbb{T}.$$

It was shown in [27; the proof of Theorem 2], [28; the proof of Theorem 3] that $f_0 \in L_p(\mathbb{T})$, $g_0 \in L_q(\mathbb{T})$ and $E_{n-1}(f_0)_p \leq C_9(p,\alpha)n^{-\alpha}$, $E_{n-1}(g_0)_q \leq C_9(q,\beta)n^{-\beta}$, $n \in \mathbb{N}$. By Theorem A, the convolution

$$(f_0 * g_0)(x) = \sum_{n=1}^{\infty} n^{-(\gamma+1/p'+1/q')} e^{inx}, \quad x \in \mathbb{T}$$

belongs to $L_r(\mathbb{T})$ for $r = pq/(p+q-pq)$. Taking into account $r-1-r(1/p'+1/q') = r[1-(1/p'+1/q')]-1 = r[(1/p+1/q)-1]-1 = r(1/r)-1 = 0$, we have by Lemma 2 that

$$\begin{aligned} C_3(k,r)\omega_k(f_0 * g_0; \pi/n)_r &\geq n^{-k}\left(\sum_{\nu=1}^{n} \nu^{rk+r-2}\left|c_\nu(f_0 * g_0)\right|^r\right)^{\frac{1}{r}} \\ &= n^{-k}\left(\sum_{\nu=1}^{n} \nu^{r(k-\gamma)-1+r-1-r(1/p'+1/q')}\right)^{\frac{1}{r}} \\ &= n^{-k}\left(\sum_{\nu=1}^{n} \nu^{r(k-\gamma)-1}\right)^{\frac{1}{r}} \\ &\geq \begin{cases} C_{12}(r,k-\gamma)n^{-\gamma} & (\gamma < k), \\ n^{-k}(\ln(n+1))^{1/r} & (\gamma = k), \\ C_{13}(r,k-\gamma)n^{-k} & (\gamma > k), \end{cases} \end{aligned}$$

whence the estimation (ii) follows in the case $1 < r \leq 2$. For $\gamma = k$, taking into account $r-1-r(1/p'+1/q') = 0$, we obtain by Lemma 2 that

$$
\begin{aligned}
C_3(k+1,r)\omega_{k+1}(f_0*g_0;\pi/n)_r &\ge n^{-(k+1)}\left(\sum_{\nu=1}^{n}\nu^{r(k+1)+r-2}\left|c_\nu(f_0*g_0)\right|^r\right)^{\frac{1}{r}}\\
&= n^{-(k+1)}\left(\sum_{\nu=1}^{n}\nu^{r(k+1-\gamma)-1}\right)^{\frac{1}{r}}\\
&= n^{-(k+1)}\left(\sum_{\nu=1}^{n}\nu^{r-1}\right)^{\frac{1}{r}}\ge r^{-1/r}n^{-k},
\end{aligned}
$$

whence the estimation (iii) follows in the case $1<r\le 2$.

Consider now the case $2\le r<\infty$. Let

$$
f_0(x;\alpha)=\sum_{\nu=0}^{\infty}2^{-(\nu+1)\alpha}e^{i2^\nu x},\quad g_0(x;\beta)=\sum_{\nu=0}^{\infty}2^{-(\nu+1)\beta}e^{i2^\nu x},\quad x\in\mathbb{T}.
$$

Since $\sum_{\nu=0}^{\infty}(2^{-(\nu+1)\alpha})^2=(2^{2\alpha}-1)^{-1}$ and $\sum_{\nu=0}^{\infty}(2^{-(\nu+1)\beta})^2=(2^{2\beta}-1)^{-1}$, by [1; v. 1, Theorem 8.20, p. 345], trigonometric series considered converge almost everywhere and are Fourier series of their sums f_0 and g_0, respectively, and, for every $p,q\in(1,\infty)$, $\|f_0\|_p\le C_{14}(p)(2^{2\alpha}-1)^{-1/2}=C_{15}(p,\alpha)<\infty$ and $\|g_0\|_q\le C_{14}(q)(2^{2\beta}-1)^{-1/2}=C_{15}(q,\beta)<\infty$. Hence $f_0\in L_p(\mathbb{T})$ and $g_0\in L_q(\mathbb{T})$. Since for every $n\in\mathbb{N}$ there exists $m\in\mathbb{Z}_+$ such that $2^m\le n<2^{m+1}$, we have by [1; v. 1, Theorem 8.20, p. 345] that

$$
\begin{aligned}
E_n(f_0)_p &\le E_{2^m}(f_0)_p\le\|f_0-S_{2^m}(f_0)\|_p\le C_{14}(p)\left(\sum_{\nu=m+1}^{\infty}2^{-2(\nu+1)\alpha}\right)^{1/2}\\
&= C_{14}(p)(1-2^{-2\alpha})^{-1/2}2^{-(m+2)\alpha}=C_{15}(p,\alpha)2^{-(m+1)\alpha}\\
&\le C_{15}(p,\alpha)(n+1)^{-\alpha},
\end{aligned}
$$

whence $E_n(f_0)_p\le C_{15}(p,\alpha)(n+1)^{-\alpha}$ for every $n\in\mathbb{N}$. Since $E_0(f_0)_p\le\|f_0\|_p\le C_{15}(p,\alpha)$, we obtain that $E_{n-1}(f_0)_p\le C_{15}(p,\alpha)n^{-\alpha}$ for every $n\in\mathbb{N}$. Similarly, $E_{n-1}(g_0)_q\le C_{15}(q,\beta)n^{-\beta}$ for every $n\in\mathbb{N}$. By the formula above for Fourier coefficients of convolution,

$$
(f_0*g_0)(x)=\sum_{\nu=0}^{\infty}2^{-(\alpha+\beta)(\nu+1)}e^{i2^\nu x},x\in\mathbb{T}.
$$

Since $p,q\in(1,\infty)$, $f_0*g_0\in L_r(\mathbb{T})$ for $r=pq/(p+q-pq)$ and, since $r\ge 2$, $f_0*g_0\in L_2(\mathbb{T})$. We have that

$$
\begin{aligned}
E_{n-1}(f_0*g_0)_2 &\ge E_n(f_0*g_0)_2\ge E_{2^{m+1}}(f_0*g_0)_2\\
&= \|f_0*g_0-S_{2^{m+1}}(f_0*g_0)\|_2=\left(\sum_{\nu=m+2}^{\infty}2^{-2(\nu+1)\gamma}\right)^{1/2}\\
&= 2^{-\gamma(m+3)}(1-2^{-2\gamma})^{-1/2}=(2^{2\gamma}-1)^{-1/2}2^{-2\gamma}2^{-m\gamma}\\
&\ge (2^{2\gamma}-1)^{-1/2}2^{-2\gamma}n^{-\gamma}=C_{16}(\gamma)n^{-\gamma},
\end{aligned}
$$

whence $E_{n-1}(f_0 * g_0)_2 \geq C_{16}(\gamma)n^{-\gamma}$ for every $n \in \mathbb{N}$. Then we obtain by Lemma 3 that

$$\begin{aligned} C_5(k)\omega_k(f_0 * g_0; \pi/n)_r &\geq C_5(k)\omega_k(f_0 * g_0; \pi/n)_2 \\ &\geq n^{-k}\left(\sum_{\nu=1}^{n} \nu^{2k-1} E_{\nu-1}^2 (f_0 * g_0)_2\right)^{1/2} \\ &\geq C_{16}(\gamma)n^{-k}\left(\sum_{\nu=1}^{n} \nu^{2(k-\gamma)-1}\right)^{1/2} \\ &\geq C_{16}(\gamma)\begin{cases} C_{17}(k-\gamma)n^{-\gamma} & (\gamma < k), \\ n^{-k}(\ln(n+1))^{1/2} & (\gamma = k), \\ C_{18}(k-\gamma)n^{-k} & (\gamma > k), \end{cases} \end{aligned}$$

whence the estimation (ii) follows in the case $2 \leq r < \infty$. For $\gamma = k$ we have by Lemma 3 that

$$\begin{aligned} C_5\omega_{k+1}(f_0 * g_0; \pi/n)_r &\geq C_5\omega_{k+1}(f_0 * g_0; \pi/n)_2 \\ &\geq n^{-(k+1)}\left(\sum_{\nu=1}^{n} \nu^{2(k+1)-1} E_{\nu-1}^2 (f_0 * g_0)_2\right)^{1/2} \\ &\geq C_{16}(k)n^{-(k+1)}\left(\sum_{\nu=1}^{n} \nu^{2(k-\gamma)+1}\right)^{1/2} \\ &= C_{16}(k)n^{-(k+1)}\left(\sum_{\nu=1}^{n} \nu\right)^{1/2} \geq C_{16}(k)2^{-1/2}n^{-k}, \end{aligned}$$

where $C_5 = C_5(k+1)$, whence the estimation (iii) follows in the case $2 \leq r < \infty$.

At last we consider the case $r = \infty$. In this case $1/p + 1/q = 1$, that is $q = p'$, and therefore $1/p' + 1/q' = 1$. Let $f_0(\cdot; \alpha; p)$ and $g_0(\cdot; \beta; q)$ be functions such as in the case $1 < r \leq 2$. By Lemma 4 (i) for even k and Lemma 4 (ii) for odd k, we have that

$$\begin{aligned} C_{19}^{-1}\omega_k(f_0 * g_0; \pi/n)_\infty &\geq n^{-k}\sum_{\nu=1}^{n} \nu^k c_\nu(f_0 * g_0) = n^{-k}\sum_{\nu=1}^{n} \nu^{k-(\gamma+1/p'+1/q')} \\ &= n^{-k}\sum_{\nu=1}^{n} \nu^{k-\gamma-1} \geq \begin{cases} C_{20}(k-\gamma)n^{-\gamma} & (\gamma < k), \\ n^{-k}\ln(n+1) & (\gamma = k), \\ C_{21}(k-\gamma)n^{-k} & (\gamma > k), \end{cases} \end{aligned}$$

where $C_{19} = C_{19}(k)$, whence the estimation (ii) follows in the case $r = \infty$. For $\gamma = k$ we obtain by Lemma 4 (i) for even k and Lemma 4 (ii) for odd k (in the both of cases $\varkappa = k+2$) that

$$\begin{aligned}(C_{19}(k+1))^{-1}\omega_{k+1}(f_0 * g_0;\pi/n)_\infty &\geq n^{-(k+2)}\sum_{\nu=1}^{n}\nu^{k+2}c_\nu(f_0 * g_0)\\ &= n^{-(k+2)}\sum_{\nu=1}^{n}\nu^{k+2-(\gamma+1/p'+1/q')}\\ &= n^{-(k+2)}\sum_{\nu=1}^{n}\nu \geq n^{-(k+2)}2^{-1}n^2 = 2^{-1}n^{-k}.\end{aligned}$$

The last estimation can be also received by application of Lemma 4 (ii) for even k and of Lemma 4 (i) for odd k (in the both of cases $\varkappa = k+1$), namely

$$\begin{aligned}(C_{22}(k+1))^{-1}\omega_{k+1}(f_0 * g_0;\pi/n)_\infty &\geq n^{-(k+1)}\sum_{\nu=1}^{n}\nu^{k+1}c_\nu(f_0 * g_0)\\ &= n^{-(k+1)}\sum_{\nu=1}^{n}\nu^{k+1-(\gamma+1/p'+1/q')}\\ &= n^{-(k+1)}\sum_{\nu=1}^{n}1 = n^{-k},\end{aligned}$$

whence the estimation (iii) follows in the case $r=\infty$. Lemma 5 is proved. □

The proof of Theorem 2. First note the following. For every $\delta \in (0,\pi]$ there exists an $n \in \mathbb{N}$ such that $\pi/(n+1) < \delta \leq \pi/n$, whence we have the following estimations

$$2^{-k}\omega_k(h;\pi/n)_r \leq \omega_k(h;\delta)_r \leq \omega_k(h;\pi/n)_r,$$
$$2^{-\gamma}(\pi/n)^\gamma < \delta^\gamma \leq (\pi/n)^\gamma \text{ for every } \gamma \in (0,\infty),$$

$$\begin{aligned}\delta^k(\ln(\pi e/\delta))^{1/\theta} &\leq (\pi/n)^k(\ln(e(n+1)))^{1/\theta}\\ &= \pi^k n^{-k}(1+\ln(n+1))^{1/\theta} \leq 3^{1/\theta}\pi^k n^{-k}(\ln(n+1))^{1/\theta},\end{aligned}$$

$$n^{-k}(\ln(en))^{1/\theta} \leq (2/\pi)^k(\pi/(n+1))^k(\ln(\pi e/\delta))^{1/\theta} < (2/\pi)^k\delta^k(\ln(\pi e/\delta))^{1/\theta}.$$

Upper estimations. Let $\gamma = \alpha+\beta$. For every function $h \in E_{p,\alpha} * E_{q,\beta}$, we have that $h = f * g$ for some $f \in L_p(\mathbb{T})$ and $g \in L_q(\mathbb{T})$ with $E_{n-1}(f)_p \leq n^{-\alpha}$ and $E_{n-1}(g)_q \leq n^{-\beta}$, for every $n \in \mathbb{N}$. Hence we obtain by Theorem 1 that

$$\begin{aligned}C_1{}^{-1}\omega_k(h;\delta)_r &\leq C_1{}^{-1}\omega_k(h;\pi/n)_r \leq n^{-k}\left(\sum_{\nu=1}^{n}\nu^{\theta(k-\gamma)-1}\right)^{1/\theta}\\ &\leq \begin{cases} C_{23}n^{-\gamma} \leq C_{23}(2/\pi)^\gamma\delta^\gamma & (\gamma<k),\\ n^{-k}(\ln(en))^{1/\theta} \leq (2/\pi)^k\delta^k(\ln(\pi e/\delta))^{1/\theta} & (\gamma=k),\\ C_{24}n^{-k} \leq C_{24}(2/\pi)^k\delta^k & (\gamma>k),\end{cases}\end{aligned}$$

where $C_1 = C_1(k,r)$, $C_{23} = C_{23}(k-\gamma,\theta)$, $C_{24} = C_{24}(k-\gamma,\theta)$, and, for $\gamma = k$,

$$\begin{aligned}\omega_{k+1}(h;\delta)_r &\le \omega_{k+1}(h;\pi/n)_r \le C_1 n^{-(k+1)}\left(\sum_{\nu=1}^{n}\nu^{\theta(k+1-\gamma)-1}\right)^{1/\theta}\\ &= C_1 n^{-(k+1)}\left(\sum_{\nu=1}^{n}\nu^{\theta-1}\right)^{1/\theta} \le C_1 n^{-k} \le C_1(2/\pi)^k\delta^k,\end{aligned}$$

where $C_1 = C_1(k+1,r)$. It follows from these inequalities that the upper estimations in (i) and (ii) of Theorem 2 hold.

Lower estimations. We have by Lemma 5 (i) that $(C_9(p,\alpha))^{-1}f_0(\cdot;p;\alpha) \in E_{p,\alpha}$ and $(C_9(q,\beta))^{-1}g_0(\cdot;q;\beta) \in E_{q,\beta}$, whence

$$h_0 = (C_9(p,\alpha))^{-1}f_0 * (C_9(q,\beta))^{-1}g_0 \in E_{p,\alpha} * E_{q,\beta}.$$

So, we obtain by Lemma 5 (ii) and (iii) that

$$\begin{aligned}C_{25}\omega_k(h_0;\delta)_r &= \omega_k(f_0 * g_0;\delta)_r \ge 2^{-k}\omega_k(f_0 * g_0;\pi/n)_r\\ &\ge 2^{-k}C_{10}\begin{cases} n^{-\gamma} \ge \pi^{-\gamma}\delta^{\gamma} & (\gamma < k),\\ \frac{(\ln(n+1))^{1/\theta}}{n^k} \ge \frac{\delta^k(\ln(\pi e/\delta))^{1/\theta}}{3^{1/\theta}\pi^k} & (\gamma = k),\\ n^{-k} \ge \pi^{-k}\delta^k & (\gamma > k),\end{cases}\end{aligned}$$

where $C_{25} = C_9(p,\alpha)C_9(q,\beta)$, and

$$\begin{aligned}C_{25}\omega_{k+1}(h_0;\delta)_r &= \omega_{k+1}(f_0 * g_0;\delta)_r \ge 2^{-(k+1)}\omega_{k+1}(f_0 * g_0;\pi/n)_r\\ &\ge 2^{-(k+1)}C_{11}(k,\theta)n^{-k} \ge 2^{-(k+1)}C_{11}(k,\theta)\pi^{-k}\delta^k.\end{aligned}$$

It follows from these inequalities that the lower estimations in (i) and (ii) of Theorem 2 hold. □

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Niyazi A. Ilyasov
Institute of Mathematics and Mechanics of NAS Azerbaijan
9, F. Agayev str., AZ1141, Baku, Azerbaijan
Tel.: (99412) 439 92 74 (off.)
E-mail: nilyasov@yahoo.com

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