**Vugar E. ISMAILOV**

# ON TWO-SIDED EXACT ESTIMATES FOR THE BEST APPROXIMATION BY SUMS $\varphi(x)+\psi(y)$

**Abstract**

*Two-sided exact estimates for the best approximation of a continuous function $f(x,y)$ having the bounded mixed derivative $f_{xy}$ by sums $\varphi(x)+\psi(y)$ of continuous functions are established in special domains different from a rectangle with sides parallel to coordinate axes.*

**1. Introduction.** Let $Q$ be a compact set on $R^2$. Consider the approximation of a continuous fucntion $f(x,y)\in C(Q)$ by the manifold $D=\{\varphi(x)+\psi(y)\}$, where $\varphi(x)$, $\psi(y)$ are defined and continuous on the projections of $Q$ onto coordinate axes $x$ and $y$, respectively. The best approximation is defined as the distance from $f$ to $D$

$$E(f)=dist(f,D)=\inf_{\varphi+\psi\in D}\|f-\varphi-\psi\|_{C(Q)}.$$

To show that $E(f)$ depends also on $Q$, in some cases to avoid confusion, we write $E(f,Q)$ instead of $E(f)$.

The following notion is basic in theory of representation and approximation by superposition of functions of fewer variables:

**Definition.** *A bolt of lightning is a finite ordered subset $\{P_1,P_2,...,P_n\}$ of $Q$ with $P_i\neq P_{i+1}$ $\left(i=\overline{1,n-1}\right)$ and either $P_1=(x_1,y_1)$, $P_2=(x_1,y_2)$, $P_3=(x_2,y_2)$, $P_4=(x_2,y_3)$, ..., or $P_1=(x_1,y_1)$, $P_2=(x_2,y_1)$, $P_3=(x_2,y_2)$, ...*

A bolt of lightning with at least two distinct points is called a closed bolt if $P_nP_1\perp P_1P_2$. It is clear that if a bolt $\{P_1,P_2,...,P_n\}$ is closed then $n$ is an even number.

We associate each closed bolt of lightning $H=\{P_1,P_2,...,P_{2n}\}$ with the following functionals

$$L(f)=L(f,H)=f(P_1)-f(P_2)+...-f(P_{2n})$$

$$r(f,H)=\frac{1}{2n}L(f,H)\ .$$

In 1951 Diliberto and Straus [1] established the formula

$$E(f,Q)=\sup_{H\subset Q}|r(f,H)|, \tag{1}$$

where $Q$ is a unit square $[0,1;0,1]$ and sup is taken over all closed bolts. Later this formula was proved differently by Ofman [2], Light and Cheney [3] when $Q$ is a rectangle with sides parallel to coordinate axes. Havinson [4] proved it for more general domains satisfying the existence of the best approximating sum and the condition that any bolt $G = \{P_1, P_2, ..., P_n\}$ can be made closed by adding limited number of points $P_{n+1}, P_{n+2}, ..., P_{n+s},\ s \leq \nu$, $\nu$ doesn't depend on $G$. The author [11] proved the mentioned formula for elementary domains. Although the formula has played the main role in this field and is of great theoretical value, it is sometimes less practicable from the standpoint of computability.

Trofimov and Hariton [5] for discrete functions and Babaev [6] for all continuous functions established the exact upper and lower bounds for $E(f)$:

$$\frac{1}{4}\sup_{H\subset Q}|L(f,H)| \leq E(f,Q) \leq \frac{1}{2}\sup_{H\subset Q}|L(f,H)|\ , \tag{2}$$

where $Q$ is a rectangle with sides parallel to coordinate axes and sup is taken over all closed bolts consisting of 4 points.

In [7] the first attempt has been made for obtaining these estimates for domains different from a rectangle. Although the estimates are valid for all continuous functions, they are as well as formula (1) not always easily calculable.

A number of mathematicians paying special attention to the practical value of the problem considered some subsets of the space of continuous functions and established easily calculable formulas and estimates for $E(f)$ (see references in [8]). One of these results was obtained by Babaev [8]. He proved that for a continuous fucntion $f(x,y)$ on a unit square $S[0,1;0,1]$ with bounded $f_{xy} \in C(S)$ the following exact estimates are valid

$$\frac{1}{4}\left|L\left(f,S'\right)\right| \leq E(f,S) \leq \frac{M}{2} - \frac{1}{4}\left|L\left(f,S'\right)\right| \tag{3}$$

where $M = \max\limits_{P\in Q}|f_{xy}(P)|$ and $S'$ is a bolt formed by vertices of $S$.

The purpose of this paper is to obtain exact estimates of this kind in more general domains different from a rectangle.

**Remark.** It should be noticed that inequalities (2) and (3) established for a fucntion $f(x) = f(x_1, x_2, ..., x_n)$ of $n$ variables approximated by sums $\sum\limits_{i=1}^{n}\varphi_i(x\backslash x_i)$. But (1) is known only for a fucntion of two variables. Some attempts in proving it for the case of $n$ dimension were failed and the problem remains still open. Because of the lack of an appropriate generalized notion of a bolt the validity of many results proved by means of lightning bolt principles are still unknown in $n$-dimensional case.

**2. The main result.** Let $T = \{(x,y) : x \geq 0,\ y \geq 0,\ x+y \leq 1\}$ be a closed triangle. Consider a closed domain $Q \subset T$ with the following properties:

1) $Q = \bigcup\limits_{i=1}^{s} K_i$, where $K_i = [a_i, a_{i+1}; c_i, d_i]\,,\ i = \overline{1,s}$ are rectangles with sides parallel to coordinate axes.

2) $Q$ is convex according to coordinate axes

3) $Q$ contains the square $K = \left[0, \frac{1}{2}; 0, \frac{1}{2}\right]$.

Let $Q' = \{P_1, P_2, ..., P_{2n}\}$ be a closed bolt formed by all vertices of $Q$ and $P_1 = (0, 0)$. As we denote in section 1

$$L(f) = L(f, Q') = f(P_1) - f(P_2) + ... - f(P_{2n})$$

**Theorem.** *Let $f(x, y)$ be from $C(Q)$ and there exists $f_{xy} \in C(Q)$. Then the following two-sided and exact estimates are valid*

$$\frac{1}{2n}|L(f)| \leq E(f, Q) \leq \frac{M}{16} + \frac{1}{4}M \cdot L(g) - \frac{1}{4}|L(f)|,$$

*where*

$$M = \max_{P \in Q} |f_{xy}(P)| \quad \text{and} \quad g = g(x, y) = x \cdot y .$$

**Proof.** Denote $f_1 = f + Mg$. It is obvious that

$$(f_1)_{xy} = f_{xy} + M \geq 0 .$$

$f_1$ satisfies the conditions of the theorem proven in [10].

Then

$$E(f_1) \leq \frac{1}{4}L(f_1) = \frac{1}{4}L(f) + \frac{1}{4}M \cdot L(g) . \tag{4}$$

We need the following obvious lemma

**Lemma.** *Let $X$ be a linear normed space, $F$ be a subspace of $X$. The following inequalities are valid for an element $x \in X$, $x = x_1 + x_2$*

$$|E(x_1) - E(x_2)| \leq E(x) \leq E(x_1) + E(x_2) ,$$

*where*

$$E(x) = E(x, F) = \inf_{y \in F} \|x - y\| .$$

By this lemma

$$E(f) - E(Mg) \leq E(f_1)$$

and therefore

$$E(f) - M \cdot E(g) \leq E(f_1) . \tag{5}$$

Previously we found the exact value of the best approximation for the function $g(x, y) = x \cdot y$ on $T$ (see [11])

$$E(g, T) = E(g, K) = \frac{1}{16} .$$

As by the condition $K \subset Q \subset T$, it is clear that

$$E(g, Q) = \frac{1}{16} .$$

Taking it into account in (5) we obtain

$$E(f)-\frac{M}{16}\leq E(f_1).$$

Considering this inequality with (4) yields

$$E(f)\leq\frac{M}{16}+\frac{1}{4}M\cdot L(g)+\frac{1}{4}L(f)\ . \tag{6}$$

Denote $f_2=Mg-f$. Then $(f_2)_{xy}=M-f_{xy}\geq 0$. Applying the theorem from [10] now to the fucntion $f_2$ we obtain

$$E(f_2)\leq\frac{1}{4}L(f_2)=\frac{1}{4}ML(g)-\frac{1}{4}L(f). \tag{7}$$

By the foregoing lemma

$$E(f)-E(Mg)\leq E(f_2)$$

or after considering that $E(Mg)=\dfrac{M}{16}$,

$$E(f)-\frac{M}{16}\leq E(f_2)\ .$$

This inequality with (7) gives

$$E(f)\leq\frac{M}{16}+\frac{1}{4}ML(g)-\frac{1}{4}L(f)\ . \tag{8}$$

We obtain from (6) and (8) that

$$E(f)\leq\frac{M}{16}+\frac{1}{4}ML(g)-\frac{1}{4}|L(f)|\ .$$

The upper estimate has been proved.

The proof of the lower estimate is not difficult. On the one hand one can write $L(f)=L(f-\varphi-\psi)$. On the other hand $L(f-\varphi-\psi)\leq 2n\|f-\varphi-\psi\|_{C(Q)}$. Hence

$$\frac{1}{2n}L(f)\leq E(f,Q)\ .$$

To prove the exactness of the upper estimate it is sufficient to take $f(x,y)=xy$.

It should be remarked that the lower estimate is true in the whole space of continuous functions. Show that there exists such a continuous fucntion $\tilde{f}(x,y)$ that $\dfrac{1}{2n}L\left(\tilde{f}\right)=E\left(\tilde{f},Q\right)$. By Urison's great lemma [9] there always exists a continuous fucntion $\tilde{f}(x,y)$ with the following properties

$$\tilde{f}(x,y)=\begin{cases}1, & (x,y)=P_{2k-1},\quad k=\overline{1,n}\\ -1, & (x,y)=P_{2k},\quad k=\overline{1,n}\end{cases}$$

and $\left|\tilde{f}(x,y)\right|<1,\quad (x,y)\in Q\backslash\{P_1,P_2,...,P_{2n}\}$.

By Havinson's result (see [4], theorem 1)

$$E\left(\tilde{f}, Q\right) = 1.$$

On the other hand

$$\frac{1}{2n} L\left(\tilde{f}\right) = 1.$$

The exactness of the lower estimate has been proved.

## References

[1]. Diliberto S.P., Straus E.G. *On the approximation of a function of several variables by the sum of functions of fewer variables.* Pacific J. Math., 1, 1951, pp.195-210.

[2]. Ofman Ju.P. *On the best approximation of functions of two variables by functions of the form $\varphi(x) + \psi(y)$.* Izv. AN SSSR, 25, No2, 1961, pp.239-252. (Russian)

[3]. Light W.A., Cheney E.W. *On the approximation of a bivariate fucntion by the sum of univariate functions.* J. Approx. Theory., 29, 1980, pp.305-322.

[4]. Havinson S.Ja. *A Chebyshev theorem for the approximation of a function of two variables by sums of the type $\varphi(x) + \psi(y)$.* Izv. AN SSSR, 33, No3, 1969, pp.650-666. (Russian)

[5]. Trofimov V.N., Hariton L.R. *On error of the uniform approximation of a function of two variables by the sum of functions of one variable.* Izv. Vuzov, Matematika, No8, 1979, pp.70-73. (Russian)

[6]. Babaev M.-B.A. *Extremal properties and two-sided estimates in approximation by sums of fucntions of fewer variables.* Mat. zametki, 36, No5, 1984, pp.647-659. (Russian)

[7]. Babaev M.-B.A., Ismailov V.E. *Two-sided estimates for the best approximation in domains different from the parallelepiped.* Functiones Et Approximato, UAM, 25, 1997, pp.121-128.

[8]. Babaev M.-B.A. *On the exact estimates for the approximation of functions of several variables by sums of functions of fewer variables.* Mat. zametki, 12, No1, 1972, pp.105-114. (Russian)

[9]. Alexandrov P.S. *Introduction to the theory of sets and general topology.* M.: "Nauka", 1977, 368 p. (Russian)

[10]. Ismailov V.E. *Two-sided estimates for the best approximation in domains consisting of the union of rectangles.* Transactions of Azerb. Acad. of Sciences, v.IV (XII), 1996, pp.3-10. (Russian)

[11]. Ismailov V.E. *Theorem on lightning bolts for elementary domains.* Proceedings of IMM of NASA, v. XVII (XXV), 2002, pp.78-86.

**Vugar E. Ismailov**
Institute of Mathematics and Mechanics of NAS of Azerbaijan.
9, F.Agayev str., AZ1141, Baku, Azerbaijan.
Tel.: (99412) 439 47 20 (off.)

Received May 19, 2003; Revised September 11, 2003.
Translated by the author.