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BASES FROM THE EXPONENTS IN L_p

Abstract

A system of exponents depending on some polynomial is considered in the paper. The necessary and sufficient condition of basicity of this system in

$$L_p\left(-\pi+\delta,\pi+\delta\right), \quad \delta\in R$$

is established.

By solving many problems of mathematical physics and mechanics there arises a question on investigation of basicity properties of systems of exponents, cosines and sines of the form

$$1\cup\left\{e^{\pm i\left[\sqrt[m]{P_m(n)}\cdot t+\beta\cdot signt\right]}\right\}_{n\in N} \tag{1}$$

$$\left\{\cos\sqrt[m]{P_m\left(n\right)}\cdot t;\sin\sqrt[m]{P_m\left(n+1\right)}\cdot t\right\}_{n\in N\cup\{0\}}, \tag{2}$$

in the spaces $L_p\left(-\pi+\delta,\pi+\delta\right),\ 1<p<+\infty,\ \delta\in R$, where$P_m\left(n\right)$ is a polynomial of the n-th power

$$P_m\left(n\right)\equiv a_m\cdot n^m+...+a_0,\quad a_m\neq 0,$$

with the complex $a_i\in C,\ i=\overline{0,m}$ coefficients.

Proceeding from the noted necessity in Yu.A.Kazmin paper [1] the system of cosines and sines

$$\left\{\cos\sqrt{n^2+\alpha t}\right\}_{n\geq 0}\cup\left\{\sin\sqrt{n^2+\alpha t}\right\}_{n\geq 1},$$

is considered, where $\alpha\in C$ is a complex parameter. The necessary and sufficient condition on the parameter α is found when this system forms the Riesz basis in $L_2\left(-\pi+\delta,\pi+\delta\right),\ \delta\in R$. For this at first the similar questions for single systems $\left\{\cos\sqrt{n^2+\alpha t}\right\}_{n\geq 0}$ and $\left\{\sin\sqrt{n^2+\alpha t}\right\}_{n\geq 1}$ in $L_2\left(0,\pi\right)$ are studied.

Then in B.T.Bilalov [2-4] papers these results transferred to the Banach case $L_p\left(-\pi,\pi\right)$ for the exponent systems $\left\{e^{\pm i\lambda_n t}\right\}$ at definite conditions on complex sequence of the numbers $\{\lambda_n\}\subset C$. At the present paper we purpose to get the necessary and sufficient conditions of basicity (Riesz basicity) of exponents of systems, cosines and sines (1), (2) in $L_p(-\pi+\delta,\pi+\delta)$.

Remark 1. It is well known that the classical exponent system $\left\{e^{int}\right\}_{-\infty}^{+\infty}$ forms the basis in $L_p(-\pi+\delta,\pi+\delta)$ for $\forall\delta\in R$. As it follows from the results of the paper [1] this effect doesn't hold in general case for systems (1) and (2).

By investigating the stated questions the following lemma constructed in [5] is very important.

Lemma. *Let E_x, E_y be Banach spaces, $F: E_x\to E_y$ be some Fredholm operator, and $\{x_n\}_{n\in N}\subset E_x$ be a basis in E_x. Then $\{Fx_n\}_{n\in N}$ either is the basis in E_y, or it isn't simultaneously complete and minimal in E_y.*

First let's prove the theorem.

Theorem 1. *The system of exponents*

$$\left\{e^{i[(n+\alpha signn)t+\beta signn\cdot signt]}\right\}_{-\infty}^{+\infty} \tag{3}$$

1. at $|\delta|\geq\pi$ forms a basis (at $p=2$ the Riesz basis) in $L_p(-\pi+\delta,\pi+\delta)$ only in case if $\alpha\in\left(-\frac{1}{2q},\frac{1}{2p}\right)$.

2. at $\delta\in(-\pi,\pi)$ the previous assertion holds if $\alpha+\frac{\beta}{\pi}\in\left(-\frac{1}{2q},\frac{1}{p}\right)$ at fulfilment of the inequality $-\frac{1}{4p}<\beta<\frac{1}{4q}$ where $q:\frac{1}{p}+\frac{1}{q}=1$ is a conjugate number.

Proof.

1. Let $|\delta|\geq\pi$. Then at $t\in(-\pi+\delta,\pi+\delta)$ it is obvious that the function *signt* is of fixed sign and, thus, the basicity in $L_p(-\pi+\delta,\pi+\delta)$ of system (3) is equivalent to the basicity of the system $\left\{e^{i(n+\alpha signn)t}\right\}_{-\infty}^{+\infty}$ in $L_p(-\pi,\pi)$, as a result the assertion of this point follows from the results of the paper [7].

2. Let $\delta\in(-\pi,\pi)$. By virtue of elementary substitutions in appropriate expressions the question on basicity of system (3) in $L_p(-\pi+\delta,\pi+\delta)$ leads to the question on basicity of the exponents system

$$\left\{e^{i[(n+\alpha signn)t+\beta signn\cdot sign(t+\delta)]}\right\}_{-\infty}^{+\infty} \tag{4}$$

in $L_p(-\pi,\pi)$. Further applying the results of [6] to this system we get the required. In fact, in case of minimality, denoting by $\left\{h_n^{\delta}(t)\right\}_{-\infty}^{+\infty}$ the biorthogonal to (4) system it is easy to note that the system $\{h_n(t)\}_{-\infty}^{+\infty}$ will be biorthogonal to (3) where $h_n(t)\equiv e^{i(n+\alpha sign)\delta}\cdot h_n^{\delta}(t-\delta)$, $t\in(-\pi+\delta,\pi+\delta)$.

Having taken $\forall f\in L_p(-\pi+\delta,\pi+\delta)$ let's consider the partial sum

$S_N(t) = \sum_{-N}^{N} f_n e^{i[(n+\alpha sign n)t+\beta sign n\cdot sign t]}$, where $f_n = \int_{-\pi+\delta}^{\pi+\delta} f(t)\cdot \overline{h_n(t)}dt, n = 0, \pm 1, ..., (\bar{\cdot})$ is a complex conjugation. It is easy to show that $S_n(t) \to f(t)$ in $L_p(-\pi+\delta, \pi+\delta)$ if system (4) forms the basis in $L_p(-\pi, \pi)$.

The theorem is proved.

Now we move to the investigation of basis properties of systems (1) and (2). Let's consider system (1). At first we assume that $a_m = 1$. Let $a_{m-k} = 0$ at $k = \overline{1, l=1}$ and $a_{m-l} \neq 0$, where $1 \leq l \leq m$. Then, it is obvious that

$$\frac{[P_m(n)]^{1/m}}{n} = \left(1 + a_{m-e}\cdot n^{-l} + ... + a_0 n^{-m}\right)^{1/m} =$$

$$= \left(1 + \underline{\underline{0}}\left(n^{-l}\right)\right)^{1/m} = 1 + \underline{\underline{0}}\left(n^{-l}\right)$$

Thus, $\sqrt[m]{P_m(n)} = n\left[1 + \underline{\underline{O}}\left(n^{-l}\right)\right]$, $n \in N$. More exactly we have:

$$\sqrt[m]{P_m(n)} = n\left[1 + a_{m-l}\cdot n^{-l} + \underline{\underline{O}}\left(n^{-l-1}\right)\right], \quad n \to \infty,$$

i.e.,

$$\sqrt[m]{P_m(n)} = n + a_{m-l}\cdot n^{-l+1} + \underline{\underline{O}}\left(n^{-l}\right), \quad n \to \infty.$$

Separately, we'll consider the following cases

1. $l > 1$, $\beta = 0$. In this case $\sqrt[m]{P_m(n)} = n + \underline{\underline{O}}\left(\frac{1}{n}\right)$. Consider the integral operator:

$$(I+K)\,y(x) = y(x) + \int_{-\pi+\delta}^{\pi+\delta} K(x;t)\,y(t)\,dt,$$

with the series

$$K(x;t) \equiv \sum_{\pm k=1}^{\infty}\left[e^{\pm i\sqrt[m]{P_m(|k|)}x} - e^{ikx}\right]e^{\pm ikt},$$

where $(\pm)$ corresponds the positive and negative values of the index k. It is easy to note that the inequality

$$\sum_{\pm k=1}^{\infty}\left[e^{\pm i\sqrt[m]{P_m(|k|)}x} - e^{ikx}\right]^p \leq C < +\infty,$$

is true, where C is an absolute constant. Further, allowing for Hausdorff-Young inequality for the system of exponents we obtain:

$$\int_{-\pi+\delta}^{\pi+\delta} |K(x;t)|^p\,dt \le C < +\infty,\ \forall x \in (-\pi+\delta, \pi+\delta)$$

Consequently, the integral operator $K : L_p \to L_p$ is completely continuous and so $(I+K) : L_p \to L_p$ is Fredholm. It is clear that $(I+K)\left[e^{ikx}\right] = e^{\pm i\sqrt[m]{P_m(|k|)x}},\ k = \pm 1, \pm 2, ...;\ (I+K)[1] \equiv 1$. Let

$$P_m(n) \neq 0,\ \forall n \in N \text{ and } P_m(k) \neq P_m(l),\ \forall k \neq l. \tag{5}$$

Then from the lemma it follows the basicity (at $p = 2$ Riesz basicity) in $L_p(-\pi+\delta, \pi+\delta)$ of system (1). If $P_m(n_0) = 0$ at some $n_0 \in N$, then again by this lemma this system is neither complete nor minimal system in $L_p(-\pi+\delta, \pi+\delta)$.

2) $l = 1$. It this case we have asymptotics:

$$\sqrt[m]{P_m(n)} = n + a_{m-1} + \underline{\underline{O}}\left(\frac{1}{n}\right),\ n \to \infty.$$

In this case we apply theorem 1. Choose the cases 2_1) and 2_2)

2_1) Let $|\delta| \ge \pi$ and $a_{m-1} \in \left(-\frac{1}{2q}, \frac{1}{2p}\right)$. In this case the system of exponents $1 \cup \left\{e^{i(n+a_{m-1}\cdot signn)t}\right\}_{n\neq 0}$ forms the basis in $L_p(-\pi+\delta, \pi+\delta)$ and by $\left\{h_n^+(t); h_{n+1}^-(t)\right\}_{n\ge 0}$ we'll denote biorthogonal to it system.

Considering the integral operator with the series

$$K(x;t) = \sum_{k=1}^{\infty}\left[e^{i\sqrt[m]{P_m(k)x}} - e^{i(k+a_{m-1})x}\right]\overline{h_k^+}(t) +$$

$$+\sum_{k=1}^{\infty}\left[e^{-i\sqrt[m]{P_m(k)x}} - e^{-i(k+a_{m-1})x}\right]\overline{h_k^-}(t)$$

where $(\bar{\cdot})$ is a complex conjugation, analogously to the case 1) we construct the basicity of system (1) in $L_p(-\pi+\delta, \pi+\delta)$ if condition (5) holds. If this condition is broken, then it is not complete and minimal in $L_p(-\pi+\delta, \pi+\delta)$, moreover has the finite effect.

2_2) Let $\delta \in (-\pi, \pi)$ and $\beta \in \left(-\frac{1}{4p}, \frac{1}{4q}\right)$. Considering the exponent system $1 \cup \left\{e^{i[(n+a_{m-1}signn)t+\beta signn\cdot signt]}\right\}$ and applying previous scheme we obtain that if $a_{m-1} + \frac{\beta}{\pi} \in \left(-\frac{1}{2q}, \frac{1}{2p}\right)$, then system (1) forms the basis in $L_p(-\pi+\delta, \pi+\delta)$ iff condition (5) is fulfilled.

The case 1) and $\beta \neq 0$ are proved analogously. As a result we have the following theorem.

Theorem 2. *Let $a_m = 1$. Then*

1) at $l > 1$, $\beta = 0$ system (1) forms the basis (at $p = 2$ Riesz basis) in $L_p(-\pi+\delta, \pi+\delta)$ only in that case if (5) holds.

2) at $l = 1$: 2_1) $|\delta| \geq \pi$ and $a_{m-1} \in \left(-\frac{1}{2q}, \frac{1}{2p}\right)$ it forms the basis in $L_p(-\pi+\delta, \pi+\delta)$ only by fulfilling condition (5); 2_2) $|\delta| < \pi$, $\beta \in \left(-\frac{1}{4p}, \frac{1}{4q}\right)$ and $a_{m-1} + \frac{\beta}{\pi} \in \left(-\frac{1}{2q}, \frac{1}{2p}\right)$ the assertion 2_1) is true.

Remark 2. Using one result of Levinson we can construct the similar results for the exponents system

$$\left\{ e^{i\left[\sqrt[m]{P_m(|n|)}\cdot t + \beta sign t\right] sign n} \right\}_{n=-\infty}^{+\infty}$$

Remark 3. At $a_m \neq 1$ it is necessary to note the cases $a_m = 0$ and $a_m \neq 0$. At $a_m = 0$ the basicity doesn't hold on any interval of real axis. At $a_m \neq 0$ the analogous questions lead to the space $L_p(-a+\delta_1, a+\delta_1)$ at some a and δ_1.

Remark 4. Allowing for the definition connection between the exponent cosine and sines functions, we can obtain analogous results also for system (2).

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