Tahir A. KHANIYEV

ABOUT MOMENTS OF GENERALIZED RENEWAL PROCESS

Abstract

In this paper an asymptotical behavior as $t \to \infty$ of the three moments of generalized renewal process $T_{N(t)} = \sum_{i=1}^{N(t)} \xi_i$ where $N(t) = \inf\{n \geq 1 : \ T_n > t\}$ and $\{\xi_i\}$, $i \geq 1$ sequence of independent identically distributed positive random variables is investigated.

Many problems of reliability, queuing, stoke control, risk, insurance theories, mathematical biology etc. by the way of boundary problems are reduced to analysis of the probabilistic characteristics of random walks. In the literature there are not few activities dedicated to this subjects (see for example [1] - [7]).

Recently roughly develops asymptotic method of investigation of the probability characteristics of boundary functionals. In this connection in the given activity is retrieved asymptotic expansion of the third order for the made three moments of the generalized renewal process.

Let $\xi_1, \xi_2, \xi_3, ...$ be a sequence of independent identically distributed random variables receiving only positive values from distribution functions $F_\xi(x)$, $x > 0$. Let's assume

$$T_n = \sum_{i=1}^{n} \xi_1, \ n \geq 1, \ T_0 = 0, \ N(t) = \inf\{n \geq 1 : T_n > t\}, \ (t > 0)$$

$T_{N(t)} = \sum_{i=1}^{N(t)} \xi_i$ is called as a generalized renewal process.

Our purpose is to investigate an asymptotical behavior of the made three moments $T_{N(t)}$ as $t \to \infty$. For this purpose we shall enter the following transformations:

$$\Psi(\lambda, k) = \int_0^\infty e^{-\lambda t} E\left(e^{-kT_{N(t)}}\right) dt \ , \quad \lambda > 0, \ k \geq 0, \ \varphi(\alpha) = E\left(e^{-\alpha \xi_1}\right), \ \alpha \geq 0 \ .$$

Let's formulate the following result.

Theorem 1. The double transformation $\Psi(\lambda, k)$ expresses through a Laplace-Stieltjes transformation $(\varphi(\alpha))$ of distribution function random variable ξ_1 as follows:

$$\Psi(\lambda, k) = \frac{\varphi(k) - \varphi(\lambda + k)}{\lambda(1 - \varphi(\lambda + k))} \ .$$

Proof. Outgoing from definitions of transformation $\Psi(\lambda, k)$ we have:

$$\Psi(\lambda, k) = \int_{t=0}^{\infty} e^{-\lambda t} \int_{x=t}^{\infty} e^{-kx} \sum_{n=1}^{\infty} \int_{s=0}^{t} P\{T_{n-1} \in ds\} P\{s < t < s + \xi_1; \ s + \xi_1 \in dx\} =$$

$$= \int_{t=0}^{\infty} e^{-\lambda t} \int_{s=0}^{t} dU(s) \int_{x=t}^{\infty} e^{-kx} f_{\xi_1}(x-s)\,dx = \tilde{L}_1(\lambda) L_2^*(\lambda) \ ,$$

where $L_1(t) = \int_t^{\infty} e^{-kx} dF_{\xi}(x)$, $L_2\{dt\} = e^{-kt} dU(t)$ and $U(t) = \sum_{n=0}^{\infty} F_{\xi}^{*n}(t)$.

Through $\tilde{L}_1(\lambda)$ and $L_1^*(\lambda)$ the Laplace transformations and Laplace-Stieltjes of a function $L_1(t)$, accordingly are above indicated. It is uneasy to see, that

$$L_2^*(\lambda) = \int_0^{\infty} e^{-\lambda t} L_2\{dt\} = U^*(\lambda+k), \quad \tilde{L}_1(\lambda) = \frac{1}{\lambda}\varphi(k) - \frac{1}{\lambda}\varphi(k+\lambda)$$

Therefore,

$$\Psi(\lambda,k) = \frac{\varphi(k) - \varphi(k+\lambda)}{\lambda(1-\varphi(\lambda+k))} \ .$$

Thus, Theorem 1 is proved.

Using outcome of Theorem 1 it is uneasy to deduce the precise formulas for Laplace transformations of the three moments . Namely, it is possible to formulate the following theorem.

Theorem 2. *If exist and the three moments of a random variable ξ_1 are finite, the following equalities are correct:*

a) $\int_0^{\infty} e^{-\lambda t} E\left(T_{N(t)}\right) dt = m_1 \tilde{U}(\lambda)$,

b) $\int_0^{\infty} e^{-\lambda t} E\left(T_{N(t)}^2\right) dt = m_2 \tilde{U}(\lambda) + 2m_1 \tilde{U}(\lambda) U^*(\lambda) D_1^*(\lambda)$,

c) $\int_0^{\infty} e^{-\lambda t} E\left(T_{N(t)}^3\right) dt = 6m_1 \tilde{U}(\lambda) U^{*2}(\lambda) D_1^{*2}(\lambda) + 3m_1 \tilde{U}(\lambda) U^*(\lambda) D_2^*(\lambda) +$

$$+3m_2 \tilde{U}(\lambda) U^*(\lambda) D_1^*(\lambda) + m_3 \tilde{U}(\lambda) \ ,$$

where $m_k = E\left(\xi_1^k\right),\ k \geq 1,\ U(t) \equiv U_{\xi}(t),\ D_i^*(\lambda) = E\left(\xi_1^i e^{-\lambda \xi_1}\right),\ i \geq 1.$

Proof. From Theorem 1 it follows, that

$$\Psi(\lambda,k) = \frac{\varphi(k) - \varphi(k+\lambda)}{\lambda[1-\varphi(\lambda+k)]} + \frac{1}{\lambda},\ (\lambda > 0)$$

As on a condition of the theorem $E\left(\xi^3\right) < +\infty$, the following asymptotic expansion (takes place, see [3]):

$$\varphi(\alpha) = E\left(e^{-\alpha\xi_1}\right) = 1 - \alpha m_1 + \frac{\alpha^2}{2} m_2 - \frac{\alpha^3}{3!} m_3 + o\left(\alpha^3\right) \ .$$

On the other hand, for any $\lambda > 0$, as $k \to 0$ the following asymptotic expansion takes place:

$$\varphi(k+\lambda) = E\left(e^{-k\xi_1} e^{-\lambda\xi_1}\right) = \varphi(\lambda) - kE\left(\xi_1 e^{-\lambda\xi_1}\right) +$$

$$+\frac{k^2}{2}E\left(\xi^2 e^{-\lambda\xi_1}\right)-\frac{k^3}{6}E\left(\xi^3 e^{-\lambda\xi_1}\right)+o\left(k^3\right) \ .$$

From here we shall receive, for any $\lambda>0$

$$1-\varphi\left(k+\lambda\right)=1-\varphi\left(\lambda\right)+kD_1^*\left(\lambda\right)-\frac{k^2}{2}D_2^*\left(\lambda\right)+\frac{k^3}{6}D_3^*\left(\lambda\right)+o\left(k^3\right)=$$

$$=\left[1-\varphi\left(\lambda\right)\right]\left\{1+k\frac{D_1^*\left(\lambda\right)}{1-\varphi\left(\lambda\right)}-\frac{k^2}{2}\frac{D_2^*\left(\lambda\right)}{1-\varphi\left(\lambda\right)}+\frac{k^3}{6}\frac{D_3^*\left(\lambda\right)}{1-\varphi\left(\lambda\right)}+o\left(k^3\right)\right\} \ .$$

Therefore,

$$\Psi\left(\lambda,k\right)=\frac{1}{\lambda}-\frac{m_1k}{\lambda\left(1-\varphi\left(\lambda\right)\right)}\left[1-k\frac{m_{21}}{2}+k^2\frac{m_{31}}{6}+o\left(k^2\right)\right]\times$$

$$\times\left\{1-kU^*\left(\lambda\right)D_1^*\left(\lambda\right)+\frac{k^2}{2}\left[U^*\left(\lambda\right)D_2^*\left(\lambda\right)+2U^*\left(\lambda\right)D_1^*\left(\lambda\right)\right]-\right.$$

$$\left.-\frac{k^3}{6}\left[U^*\left(\lambda\right)D_3^*\left(\lambda\right)+6U^{*2}\left(\lambda\right)D_1^*\left(\lambda\right)D_2^*\left(\lambda\right)+6U^{*3}\left(\lambda\right)D_1^{*3}\left(\lambda\right)\right]+o\left(k^3\right)\right\}=$$

$$=\frac{1}{\lambda}-m_1k\tilde{U}\left(\lambda\right)+\frac{m_1k^2}{2}\tilde{U}\left(\lambda\right)\left[m_{21}+2U^*\left(\lambda\right)D_1^*\left(\lambda\right)\right]-$$

$$-\frac{3m_1k^3}{6}\tilde{U}\left(\lambda\right)\left[U^*\left(\lambda\right)D_2^*\left(\lambda\right)+2U^{*2}\left(\lambda\right)D_1^{*2}\left(\lambda\right)+m_{21}U^*\left(\lambda\right)D_1^*\left(\lambda\right)+\frac{m_{31}}{3}\right]+o\left(k^3\right) \ .$$

On the other hand, for any $\lambda>0$, as $k\to 0$ the following asymptotic expansion takes place:

$$\Psi\left(\lambda,k\right)=\frac{1}{\lambda}-k\int\limits_0^\infty e^{-\lambda t}E\left(e^{-kT_{N(t)}}\right)dt+$$

$$+\frac{k^2}{2}\int\limits_0^\infty e^{-\lambda t}E\left(T_{N(t)}^2\right)dt-\frac{k^3}{6}\int\limits_0^\infty e^{-\lambda t}E\left(T_{N(t)}^3\right)dt+o\left(k^3\right).$$

Comparing the previous two asymptotic expansions we shall receive the statement of the theorem.

Thus, Theorem 2 is proved.

Applying the reverse Laplace transformation in Theorem 2, it is uneasy to receive the following inequalities:

Corollary 1. *In conditions of the Theorem 2 the following inequalities are correct:*

1) $E\left(T_{N(t)}\right)=m_1U\left(t\right),$

2) $E\left(T_{N(t)}^2\right)=m_2U\left(t\right)+2m_1U\left(t\right)*U\left(t\right)*D_1\left(t\right),$

3) $E\left(T_{N(t)}^3\right)=6m_1U^{*3}\left(t\right)*D_1^{*2}\left(t\right)+3U^{*2}\left(t\right)\left[m_1D_2\left(t\right)+m_2D_1\left(t\right)\right]+m_3U\left(t\right)$,

where $D_k\left(t\right)=\int\limits_0^t s^k dF\left(s\right), \ k=1,2$.

In some problems of a renewal theory the finding of asymptotic expansion for the moments of a generalized renewal process is required:

Theorem 3. *If $E\xi_1^3 < \infty$, then as $t \to \infty$ for the three moments of the third order for the three moments $T_{N(t)}$ it is possible to write the following asymptotic formulas:*

1) $E\left(T_{N(t)}\right) = t + \dfrac{m_2}{2m_1} + o(1)$, 2) $E\left(T_{N(t)}^2\right) = t^2 + \dfrac{m_2}{m_1}t + \dfrac{m_3}{3m_1} + o(1)$,

3) $E\left(T_{N(t)}^3\right) = t^3 + \dfrac{3m_2}{2m_1}t^2 + \dfrac{m_3}{m_1}t + o(1)$.

Proof. 1)The proof of the made formula directly follows from Wald's identity and updated estimations for the renewal function (see, for example, [4]).

2) It is uneasy to see, that as $\lambda \to 0$

$$U^{*2}(\lambda) = \frac{1}{(1-\varphi(\lambda))^2} = \frac{1}{\lambda^2 m_1^2}\left\{1 + \lambda m_{21} + \frac{\lambda^2}{12}\left[9m_{21}^2 - 4m_{31}\right] + o\left(\lambda^2\right)\right\} .$$

Provided that $m_3 < \infty$, directly from definition of a function $D_1^*(\lambda)$ it follows, that

$$D_1^*(\lambda) = E\left(\xi_1 e^{-\lambda\xi_1}\right) = m_1\left\{1 - \lambda m_{21} + \frac{\lambda^2}{2}m_{31} + o\left(\lambda^2\right)\right\} .$$

Therefore,

$$U^{*2}(\lambda) D_1^*(\lambda) = \frac{1}{\lambda^2 m_1}\left\{1 + \frac{\lambda^2}{12}\left[2m_{31} - 3m_{21}^2\right] + o\left(\lambda^2\right)\right\} .$$

Applying the Tauber-Abelian theorem [2], we have:

$$E\left(T_{N(t)}^2\right) = t^2 + tm_{21} + \frac{1}{3}m_{31} + o(1), \quad as \quad t \to \infty.$$

3) With allowance for Corollary 1, we have

$$E\left(T_{N(t)}^3\right) = 6m_1 U^{*3}(t) * D_1(t) +$$

$$+ 3m_1 U^{*2}(t) * D_2(t) + 3m_2 U^{*2}(t) D_1(t) + m_3 U(t) . \qquad (1)$$

Let's calculate the members of the sum (1), separately:

$$D_2^*(\lambda) = E\left(\xi_1^2 e^{-\lambda\xi_1}\right) = m_2\left\{1 - \lambda m_{32} + \frac{\lambda^2}{2}m_{42} + o\left(\lambda^2\right)\right\} ,$$

where $m_{32} = \dfrac{m_3}{m_2}$; $m_{42} = \dfrac{m_4}{m_2}$.

It was above showen, that

$$U^{*2}(\lambda) = \frac{1}{\lambda^2 m_1^2}\left\{1 + \lambda m_{21} + \frac{\lambda^2}{12}\left[9m_{21}^2 - 4m_{31}\right] + o\left(\lambda^2\right)\right\} . \qquad (2)$$

From equalities (1) and (2) we have:

$$U^{*2}(\lambda) D_2^*(\lambda) = \frac{m_2}{\lambda^2 m_1^2}\left\{1 + \lambda\left[m_{21} - m_{32}\right] + o(\lambda)\right\} . \qquad (3)$$

From here we shall receive

$$\frac{3m_1}{\lambda} U^{*2}(\lambda) D_2^*(\lambda) = \frac{3m_{21}}{\lambda^3} + \frac{3m_{21}}{\lambda^2}\left[m_{21} - m_{32}\right] + o\left(\frac{1}{\lambda^2}\right).$$

Thus, with allowance for Tauber-Abelian theorem, as $t \to \infty$ we have:

$$3m_1 U^{*2}(t)^* D_2(t) = \frac{3t^2}{2} m_{21} + 3t m_{21}\left[m_{21} - m_{32}\right] + o(t). \qquad (4)$$

Now similarly we shall calculate the third member of the sum in (1)

$$U^{*2}(\lambda) M_1^*(\lambda) = \frac{1}{\lambda^2 m_1}\left\{1 + \frac{\lambda^2}{12}\left[2m_{31} - 3m_{21}^2\right] + o\left(\lambda^2\right)\right\}.$$

Therefore, as $t \to \infty$ the asymptotic expansion takes place:

$$3m_2 U^{*2}(t)^* D_1(t) = \frac{3t^2}{2} m_{21} + O(1)\ . \qquad (5)$$

At last, we shall calculate the last member of the sum (1):

$$m_3 U(t) = m_3\left[\frac{t}{m_1} + O(1)\right] = t m_{31} + O(1)\ . \qquad (6)$$

Similarly, it is possible to show, that

$$6m_1 U^{*3}(t)^* D_1^{*2}(t) = t^3 - \frac{3t^2}{2} m_{21} + 3t\left[m_{31} - m_{21}^2\right] + O(1). \qquad (7)$$

Substituting decomposing (4), (5), (6) and (7) in the formula (1), we shall receive

$$E\left(T_{N(t)}^3\right) = t^3 + \frac{3t^2}{2} m_{21} + t\left\{4m_{31} - 3\frac{m_2}{m_1}\frac{m_3}{m_2}\right\} + O(1) = t^3 + \frac{3}{2} m_{21} t^2 + t m_{31} + O(1).$$

Thus, Theorem 3 is proved.

References

[1]. Rogozin, B. A. *On the distribution of the first jump.* Theory Probab. Appl. 1964, v.9, No3, pp.498-545.

[2]. Feller W. *An Introduction to Probability Theory and Its Applications II.* 1971, 2nd Ed. New York, 680 p.

[3]. Lukacs E. *Characteristics Function.* Griffin, London, 1970, p.385.

[4]. Kovalenko, I.N., Kuznetchov N., Shurenkov V.M., *Stochastic Processes.* Kiev: "Naukova Dumka", 1983, 366 p.

[5]. Alsmeyer G. *On generalized renewal measures and certain first passage times.* The Annals of Probability, 1992, v.20, No3, pp.1229-1247.

[6]. Lotov V.I. *On some boundry crossing problems for Gaussian random walks.* The Annals of Probability, 1996, v.24, No4, pp.2154-2171.

[7]. Khaniev T.A., Unver I., Maden S. *On the semi-Markovian random walk with two reflecting barriers.* Stochastic Analysis and Appl., USA, 2001, v.19, No5, pp.799-819.

Tahir A. Khaniyev
Karadeniz Technical University,
Faculty of Arts and Sciences
Trabzon, 61080, Turkey
tel.: 90 (462) 377 23 90
E-mail: khaniyevtahir@yahoo.com

Received September 26, 2003; Revised January 12, 2004.
Translated by the author.