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ON A UNIQUENESS OF STRONG SOLUTION OF DIRICHLET PROBLEM FOR SECOND ORDER QUASILINEAR PARABOLIC EQUATIONS

Abstract

It is considered a first boundary value problem for parabolic equations of the second order in nondivergent form whose principal part satisfies the Cordes condition. A uniquenss of strong (almost everywhere) solution of the problem is proved.

Let E_n and $R_{n+1}-n$ and $n+1$ be Euclidian spaces of the points $x=(x_1,...,x_n)$, $(t,x)=(t,x_1,...,x_n)$, respectively. Ω is a bounded convex domain in E_n with boundary $\partial\Omega$ belonging to the class C^2, Q_T is the cylinder $\Omega\times(0,T)$, S_T is its lateral surface $\partial\Omega\times(0,T)$, $\Gamma(Q_T)=\Omega\cup S_T, 0<T<\infty$. In Q_T consider the first boundary value problem

$$Lu=\sum_{i,j=1}^{n}a_{ij}(t,x,u)\,u_{ij}-u_t=f(t,x)\,;(t,x)\in Q_T, \tag{1}$$

$$u|_{\Gamma(Q_T)}=0, \tag{2}$$

where $u_t=\dfrac{\partial u}{\partial t}, u_i=\dfrac{\partial u}{\partial x_i}, u_{ij}=\dfrac{\partial^2 u}{\partial x_i\partial x_j};\ i,j=1,...,n, \|a_{ij}(t,x,z)\|$ is a real symmetric matrix whose elements are measurable in Q_T at any fixed $z\in E_1$, moreover

$$\mu|\xi|^2\le\sum_{i,j=1}^{n}a_{ij}(t,x,z)\,\xi_i\xi_j\le\mu^{-1}|\xi|^2;\ (t,x)\in Q_T, z\in E_1,\xi\in E_n; \tag{3}$$

$$\sigma=\underset{\substack{(t,x)\in Q_T\\ z\in E_1}}{ess\sup}\frac{\sum\limits_{i,j=1}^{n}a_{ij}^2(t,x,z)}{\left[\sum\limits_{i=1}^{n}a_{ii}(t,x,z)\right]^2}<\frac{1}{n-1}. \tag{4}$$

Here $\mu\in(0,1]$ is a constant. Condition (4) is said to be Cordes condition and is understood to within non-degenerate transformation in the following sense: the domain Q_T may be covered by a finite number of subdomains $Q^1,...,Q^l$, so that at each Q^i one can make non-degenerate linear transformation of coordinates at which the coefficients of the operator L satisfy condition (4) in the image $Q^i;\ i=1,...,l$.

The goal of the paper is to prove a uniqueness of strong (almost everywhere) solution of a first boundary value problem (1)-(2) $f(t,x)\in L_2(Q_T)$ at $n=1,2$. Indicate the papers [1-4] where analogous results were obtained for linear parabolic equations. We mention also the papers [5-7] where some classes of abovementioned equations with discontinuous coefficients were considered. Note the papers [8-11] where problems of strong solvability of boundary value problems were searched for second order elliptic equations of nondivergent form. In [12] the existence of strong solution of the first boundary value problem (1)-(2) was established for several general class equations than (1). Mention also the papers [13-14] where the solvability of boundary-value problems were studied for nonlinear second order parabolic equations. There with in [14] the solvability of the first boundary value problem was proved under more rigid condition than condition (4).

Now let's agree to some denotation. By $W_p^{2,1}(Q_T)$ we'll denote a Banach space of functions $u(t,x)$, given on Q_T with finite norm

$$\|u\|_{W_p^{2,1}(Q_T)} = \left(\int\limits_{Q_T}\left(|u|^p + \sum_{i=1}^n |u_i|^p + \sum_{i,j=1}^n |u_{i,j}|^p + |u_t|^p\right) dtdx\right)^{1/p},$$

where $p \in (1,\infty)$. By $\dot{W}_p^{2,1}(Q_T)$ we denote a subspace $W_p^{2,1}$ where the totality of all functions $u(t,x) \in C^\infty\left(\bar{Q}_T\right)$ vanishing at $\Gamma(Q_T)$ is a dense set.

The function $u(t,x) \in \dot{W}_p^{2,1}(Q_T)$ is said to be a strong solution of the first boundary value problem (1)-(2) (at $f(t,x) \in L_p(Q_T)$), if it satisfies equation (1) almost everywhere in Q_T.

Everywhere $C(...)$ means that the positive constant C depends only on the content of parenthesis.

Theorem 1. *(see [3]). For any function $u(t,x) \in W_p^{2,1}(Q_T)$ the estimations*

$$\|u_i\|_{L_{q_1}(Q_T)} \le C_1(p,n)\|u\|_{W_p^{2,1}(Q_T)} \ \ \text{if}\ 1 < p < n+2, 1 \le q_1 \le \frac{(n+2)p}{n+2-p},$$

$$\|u\|_{L_{q_2}(Q_T)} \le C_2(p,n)\|u\|_{W_p^{2,1}(Q_T)} \ \ \text{if}\ 1 < p < \frac{n+2}{2},$$

$$1 \le q_2 \le \frac{(n+2)p}{n+2-2p}$$

hold. It follows from this theorem that for any function $u(t,x) \in W_2^{2,1}(Q_T)$ at $n = p = 2$ and $q \in [1,\infty)$ it is valid the estimation

$$\|u\|_{L_q(Q_T)} \le C_3(q)\|u\|_{W_2^{2,1}(Q_T)}. \tag{5}$$

Theorem 2. *(see [11]). Let conditions (3)-(4) and the condition*

$$\left|a_{ij}\left(t,x,z^1\right) - a_{ij}\left(t,x,z^2\right)\right| \le H_1\left|z^1 - z^2\right|,$$
$$(t,x) \in Q_T,\ z^1, z^2 \in E_1,\ i,j = 1,...,n, \tag{6}$$

with some non-negative constant H be fulfilled for the coefficients of the operator L. Then there exists $p_1(\mu,\sigma,n) \in \left(\frac{5}{3},2\right)$ such that at any $p \in [p_1,2]$ for any function $u(t,x) \in \dot{W}_p^{2,1}(Q_T)$ it is valid the estimation

$$\|u\|_{W_p^{2,1}(Q_T)} \le C_4(\mu,\sigma,n,\Omega)\|Lu\|_{L_p(Q_T)}. \tag{7}$$

In this case for any $A>0$ there exists such $T_A=T_A(\mu,\sigma,n,\Omega,H,A)$ that if $T\le T_A$, then the first boundary value problem (1)-(2) has a strong solution from the space $\dot W_2^{2,1}(Q_T)$, at any function $f(t,x)\in L_2(Q_T)$, as soon as $\|f\|_{L_2(Q_T)}\le A$.

Theorem 3. *(see [11]). Let conditions (1)-(2) and (6) be fulfilled for the coefficients of the operator L. Then for any $s\in(2,\infty)$ at $n=2, s\in[2,\infty)$ at $n=1$ and $A>0$ there exists such $T'_A=T'_A(\mu,\sigma,n,\Omega,H_1,s,A)$ that if $T\le T'_A$, $f(t,x)\in L_s(Q_T)$ and $\|f\|_{L_s(Q_T)}\le A$ then the first boundary value problem has a strong solution $u(t,x)\in\dot W_2^{2,1}(Q_T)$.*

Proof. We'll choose such a constant T'_A that $T'_A\le T_A$. Therefore, by theorem 2 only the uniqueness is to be proved. Let $u^1(t,x)$ and $u^2(t,x)$ be two strong solutions of the first boundary value problem (1)-(2) from the space $\dot W_2^{2,1}(Q_T)$,

$$L_1\equiv\sum_{i,j=1}^n a_{ij}\left(t,x,u^1(t,x)\right)\frac{\partial^2}{\partial x_i\partial x_j}-\frac{\partial}{\partial t}.$$

We have

$$L_{(1)}\left(u^1-u^2\right)=\sum_{i,j=1}^n a_{ij}\left(t,x,u^1(t,x)\right)u^1_{ij}-u^1_t-$$

$$-\sum_{i,j=1}^n\left[a_{ij}\left(t,x,u^1\right)-a_{ij}\left(t,x,u^2\right)\right]u^2_{ij}-f(t,x)=$$

$$=-\sum_{i,j=1}^n\left[a_{ij}\left(t,x,u^1\right)-a_{ij}\left(t,x,u^2\right)\right]u^2_{ij}=F(t,x). \tag{8}$$

On the other hand, by (6)

$$|F(t,x)|\le H_1\left|u^1-u^2\right|\sum_{i,j=1}^n\left|u^2_{ij}\right|. \tag{9}$$

First we consider the case $n=2$. Let $q_1=\dfrac{4p_1}{4-2p_1}$. Applying theorems 1 and 2 using inequality (5) and Hőlder inequality from (8) and (9) we deduce

$$\left\|u^1-u^2\right\|_{L_{q_1}(Q_T)}\le C_3(p,n)\left\|u^1-u^2\right\|_{W^{2,1}_{p_1}(Q_T)}\le$$

$$\le C_3C_4\|F\|_{L_{p_1}(Q_T)}\le C_3C_4H_1\left[\int_{Q_T}\left|u^1-u^2\right|^{p_1}\left(\sum_{i,j=1}^n\left|u^2_{ij}\right|\right)^{p_1}dtdx\right]^{1/p_1}\le$$

$$\le C_3C_4H_1\left\|u^1-u^2\right\|_{L_{q_1}(Q_T)}\left(\int_{Q_T}\left(\sum_{i,j=1}^n\left|u^2_{ij}\right|\right)^2dtdx\right)^{1/2}\le \tag{10}$$

$$\le 4C_3C_4H_1\left\|u^1-u^2\right\|_{L_{q_1}(Q_T)}\left\|u^2\right\|_{W_2^{2,1}(Q_T)}\le$$

$$\leq 4C_3C_4^2H_1 \left\|u^1-u^2\right\|_{L_{q_1}(Q_T)} \|f\|_{L_2(Q_T)} \leq$$

$$\leq 4C_3C_4^2H_1 \left\|u^1-u^2\right\|_{L_{q_1}(Q_T)} \|f\|_{L_s(Q_T)} (T \times mes\Omega)^{\frac{s-2}{2s}} \leq$$

$$\leq 4C_3C_4^2H_1A \left(T'_A \times mes\Omega\right)^{\frac{s-2}{2s}} \left\|u^1-u^2\right\|_{L_{q_1}(Q_T)}.$$

Let T' be such that

$$4C_3C_4^2H_1A \left(T' \times mes\Omega\right)^{\frac{s-2}{2s}} = \frac{1}{2}.$$

Choose $T'_A = \min\{T_A, T'\}$. Then if follows from (10) that

$$\left\|u^1-u^2\right\|_{L_{q_1}(Q_T)} \leq \frac{1}{2}\left\|u^1-u^2\right\|_{L_{q_1}(Q_T)},$$

i.e. $u^1(t,x) = u^2(t,x)$ a.e. in Q_T.

Now, let $n=1$. Then by the embedding theorem [3] for any function $u(t,x) \in W_p^{2,1}(Q_T)$, at every $p \in (1,5;\infty)$ it is valid estimation

$$\sup_{Q_T} |u| \leq C_5(p)\|u\|_{W_p^{2,1}},$$

and in this case applying theorems 1 and 2 and using Hőlder inequality, we deduce from (8) and (9)

$$\sup_{(t,x)\in Q_T} \left|u^1-u^2\right| \leq C_5 \left\|u^1-u^2\right\|_{W_{p_1}^{2,1}(Q_T)} \leq$$

$$\leq C_4C_5\|F(t,x)\|_{L_{p_1}(Q_T)} \leq C_5C_4H_1\left(\int_{Q_T}\left|u^1-u^2\right|^{p_1}\left|u_{11}^2\right|^{p_1}dtdx\right)^{1/p_1} \leq$$

$$\leq C_5C_4H_1\left(T' \times mes\Omega\right)^{\frac{2-p_1}{2p_1}} \sup_{(t,x)\in Q_T}\left|u^1-u^2\right| \left\|u^2\right\|_{W_2^{2.1}(Q_T)} \leq$$

$$\leq C_5C_4^2H_1 (T \times mes\Omega)^{\frac{2-p_1}{2p_1}} \sup_{(t,x)\in Q_T}\left|u^1-u^2\right| \|f\|_{L_{p_1}(Q_T)} \leq$$

$$\leq C_5C_4^2H_1A\left(T'_A \times mes\Omega\right)^{\frac{2-p_1}{2p_1}} \sup_{(t,x)\in Q_T}\left|u^1-u^2\right|. \quad (11)$$

Let T'' be such that

$$C_5C_4^2H_1A\left(T'' \times mes\Omega\right) = \frac{1}{2}.$$

Then, choosing $T\prime = \min\{T_A, T''\}$. We deduce from (11)

$$\sup_{(t,x)\in Q_T}\left|u^1-u^2\right| \leq \frac{1}{2}\sup_{(t,x)\in Q_T}\left|u^1-u^2\right|$$

i.e. $u^1(t,x) = u^2(t,x)$ i.e. in Q_T. The theorem is proved.

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