Faig B. GUSEYNOV

ASYMPTOTICS AS $t\to +\infty$ OF SOLUTION OF CAUCHY PROBLEM FOR SOBOLEV-GALPERN TIME DERIVATIVE OF THE FIRST ORDER EQUATION

Abstract

The behaviour as $t\to +\infty$ of the solution of the following Cauchy problem

$$Q\left(i\frac{\partial}{\partial x}\right)\frac{\partial u(x,t)}{\partial t}=P\left(i\frac{\partial}{\partial x}\right)u(x,t),$$

$$u(x,0)=\psi(x),$$

where $\operatorname{Re}\dfrac{P(\sigma)}{Q(\sigma)}\le c_0$ *at* $\sigma\in(-\infty,+\infty)$ *is studied in the paper.*

The equations unsolved with respect to higher time derivative first was studied in Poincare's known paper [1] in 1985. Further, this class of equations repeatedly drew attention of mathematicians and mechanical engineers. In 1939 Rossby in [2] obtained the equality

$$\Delta_2 D_t(x,t)+\beta D_{x_2}u(x,t)=0,$$

where Δ_2 is a Laplace operator with respect to (x_1,x_2), $D_{x_j}=\dfrac{\partial}{\partial x_j}$.

In the forties of the last century by solving the problem on small oscillations of rotating fluid S.L.Sobolev in [3] extracted the following equality

$$\Delta_3 D_t^2 u+\omega^2 D_{x_3}^2 u=f(x,t),$$

where Δ_3 is a Laplace operator with respect to (x_1,x_2,x_3), and ω is an angular velocity. For this equation S.L.Sobolev studied the Cauchy problem, first and second boundary value problems in a cylindric domain and also formulated a number of new problems of mathematical physics. A general case of a system of equations of the noted type with many variables was first studied by S.A.Galpern in [4] in a class of functions integrable with square with respect to x together with some number of derivatives. The Cauchy problem for systems unsolved with respect to time derivative in a class of distributions were studied in [5]. The behaviour as $t\to+\infty$ of the solution of the Cauchy problem for some equations unsolved with respect to time derivative was studied in [6]. In this paper we'll study the behaviour as $t\to+\infty$ of the solution of the Cauchy problem for the equation not considered in [6].

Consider the following Cauchy problem

$$Q\left(i\frac{\partial}{\partial x}\right)\frac{\partial u(x,t)}{\partial t}=P\left(i\frac{\partial}{\partial x}\right)u(x,t),\tag{1}$$

$$u(x,0)=\psi(x),\tag{2}$$

where $Q(s)$ and $P(s)$ are some polynomials with respect to s, $\psi(x)$, is a sufficiently smooth function that will be defined below. The Green function of problem (1)-(2) is the function $G(x,t)$ that is the solution of the problem

$$Q\left(i\frac{\partial}{\partial x}\right)\frac{\partial G(x,t)}{\partial t}=P\left(i\frac{\partial}{\partial x}\right)G(x,t), \tag{3}$$

$$G(x,0)=\delta(x), \tag{4}$$

where $\delta(x)$ is a Dirac function.

We'll assume that for

$$\lambda(s)=\frac{P(s)}{Q(s)}$$

the condition

$$\operatorname{Re}\lambda(\sigma)\leq\tilde{c},\quad \sigma\in R_1\equiv(-\infty,+\infty) \tag{5}$$

is satisfied, i.e. equation (1) is correct by Petrowskii. Expand $\lambda(s)$ in series in the neighbourhood of the point at infinity

$$\lambda(s)=\alpha_\nu s^\nu+\alpha_{\nu-1}s^{\nu-1}+... \tag{6}$$

In [7] it is proved that by condition (5) the following cases are possible in expansion (6): 1^0. $\operatorname{Re}\alpha_\nu<0$, ν -is even, $\nu>0$;

2^0. $\operatorname{Re}\alpha_\nu=\operatorname{Re}\alpha_{\nu-1}=...=\operatorname{Re}\alpha_{p+1}=0$, $\operatorname{Re}\alpha_p<0$, $p>0$, p is even;

3^0. $\operatorname{Re}\alpha_\nu=\operatorname{Re}\alpha_{\nu-1}=...=\operatorname{Re}\alpha_1=0$, $\quad a)$ $\nu\geq 2$; $\quad b)$ $\nu<2$.

Expand $\lambda(s)$ also in the neighbourhood of the real root σ_μ^* of the polynomial $Q(s)$

$$\lambda(s)=\sum_{j=\nu}^{0}\frac{\alpha_{j_\mu}}{(s-\sigma_\mu)}j_\mu+\frac{P_1(s)}{Q_1(s)}, \tag{7}$$

where $\dfrac{P_1(s)}{Q_1(s)}$ -is a regular part of expansion (7). The following cases are possible:

1^0. $\operatorname{Re}\alpha_{\nu_\mu}<0$, $\nu_\mu>0$, ν_μ is even;

2^0. $\operatorname{Re}\alpha_{\nu_\mu}=...=\operatorname{Re}\alpha_{(p+1)_\mu}=0$, $\operatorname{Re}\alpha_{p_\mu}<0$, $p_\mu>0$, p_μ is even

3^0. $\operatorname{Re}\alpha_{\nu_\mu}=...=\operatorname{Re}\alpha_{1_\mu}=0$.

For the Green function of Cauchy problem (1)-(2) is represented as

$$G(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{t\lambda(\sigma)+ix\sigma}d\sigma, \tag{8}$$

and the solution of Cauchy problem (1)-(2) is defined by the formula

$$u(x,t)=G(x,t)*\psi(x), \tag{9}$$

[8] (pp.130-180) where the sign $*$ means the convolution of two functions.

By $\mathbf{C}$ we denote a complex plane.

Definition. *The point $S_0=\sigma+i\tau_0\in\mathbf{C}$ is said to be the l -th order saddle-point of the analytic function $f(s)$, if*

$$f'(s_0)=...=f^{(l)}(s_0)=0,\quad f^{(l+1)}(s_0)\neq 0.$$

The saddle-point is said to be ordinary if $l = 1$.

Lemma 1. *The saddle-points of the function $\lambda(s)$ may not coincide with the zeros of the polynomial $Q(s)$.*

Proof. *Without losing generality we'll assume that $P(s)$ and $Q(s)$ have no common zeros. Let s_0 be the saddle-point of the function $\dfrac{P(s)}{Q(s)}$.*

Then

$$\frac{P'(s_0)Q(s_0) - P(s_0)Q'(s_0)}{Q^2(s_0)} = 0. \tag{10}$$

In order that equality (10) have sense, $Q(s_0) \neq 0$. If $Q(s_0) = 0$ and $Q'(s_0) \neq 0$ then in order that equality (8) have sense, the numerator in this expression must have a zero of order not less than one. Then from (8) we get

$$P'(s_0)Q(s_0) - P(s_0)Q'(s_0) = 0.$$

Hence, we get that $P(s_0) = 0$. If at the point s_0 $Q(s_0) = 0$, $Q'(s_0) = 0$, $Q''(s_0) \neq 0$, then acting as above we get

$$P''(s_0)Q(s_0) - Q''(s_0)P(s_0) = 0.$$

Hence, we get that $P(s_0) = 0$, i.e. we get contradiction. Continuing by such a way we get that the polynomials $P(s)$ and $Q(s)$ have no common zeros. The lemma is proved.

For the solution of the Cauchy problem (1)-(2) the following theorem holds.

Theorem 1. *Let x be a variable, σ_j $(j = 1, ..., \nu_2)$ be ordinary real saddle-points of $\lambda(s)$,*

$$\operatorname{Re}\lambda(c_0) = \max_{1\le j\le \nu_2} \operatorname{Re}\lambda(\sigma_j).$$

1. In case $1^0(6)$, $2^0(6)$ and $1^0(7)$, $2^0(7)$

$$|\psi(x)| \le \frac{C}{(1+|x|)^{2+\varepsilon}}.$$

2. In case $3^0(6)$, $3^0(4)$

$$\left|\frac{d^\alpha}{dx^\alpha}\psi(x)\right| \le \frac{C}{(1+|x|)^{2+\varepsilon}},\ \alpha = 0, 1, 2.$$

Then as $t \to +\infty$ for the solution of the Cauchy problem (1)-(2) it holds the asymptotics

$$u(x,t) = \frac{ie^{t\lambda(c_0)+ixc_0}}{\sqrt{2\pi t\lambda''(c_0)}}\upsilon(c_0) + W(x,t) + O\left(t^{-\frac{3}{2}}\right),$$

where $\upsilon(c_0)$ is the value of Fourier transform of the initial function $\psi(x)$ at the point $\sigma = c_0$ and for $W(x,t)$ it holds the estimation

$$|W(x,t)| \le \frac{C}{t}e^{t\tilde{c}} \tag{11}$$

uniformly with respect to x at each segment from R_1.

Proof. Write the following expansion of the unit

$$1 \equiv \varphi_{-R}(\sigma) + \varphi_R(\sigma) + \sum_{\mu=1}^{\nu_1} \varphi_{1,\mu}(\sigma) + \sum_{j=1}^{\nu_2} \varphi_{2,j}(\sigma) + \sum_{k=1}^{\nu_3} \varphi_{3,k}(\sigma),$$

where $\varphi_{-R}(\sigma), \varphi_R(\sigma)$ are infinitely differentiable functions equal to a unit at $x \leq -R$ and $x \geq R$, respectively; $\varphi_{1,\mu}(\sigma) \quad \nu = (1,2,...,\nu_1)$ are finite infinitely differentiable functions equal to a unit in the neighbourhood $\left(\sigma^*_{-\nu} - \delta,\ \sigma^*_\nu + \delta\right)$ of real zeros σ^*_μ of the polynomial $Q(s)$, and $\varphi_{2,j}(\sigma) \quad (j = 1,2,...,\nu_2)$ are finite infinitely differentiable functions equal to unit in the neighbourhood $(\sigma_j - \delta, \sigma_j + \delta)$ of real saddle points $\lambda(s)$ and $\varphi_{3,k}(\sigma)$ are finite infinitely differentiable functions with supports in $\left(\sigma^*_j, \sigma_\mu\right), (k = 1,2,...,\nu_3)$. Then the Green function has the following representation:

$$G(x,t) = G_{-R}(x,t) + G_R(x,t) + \sum_{\mu=1}^{\nu_1} G_{1,\mu}(x,t) +$$

$$+ \sum_{j=1}^{\nu_2} G_{2,j}(x,t) + \sum_{k=1}^{\nu_3} G_{3,k}(x,t),$$

where

$$G_{-R}(x,t) = \frac{1}{2\pi} \int\limits_{-\infty}^{-R+\delta} e^{t\lambda(\sigma)+ix\sigma} \varphi_{-R}(\sigma)\, d\sigma,$$

$$G_R(x,t) = \frac{1}{2\pi} \int\limits_{R-\delta}^{-\infty} e^{t\lambda(\sigma)+ix\sigma} \varphi_R(\sigma)\, d\sigma,$$

$$G_{1,\mu}(x,t) = \frac{1}{2\pi} \int\limits_{\sigma^*_\nu-\delta}^{\sigma^*_\nu+\delta} e^{t\lambda(\sigma)+ix\sigma} \varphi_{1,\mu}(\sigma)\, d\sigma, \quad \mu = 1,2,...,\nu_1, \tag{12}$$

$$G_{2,j}(x,t) = \frac{1}{2\pi} \int\limits_{\sigma_j-\delta}^{\sigma_j+\delta} e^{t\lambda(\sigma)+ix\sigma} \varphi_{2,j}(\sigma)\, d\sigma, \quad j = 1,2,...,\nu_2,$$

$$G_{3,k}(x,t) = \frac{1}{2\pi} \int\limits_{\sigma^*_j}^{\sigma_\mu} e^{t\lambda(\sigma)+ix\sigma} \varphi_{3,k}(\sigma)\, d\sigma, \quad k = 1,2,...,\nu_3.$$

Now let's study the behaviour of each addend in (12) as $t \to +\infty$. Since $G_R(x,t)$ and $G_{-R}(x,t)$ are studied similarly, we study one of them.

Case $1^0(6)$. In this case

$$G_R(x,t) = \frac{1}{2\pi} \int\limits_{R-\delta}^{+\infty} e^{t\sigma^\nu\left(\alpha_\nu + O\left(\frac{1}{\sigma}\right)\right)+ix\sigma} \varphi_R(\sigma)\, d\sigma,$$

where $\operatorname{Re}\alpha_\nu<0,\ \nu\geq 2,\ \nu$ is an even number. Estimating $G_R(x,t)$ by modulus and considering the properties of $\varphi_R(x)$ we get

$$|G_R(x,t)|\leq C\int\limits_{R-\delta}^{+\infty} e^{t\sigma^\nu(\operatorname{Re}\alpha_\nu+\delta_1)}d\sigma \tag{13}$$

uniformly with respect to $x\in R_1$ where δ_1 is a sufficiently small number. Represent the integral in (13) in the form

$$\frac{C}{\nu t\left(\operatorname{Re}\alpha_\nu+\delta_1\right)}\int\limits_{R-\delta}^{+\infty}\frac{1}{\sigma^{\nu-1}}de^{t\sigma^\nu(\operatorname{Re}\alpha_\nu+\delta_1)}=\frac{C\left(R-\delta\right)^{-\nu+1}}{\nu t\left(\left|\operatorname{Re}\alpha_\nu\right|-\delta_1\right)}e^{tR^\nu(\operatorname{Re}\alpha_\nu+\delta_1)}.$$

Thus,

$$|G_R(x,t)|\leq\frac{C\left(R-\delta\right)^{-\nu+1}}{\nu t\left(\left|\operatorname{Re}\alpha_\nu\right|-\delta_1\right)}e^{tR^\nu(\operatorname{Re}\alpha_\nu+\delta_1)} \tag{14}$$

uniformly with respect to $x\in R_1$. Similarly we get

$$|G_{-R}(x,t)|\leq\frac{C\left(R-\delta\right)^{-\nu+1}}{\nu t\left(\left|\operatorname{Re}\alpha_\nu\right|-\delta_1\right)}e^{tR^\nu(\operatorname{Re}\alpha_\nu+\delta_1)} \tag{15}$$

uniformly with respect to $x\in R_1$. It follows from (14) and (15) that as $t\to+\infty$ $G_{-R}(x,t)\,,\ G_R(x,t)$ exponentially decrease.

Case $2^0(6)$ is considered in a similar way. In this case we have

$$\left\{|G_R(x,t)|\,,\ |G_{-R}(x,t)|\right\}\leq\frac{C\left(\nu,\alpha_{p,\delta_1}\right)}{t}e^{t(R-\delta)^p(\operatorname{Re}\alpha_p+\delta_1)}. \tag{16}$$

Now, let's consider case $3^0(6)$. Then the integrals in expressions $G_R(x,t)$ and $G_{-R}(x,t)$ converge, but not absolutely. Regularize the integrals in $G_{\pm R}(x,t)$. For this represent $G_R(x,t)$ in the form:

$$G_R(x,t)=\frac{1}{2\pi}\left[\int\limits_{R-\delta}^{R+1}+\int\limits_{R+1}^{+\infty}\right]e^{it(\operatorname{Im}\alpha_\nu\sigma^\nu+\ldots+\operatorname{Im}\alpha_1\sigma)+ix\sigma}\varphi_R(\sigma)\,d\sigma\equiv$$

$$\equiv G_R^{(1)}(x,t)+G_R^{(2)}(x,t)\,.$$

In the expression $G_R^{(2)}(x,t)$ we arrive at the complex plane $s=re^{i\theta}$, depending on the evenness and odness of ν at the upper or lower half-plane. Let n be an even number. Then for the integrand function in the expression $G_R^{(2)}(x,t)$ we have

$$\left|e^{it\operatorname{Im}\alpha_\nu s^\nu\left(1+O\left(\frac{1}{s}\right)\right)}\right|=e^{-t\operatorname{Im}\alpha_\nu r^\nu\sin\pi\varepsilon\left(1+O\left(\frac{1}{s}\right)\right)}.$$

Therefore the integrand function in the expression $G_R^{(2)}(x,t)$ as $s\to\infty$ with respect to the ray $\infty e^{i(\pi+\varepsilon)}$ ($\varepsilon>0$ is sufficiently small angle) exponentially decreases. In the expression $G_R(x,t)$ now we can integrate by parts as much as desired times. In this the contributions in $G_R^{(1)}(x,t)+G_R^{(2)}(x,t)$ from the point $s=R+1$ will be mutually annihilated, and the contribution from the point $s = R - \delta$ will be zero, since $\varphi_R(\sigma)$ at the point $s = R - \delta$ will equal zero together with derivatives of any order. Hence we obtain

$$|G_R(x,t)| \le \frac{C(1+|x|)^l}{t^l}. \tag{17}$$

In the case $\operatorname{Im}\alpha_\nu < 0$ we'll arrive at the upper half-plane $\tau > 0$. The case of odd ν is considered similarly with the difference that at $\operatorname{Im}\alpha_\nu < 0$ we'll arrive at the lower half-plane $\tau < 0$.

Thus, at all cases estimation (17) holds.

In a similar way for $G_{-R}(x,t)$ we get the estimation

$$|G_{-R}(x,t)| \le \frac{C(1+|x|)^l}{t^l}. \tag{18}$$

In case 3^0(b) from (6) the Green function $G(x,t)$ of the Cauchy problem (1)-(2) is not an ordinary function since the integral in (8) doesn't converge in ordinary sense. Therefore, using distributions theory [8] (p.194) we represent the integral in the expression $G(x,t)$ in the form

$$G(x,t) = \frac{1}{2\pi}\left(1 - \frac{d^2}{dx^2}\right)\int\limits_{-\infty}^{+\infty}\frac{1}{1+\sigma^2}e^{t\lambda(\sigma)+ix\sigma} \equiv$$

$$\equiv \frac{1}{2\pi}\left(1 - \frac{d^2}{dx^2}\right)G^0(x,t).$$

To $G^0(x,t)$ we can apply formulae (13), (15), (17), (18). Now consider

$$G_{1,\mu}(x,t) = \frac{1}{2\pi}\int\limits_{\sigma_\mu^*-\delta}^{\sigma_\mu^*+\delta} e^{t\lambda(\sigma)+ix\sigma}\varphi_{1,\mu}(\sigma)\,d\sigma, \quad \nu = 1,2,..,\nu_1.$$

In case 1^0(7) we have

$$G_{1,\mu}(x,t) = \frac{1}{2\pi}\int\limits_{\sigma_\mu^*-\delta}^{\sigma_\mu^*+\delta} e^{t\frac{\alpha_{\nu_\mu}}{(s-\sigma_\mu^*)^{\nu_\mu}}(1+O(\delta))}\varphi_{1,\mu}(\sigma)\,d\sigma,$$

where $\operatorname{Re}\alpha_{\nu_\mu} < 0$ and ν_μ is an even number. By evenness of ν_μ and symmetricity of $\varphi_{1,\mu}(\sigma)$ we have

$$G_{1,\mu}(x,t) = \frac{1}{\pi}\int\limits_{\sigma_\mu^*-\delta}^{\sigma_\mu^*+\delta} e^{t\frac{\alpha_{\nu_\mu}}{(s-\sigma_\mu^*)^{\nu_\mu}}(1+O(\delta))+ix\sigma}\varphi_{1,\mu}(\sigma)\,d\sigma. \tag{19}$$

In (19) we make substitution

$$\left(s - \sigma_\mu^*\right)^{-\nu_\mu} = \tau. \tag{20}$$

Then

$$G_{1,\mu}(x,t)=\frac{\nu_\mu}{\pi}e^{ix\sigma_\mu^*}\int\limits_{\delta^{-\nu_\mu}}^{+\infty}e^{\alpha_{\nu_\mu}t\tau(1+O(\delta))+ix\tau^{-\nu_\mu^{-1}}}\tau^{-\nu_\mu^{-1}-1}\varphi_{1,\mu}\left(\sigma_\mu^*+\tau^{-\nu_\mu^{-1}}\right)d\tau.$$

Estimating by modulus we get

$$|G_{1,\mu}(x,t)|\le\frac{\nu_\mu^{-1}}{\pi}\delta^{1+\nu_\mu^{-1}}\int\limits_{\delta^{-\nu_\mu}}^{+\infty}e^{\mathrm{Re}\,\alpha_{\nu_\mu}t\tau}d\tau=\frac{\nu_\mu^{-1}\delta^{1+\nu_\mu^{-1}}}{\pi\left|\mathrm{Re}\,\alpha_{\nu_\mu}\right|t}e^{t\,\mathrm{Re}\,\alpha_{\nu_\mu}\delta^{-\nu_\mu}}.$$

Hence we have

$$|G_{1,\mu}(x,t)|\le\frac{\nu_\mu\delta^{1+\nu_\mu^{-1}}}{\pi\left|\mathrm{Re}\,\alpha_{\nu_\mu}\right|t}e^{t\,\mathrm{Re}\,\alpha_{\nu_\mu}\delta^{-\nu_\mu}}\tag{21}$$

uniformly with respect to $x\in R_1$. Since $\mathrm{Re}\,\alpha_{\nu_\mu}<0$, then the contribution from real zeros of the polynomial $Q(s)$ to the asymptotics of the Green function $G(x,t)$ as $t\to+\infty$ is exponentially small.

Now, consider case $2^0(8)$. In this case the estimation of $G_\mu(x,t)$ is carried out as above only with difference that in estimation (21) one should substitute ν_μ by p_μ, where $0<p_\mu<\nu_\mu, p_\mu$ is even, $\mathrm{Re}\,\alpha_{p_\mu}<0$.

Let case $3^0(7)$ hold. Then

$$G_{1,\mu}(x,t)=\frac{1}{2\pi}\int\limits_{\sigma_\mu^*-\delta}^{\sigma_\mu^*+\delta}e^{t\frac{i\,\mathrm{Im}\,\alpha_{\nu_\mu}}{(\sigma-\sigma_\mu^*)^{\nu_\mu}}(1+O(\delta))+ix\sigma}\varphi_{1,\mu}(\sigma)\,d\sigma,$$

here α_{ν_μ} may be both even and odd. Consider the case of even ν_μ, the case of odd ν_μ will be considered similarly. Represent $G_\mu(x,t)$ in the form

$$G_\mu(x,t)=\frac{1}{2\pi}\left\{\int\limits_{\sigma_\mu^*}^{\sigma_\mu^*+\delta}+\int\limits_{\sigma_\mu^*-\delta}^{\sigma_\mu^*}\right\}e^{t\frac{i\,\mathrm{Im}\,\alpha_{\nu_\mu}}{(\sigma-\sigma_\mu^*)^{\nu_\mu}}+ix\sigma[1+O(\delta)]+ix\sigma}\varphi_{1,\mu}(\sigma)\,d\sigma\equiv$$

$$\equiv G_\mu^I(x,t)+G_\mu^{II}(x,t).\tag{22}$$

Consider $G_\mu^I(x,t)$. Make in the expression $G_\mu^I(x,t)$ substitution (20). Then

$$G_\mu^I(x,t)=\frac{\nu_\mu^{-1}e^{ix\sigma_\mu^*}}{2\pi}\int\limits_{\delta^{-\nu_\mu}}^{+\infty}e^{it\,\mathrm{Im}\,\alpha_{\nu_\mu}y+ixy^{-\nu_\mu^{-1}}}y^{-\nu_\mu^{-1}-1}\varphi_{1,\mu}\left(\sigma_\mu^*+y^{-\nu_\mu^{-1}}\right)dy.\tag{23}$$

Integrating in (23) by parts once and allowing for the properties of the function $\varphi_{1,\mu}(\sigma)$ we get

$$G_\mu^I(x,t)=\frac{\nu_\mu^{-1}e^{ix\sigma_\mu}}{it\alpha_{\nu_\mu}}\int\limits_{\delta^{-\nu_\mu}}^{+\infty}e^{it\,\mathrm{Im}\,\alpha_{\nu_\mu}y+ixy^{-\nu_\mu^{-1}}}\times$$

$$\times\left\{\left[ix\nu_\mu^{-1}y^{-2\left(1+\nu_\mu^{-1}\right)}+\left(\nu_\mu^{-1}+1\right)y^{-\nu_\mu^{-1}-2}\right]\varphi_{1,\mu}\left(\sigma_\mu^*+y^{-\nu_\mu^{-1}}\right)-\right.$$

$$\left.-\nu_\mu^{-1}y^{-2\left(\nu_\mu^{-1}+1\right)}\varphi_{1,\mu}\left(\sigma_\mu^*+y^{-\nu_\mu^{-1}}\right)\right\}dy. \tag{24}$$

Here the terms outside the integral at the upper limit equal zero because of decrease of integrand function, and at the lower limit-because of finiteness of the function $\varphi_{1,\mu}(\sigma)$. Estimating (23) by modulus, we get

$$\left|G_\mu^I(x,t)\right|\le\frac{C\left(\alpha_{\nu_\mu}\right)}{t}\left(1+|x|\right).$$

Continuing the integration by parts in (23) one more $l-1$ times and estimating by modulus the obtained expression we get

$$\left|G_\mu^I(x,t)\right|\le\frac{C\left(\alpha_{\nu_\mu}\right)}{t^l}\left(1+|x|\right)^l. \tag{25}$$

We estimate the second addend by a similar way $G_\mu^{II}(x,t)$.

$$\left|G_\mu^{II}(x,t)\right|\le\frac{C\left(\alpha_{\nu_\mu}\right)}{t^l}\left(1+|x|\right)^l.$$

(22) and the last two estimates yield

$$\left|G_\mu(x,t)\right|\le\frac{C\left(\alpha_{\nu_\mu}\right)}{t^l}\left(1+|x|\right)^l. \tag{26}$$

To the integrals $G_{2,j}(x,t)$ as $t\to+\infty$ we apply the saddle method [9] (p.255). Then we obtain

$$G_{2,j}(x,t)=\frac{i}{\sqrt{2\pi t\lambda(\sigma_j)}}e^{t\lambda(\sigma_j)+ix\sigma_j}\left[1+O\left(t^{-1}\right)\right],\quad j=1,2,...,\nu_2, \tag{27}$$

uniformly with respect to $x\in R_1$.

Integrating in $G_{3,k}(x,t)$ by parts l times and considering that the terms outside the integral will be zeros by virtue of properties of $\varphi_{3,k}(\sigma)$, and estimating by modulus, we get

$$\left|G_{3,k}(x,t)\right|\le\frac{C\left(1+|x|\right)^l}{t^l}e^{t\tilde{c}}. \tag{28}$$

Putting the asymptotics $G_{\pm R}(x,t), G_{1,\mu}(x,t), G_{2,j}(x,t)$ and $G_{3,k}(x,t)$ from (16), (21), (26), (27) in formula (9) we get the proof of theorem 1.

Remark. If $\lambda(s)$ has real multiple saddle-points σ_j of multiplicity k_j, then under other conditions of theorem 1 for the solution of the Cauchy problem (1)-(2) it holds the asymptotics

$$u(x,t)=B(k_0)\,t^{-\frac{1}{k_0+1}}e^{t\lambda(c_0)+ixc_0}\upsilon(c_0)+W(x,t)+O\left(t^{-\frac{2}{k_0+1}}\right)$$

uniformly with respect to x at each segment from R_1, where k_0 is the multiplicity order of the saddle-point $\sigma=c_0$, introducing the greatest contribution to the asymptotics $u(x,t)$ as $t\to+\infty$, and for $W(x,t)$ estimation (11) holds.

If $\lambda(s)$ has no real saddle-points, then the decrease order of the solution of the Cauchy problem as $t\to+\infty$ may be increased depending on decrease order of initial function at infinity. The next theorem follows from the proof of theorem 1.

Theorem 2. *Let $\lambda(s)$ have no real saddle points and the initial function satisfy the condition*

$$\left|\frac{d^\alpha}{dx^\alpha}\varphi(x)\right|\leq\frac{C}{(1+|x|)^{l+2+\varepsilon}},\ \alpha=0,1,2,$$

in case I $\alpha=0$, $l\geq 0$ is an integer, $\varepsilon>0$ is a sufficiently small number.

Then for the solution of Cauchy problem (1), (2) at any $t>0$ it holds the estimation

$$|u(x,t)|\leq\frac{C}{t^l}e^{\tilde{c}t}$$

uniformly with respect to x at each segment from R_1.

The author expresses his deep gratitude to his supervisor corr.-member of NAS of Azerbaijan, prof. B.A. Iskenderov for the statement of the problem and his help.

References

[1]. Poincare A. *Sur I'equilibre d'une masse fluide animee d'um mouvement de rotation.* Acta Math. 1985, v.7, pp.259-380.

[2]. Rossby C.G. *Relation between variation in the intensity of the zonal circulation of the atmosphere and the displacement of the semipermanent centers of action.* J.Marine Res. 1939, v.2, No1, pp.38-55.

[3]. Sobolev S.L. *On a new problem of mathematical physics.* Izv.AN SSSR. Ser.mat. 1954, v.18, No1, pp.3-50. (Russian)

[4]. Galpern S.A. *The Cauchy problem for general systems of linear partial equations.* Trudy Mosk.Mat.obshestva. 1960, v.9, pp.401-423. (Russian)

[5]. Kostyuchenko A.G., Eskin G. I. *The Cauchy problem for Sobolev-Galpern equations.* Trudy Mosk.Mat.Obchestva. 1961, v.10. pp.273-284. (Russian)

[6]. Iskenderov B.A. *Behaviour of the Cauchy problem as $t\to +\infty$ for Sobolev Galpern equation.* Issl.spec.voprosov theorii diff.uravneniy. Baku "Elm", 1974, pp.83-94. (Russian)

[7]. Wineberg B.R. *Asymptotic of Green function for Sobolev-Galpern equation.* Soviet Doklady. 1961, v.136, No5, pp.1015-1018. (Russian)

[8]. Gelphand I.M., Shilov G.E. *Distributions and operators on them.* Issue1. M.Fiz.Mat.giz. 1959. (Russian)

[9]. Federyuk M.V. *Asymptotics, integrals, series.* M., "Nauka", 1987. (Russian)

Faig B. Guseynov
Institute of Mathematics and Mechanics of NAS of Azerbaijan.
9, F.Agayev str., AZ1141, Baku, Azerbaijan.
Tel.: (99412) 439 47 20 (off.)

Received October 11, 2004; Revised December 20, 2004.
Translated by Nazirova S.H.