Aydin I. SHAHBAZOV

NUCLEAR WEIGHTED COMPOSITION OPERATORS ON SPACE OF ANALYTIC FUNCTIONS

Abstract

Let X be a compact metric space and let $C(X)$ denote the space of all continuous complex-valued functions on X equipped with the *sup*-norm. In this paper we investigate nuclearity of weighted composition operators acting on uniformly closed subspaces of $C(X)$ with analytic structure. In particular, was given a nuclearity criterion for weighted composition operators on the disc, polydisc and Ball algebras.

1. Let X be a compact metric space and $C(X)$ denote the space of all continuous complex-valued functions defined on X equipped with the sup-norm. Let $A(X)$ be a uniformly closed subspace of $C(X)$. We will consider the operators $T: A(X) \to C(X)$ of the form $T: f \mapsto u \cdot f \circ \varphi$ (the symbol " $\circ$ " denote the composition of functions), where $u \in C(X)$ is fixed function and $\varphi: X \to X$ is a continuous selfmapping of X (in particular, we can choose the function u and selfmapping φ such that the operator T may be acting in $A(X)$, i.e., $T: A(X) \to A(X)$). The operators of these forms are called the weighted composition operators induced by function u (the weighted function) and by self mapping φ. Since the endomorphisms of any semisimple commutative Banach algebras (also, for any bounded linear operator on Banach space) can be represented as operators of these forms, so, the weighted composition operators are very interesting to study. Composition operators (i.e., the operators of the form as operator T with the weighted function $u \equiv 1$) and weighted composition operators on uniform algebras are being investigated from different points of view (such as compactness, spectrum, closedness of ranges, etc) by many authors. In particular, in [1] Kamowitz gives the compactness criterion for the weighted composition operators which acting in the disc-algebra $A(D)$ (where u, φ are in $A(D)$ and $\|\varphi\| \leq 1$) and describes its spectrum ($A(D)$ denote the uniform algebra of functions analytic on the open unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ of complex plane $\mathbb{C}$ and continuous on its closure $\overline{D}$). In [2] (see also [3]) and [4] Kamowitz results are extended to many other uniformly closed subspaces $A(X)$ of $C(X)$ including multidimensional case, i.e., generalizes Kamowitz theorem for functions of several complex variables, where Kamowitz's method is not suitable. Also, note that from method of [2] it is clear that in general case we may assume that the selfmapping $\varphi: X \to X$ is continuous only on the support set of the function u, i.e., on the set $\{x \in X : u(x) \neq 0\}$. By using the results of [2] in [5], [3] was investigated the nuclearity of weighted composition operators on $A(X)$ in case when the weight function u does not vanish anywhere on X.

In this paper we investigated the nuclearity of the weighted composition operators in general case (i.e., in case without the above mentioned condition for the weighted function u) in $A(X)$, when $A(X)$ has analytic structure and we give the nuclearity criteria for the operator T acting on the disc-algebra $A(D)$, also its generalizations for the polydisc and ball algebras.

2. Nuclear weighted composition operates on the disc-algebra.

Let E, F are Banach spaces and $L(E,F)$ is the space of all bounded linear operators acting from space E to the space F. We remember that $T \in L(E,F)$ is called a nuclear operator if we can represent it in the form $T = \sum\limits_{i=1}^{\infty} a_i \otimes f_i$, where $a_i \in E^*$ (E^* is the dual space of E and the symbol $\otimes$ denote the tensor product) and $f_i \in F$, such that $\sum\limits_{i=1}^{\infty} \|a_i\| \|f_i\| < +\infty$, i.e., if we can show that the series $\sum\limits_{i=1}^{\infty} T_i$ is absolutely convergent to operator T in $L(E,F)$, where $T_i = < \cdot, a_i > f_i$ is one-dimensional operator for all i. Consequently nuclearity operators are compact ones.

Let $A(D)$ be a disc-algebra and the weighted composition operator $T : A(D) \to A(D)$ has the form $Tf(z) = u(z) f(\varphi(z))$, for all $f \in A(D)$, where the fixed functions u, φ are in $A(D)$, and $\|\varphi\| \leq 1$. Except the easily degenerative cases, we will assume that $u \not\equiv 0$ and $\varphi \neq const$.

Theorem 1. *The weighted composition operator $T : A(D) \to A(D)$ of the form $T : f \mapsto u \cdot f \circ \varphi$ is nuclear if, and only if, there exists a constant $M > 0$, such that*

$$|u(z)| \leq M(1 - |\varphi(z)|) \quad for \ \ all \ \ z \in \overline{D}.$$

Proof. Necessity. Since the operator T is nuclear, then it has a representation in the form $T = \sum\limits_{k=1}^{\infty} b_k \otimes f_k$, where for any k, b_k is a linear bounded functional on $A(D)$, and $f_k \in A(D)$ is a fixed function, such that $\sum\limits_{k=1}^{\infty} \|b_k\| \cdot \|f_k\| < +\infty$. Then for any function $f \in A(D)$ we have

$$(Tf)(z) = \sum_{k=1}^{\infty} b_k(f) f_k(z) \tag{$*$}$$

for all $z \in \overline{D}$, and the series is absolutely convergent on $\overline{D}$. If $f(z) = \sum\limits_{k=1}^{\infty} a_k(f) z^k$, ($z \in D$), denote the Taylor series expansion inside of the open unit disc, then it is clear that the function $Tf \in A(D)$ has the representation of the form convergent series

$$Tf(z) = \sum_{k=1}^{\infty} a_k(f)\, u(z)\, \varphi(z)^k \qquad (**)$$

on the set $D_\varphi = \left\{ z \in \overline{D} : |\varphi(z)| < 1 \right\}$, where

$$a_k : f \mapsto a_k(f) = \frac{1}{2\pi i} \int\limits_{|z|=1} \frac{f(z)}{z^{k+1}} dz$$

is the bounded linear functional on $A(D)$, which for any function $f \in A(D)$ according its k-th Taylor coefficient $a_k(f)$. Since the function $Tf \in A(D)$ and on the set D_φ (which contains the open unit disc D) $(*)$ and $(**)$ coincides, then we have that the series $\sum\limits_{k=0}^{\infty} \|a_k\| \left\| u\varphi^k \right\|$ is convergent and by M (it is clear that $M > 0$) we denote its sum. But $\|a_k\| = 1$ for all k, then from this we obtain that $\sum\limits_{k=0}^{\infty} \left\| u\varphi^k \right\| = M < +\infty$. Consequently, for any $z \in \overline{D}$ we have:

$$\sum_{k=0}^{\infty} |u(z)|\, |\varphi(z)|^k = \sum_{k=0}^{\infty} \left| u(z)\, \varphi(z)^k \right| \le \sum_{k=0}^{\infty} \left\| u\varphi^k \right\| = M.$$

On the other hand, the inequality

$$\frac{|u(z)|}{1 - |\varphi(z)|} = \sum_{k=0}^{\infty} |u(z)|\, |\varphi(z)|^k \le M$$

is true for all $z \in D_\varphi$, then we have that the inequality

$$|u(z)| \le M\left(1 - |\varphi(z)|\right) \qquad (***)$$

is correct on the set D_φ. But, the set D_φ contains the open unit disc D, and the functions u and φ are in $A(D)$; so, we conclude that the inequality $(***)$ hold for all $z \in \overline{D}$.

Sufficiency. Let there exists a constant $M > 0$, such that the inequality $(***)$ hold for all $z \in \overline{D}$. From this we obtain that $u(z) = 0$ for any point $z \in \overline{D} \backslash D_\varphi$. So, for any function $f \in A(D)$ with the Taylor series expansion $f(z) = \sum\limits_{k=0}^{\infty} a_k(f)\, z^k$ on the D we have that the representation $Tf(z) = \sum\limits_{k=0}^{\infty} a_k(f)\, u(z)\, \varphi(z)^k$ is correct on the closed unit disc $\overline{D}$. In other words, this is meaning that the operator T has the form $T = \sum\limits_{k=0}^{\infty} a_k \otimes f_k$, where for any k, a_k denote the above mentioned bounded linear functional on $A(D)$ and f_k is the function of the form $f_k(z) = u(z)\, \varphi(z)^k$ on the $\overline{D}$ (it is clear that $f_k \in A(D)$ for all k). Since for any k the norm of a_k is equal one, then we obtain that:

$$\sum_{k=0}^{\infty} \|a_k\| \ \|f_k\| = \sum_{k=0}^{\infty} \|f_k\| = \sum_{k=0}^{\infty} \left\| u\varphi^k \right\| = \sum_{k=0}^{\infty} \sup_{z\in\overline{D}} |u(z)| \ |\varphi(z)|^k \leq M$$

(because for any point z in the D_φ we hav that

$$\sum_{k=0}^{\infty} |u(z)| \ |\varphi(z)|^k = \frac{|u(z)|}{1-|\varphi(z)|} \leq M$$

and for all $z \in \overline{D} \backslash D_\varphi$ we have that

$$\sum_{k=0}^{\infty} |u(z)| \ |\varphi(z)|^k = 0 < M).$$

So, the operator T is nuclear. ■

Remark 1. *In the case when $u(z) \neq 0$ everywhere on the closed unit disc, then from the theorem we receive the Theorem 4 in [5].*

3. Generalization

By using the Riemann Mapping Theorem and the method of Theorem 1 we can easily prove the analogous theorem for all domains $U \subset \mathbb{C}$ which conformally equivalent to the open unit disc. If by $A(U)$ we denote the uniform algebra of all functions which are analytic on the U and continuous on its closure $\overline{U}$, then we have following nuclearity criterion.

Theorem 2. *The weighted composition operator $T : A(U) \to A(U)$ of the form $Tf(z) = u(z) f(\varphi(z))$ (the fixed functions φ and u are in $A(U)$) is nuclear if, and only if, there exists a constant $M > 0$ such that for all $z \in \overline{U}$ we have $|u(z)| \leq M \ \ d\left(\varphi(z), \partial\overline{U}\right)$, where d denote a distance between $\varphi(z)$ and $\partial\overline{U}$-the boundary of U.*

By this way we can prove the nuclearity criterion for weighted composition operator on the polydisc-algebra $A(D^n)$ ($A(D^n)$ is the algebra of analytic functions in $D^n = \{z = (z_1, ..., z_n) \in \mathbb{C}^n : |z_k| < 1, \, 1 \leq k \leq n\}$ and continuous on its closure). If $u \in A(D^n)$ is any fixed function and $\varphi : \overline{D^n} \to \overline{D^n}$ is continuous selfmapping such that any $k-th$ coordinate function $\varphi_k \in A(D^n)$ $(1 \leq k \leq n)$, then we have following theorem (see also Theorem4.1 in [3])

Theorem 3. *The weighted composition operator $T : A(D^n) \to A(D^n)$ of the form $Tf(z) = u(z) f(\varphi(z))$ is nuclear if, and only if, there exists a constant $M > 0$, such that, for any k $(1 \leq k \leq n)$ either $\varphi_k(z) = const$, or $|u(z)| \leq M(1 - |\varphi_k(z)|)$ for all $z \in \overline{D^n}$.*

Remark 2. *If $u(z) \neq 0$ on distinguished boundary of polydisc $T^n = \{z = (z_1, ..., z_n) \in \mathbb{C}^n : |z_i| = 1 \ \ for \ \ any \ \ i = 1, ..., n\}$, then we obtain Theorem 4.1 in [3].*

In general case by using the results of [2] and Theorem 1 we can prove more general nuclearity criterion for the weighted composition operators which are acting on uniformly closed subspaces $A(X)$ with analytic structure. At first we give following definition of peak point with respect to $A(X)$ (in case when $A(X)$ is a uniform algebra we have in ordinary sense definition of peak points).

Definition. *A closed subset $E \subset X$ is called a peak set with respect to $A(X)$, if there exists a sequence $\{f_n\} \subset A(X)$, such that $\|f_n\| = f_n(x) = 1$ for all n and all $x \in E$, moreover, outside any neighborhood of the set E the sequence $\{f_n\}$ tends to zero uniformly. A peak set consisting of only one point is called peak point.*

Let $P(A(X))$ be the set of peak points with respect to $A(X)$, and X is connected metric space. In this particular case we have following nuclearity criterion:

Theorem 4. *The weighted composition operator $T : A(X) \to A(X)$ of the form $Tf(x) = u(x) f(\varphi(x))$ (the fixed function u and the selfmapping $\varphi : X \to X$ are analytic and chosen such that the operator T is acting on $A(X)$) is nuclear if, and only if, there exists a constant $M > 0$ such that $|u(x)| \leq M\rho(\varphi(x), P(A(X)))$ for all $x \in X$, where ρ is a metric on X.*

Since the Shilov's boundary of polydisc – algebra $A(D^n)$ is the torus $T^n = \{z = (z_1, ..., z_n) \in \mathbb{C}^n : |z_k| = 1, \ \ 1 \leq k \leq n\}$ and it is contained in the topological boundary ∂D^n as a proper subset for $n > 1$ so, it is clear that Theorem 3 is a corollary of Theorem 4. The case of the ball is easier than the case of polydisc. Let $A(B^n)$ be a ball-algebra, i.e., the algebra of analytic functions in the interior of the ball $B^n = \left\{ z = (z_1, ..., z_n) \in \mathbb{C}^n : \sum\limits_{k=1}^{n} |z_k|^2 < 1 \right\}$ and continuous on its closure. Since the peak set of $A(B^n)$ coincides with its topological boundary (this is the main difference of B^n with polydisc), so, we have:

Theorem 5. *The operator of weighted composition T on $A(B^n)$ in the form $T : f \mapsto u.fo\varphi$ (u and φ are analytic) is nuclear iff, there exists a constant $M > 0$ such that for all $z \in \overline{B^n}$ we have $|u(z)| \leq M(1 - \|\varphi(z)\|)$ (where $\|\cdot\|$ is the Euclidian norm).*

References

[1]. Kamowitz H. *Compact operator of the form UC_φ.* Pasific J. of Math., vol. 80, No.1, 1979, pp. 205-211.

[2] Shahbazov A.I. *On some compact operators in uniform spaces of continuous functions*. Dokl. Acad. Nauk. Azer. SSR, vol.36, No.12, 1980, pp.6-8.(Russian)

[3] Shahbazov A.I., Dehghan Y.N. *Compactness and nuclearity of weighted composition operators on uniform Spaces.* Bulletin of Ir. Math. Soc., vol. 23, No.1, 1997, pp. 49-62.

[4] Shahbazov A.I. *Spectrum of a compact operator of weighted composition in certain Banach spaces of holomorphic functions*. Jour. Sov. Math., vol. 48, No. 6, 1990, pp. 696-701.

[5] Shahbazov A.I. *Compactness and nuclearity of finite sums of weighted composition operators*. Proc. of Azerb.-Turkey Math. Simpoz., Trabzon (Turkey), 1993, pp. 113-114.

Aydin I. Shahbazov.
Institute of Mathematics and Mechanics of NAS of Azerbaijan
9, F.Agayev str., AZ1141, Baku, Azerbaijan
Tel.: (99412) 439 60 69 (off.)
E-mail: aydinshahbazov@yahoo.com

Received September 02, 2005; Revised October 19, 2005.
Translated by author.