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FINDING THE EXTREMALS OF THE OPTIMAL APPROXIMATION ON THE SET DIFFERENT FROM PARALLELEPIPED

Abstract

The suitable formula for calculation of the best approximation is found and the best approximate function in approximation of the function of m groups variables by the sums of the functions depending on $m-1$ groups of variables at the boundary of multivariate parallelepiped is constructed. The formulae are also constructed in approximation on an arbitrary set containing the domain of this parallelepiped.

The problems of exact calculation and the more, finding the extremal function are very important in approximation theory and belong to hardly solved problems of this theory.

In the paper the convenient formula for calculation of the best approximation was found and the best approximation function in approximation of the function of m groups of variables by the sums functions depending on $m-1$ groups on the boundary of multivariate parallelepiped is constructed. Such formulae are constructed in approximation on arbitrary set containing the boundary of this parallelepiped. These problems were previously solved by the author in approximation on full multivariate parallelepiped [1], and in case of functions of two variables on the boundary of rectangle with additional limitation on approximated function [2].

Let $\Pi = \Pi(a,h) = \left\{x \in R^n / a_i \leq x_i \leq a_i + h_i,\ i = \overline{1,n}\right\}$ be n-dimensional parallelepiped. Choosing the numbers $0 \leq k_0 < k_1 < ... < k_m = n$ denote $K = (k_0, k_1, ..., k_m)$, $|K| = m$. Consider the group the variables $t_j = \left(x_{k_{j-1}^{+1}}, ..., x_{k_j}\right)$, $j = \overline{1,m}$ and let $t = (t_1, ..., t_m)$. Further, denote by

$$D^m = \left\{\varepsilon = (\varepsilon_1, ..., \varepsilon_m),\quad \varepsilon_j = 0,1;\quad j = \overline{1,m}\right\}$$

the set of vertices of m-dimensional unit cube and let

$$\delta(\varepsilon) = \sum_{j=1}^{m} (1 - \varepsilon_j).$$

Consider the mapping

$$g(\xi,\tau) : D^m \to \Pi(\xi,\tau)$$

of the set D^m onto the set of vertices of n-dimensional parallelepiped $\Pi(\xi,\tau)$

$$g(\xi,\tau) = \left(\xi_1 + \varepsilon_1\tau_1, ..., \xi_{k_1} + \varepsilon_1\tau_{k_1}, ..., \xi_{k_{m-1}^{+1}} + \xi_{k_m} + \varepsilon_m\tau_{k_m}\right).$$

For arbitrary set $Q \subset \Pi(\xi,\tau)$ denote by $M_K = M_K(Q)$ the class of functions $f = f(x) : R^n \to R,\quad x \in Q$ satisfying the condition

$$\sum_{\varepsilon \in D}^{|K|} (-1)^{\delta(\varepsilon)} f(g(\xi,\tau)) \geq 0.$$

for an arbitrary parallelepiped $\Pi(\xi,\tau)$ with the vertices from Q. It is evident that if the set Q is with "meat", i.e., contains a part of positive n-dimensional measure then such parallelepipeds $\Pi(\xi,\tau)$ will be many, and these parallelepipeds may not belong to the set Q.

We'll sometimes use the notation $f(t)$, $f(t_1,...,t_m)$ instead of $f(x)$, $f(x_1,...,x_n)$ assuming that this will not lead to misunderstanding. We'll determine by

$$\Delta_{\delta_i} f(t) = \begin{cases} f(t\backslash t_i, t_i+\delta_i) - f(t), & \text{if the points } t \text{ and } (t\backslash t_i,\ t_i+\delta_i) \in Q \\ 0, & \text{if } t \text{ or } (t\backslash t_i, t_i+\delta_i) \notin Q, \end{cases}$$

the finite difference of the function f relative to the group of the variables $t_i = \left(x_{k_{i-1}^{+1}},...,x_{k_i}\right)$ with vector-step $\delta_i = \left(r_{k_{i-1}^{+1}},...,r_{k_i}\right)$ and the mixed difference f by the groups of the variables $t_1,...,t_k$ relative to the corresponding vector-steps $\delta_1,...,\delta_k$ we'll determine by

$$\Delta_{\delta_1...\delta_k} f(t) = \Delta_{\delta_k}\left(\Delta_{\delta_1...\delta_{k-1}} f(t)\right).$$

Introduce the notation

$$\begin{aligned} c_i &= \left(a_{k_{i-1}^{+1}},...,a_{k_i}\right), \qquad d_i = \left(a_{k_{i-1}^{+1}} + h_{k_{i-1}^{+1}},...,a_{k_i}+h_{k_i}\right); \\ \overline{m} &= \{1,...,m\}, \quad I_p = \{i_1,...,i_p\}, \quad J_q = \{j_1,...,j_q\}; \\ I_p, J_q &\subset \overline{m},\ I_p \cap J_q = \varnothing,\ \underline{pq} = \overline{m}\backslash(I_p \cup J_q);\ p,q = \overline{0,m}; \\ (a)_{I_p} &= \left(a_{i_1},...,a_{i_p}\right),\ (a+h)_{J_q} = \left(a_{j_1}+h_{j_1},...,a_{j_q}+h_{j_q}\right); \\ (a)_{I_0} &= (a+h)_{J_0} = \varnothing. \end{aligned}$$

Introduce the class $W_K = W_K(Q)$ of the functions f for all $x \in Q$ satisfying the conditions

$$\sum_{k=0}^{[m/2]} \sum_{I_{2k}\subset\overline{m}} \Delta_{(d_1-t_1)...(t_{i_1}-c_{i_1})...(t_{i_{2k}}-c_{i_{2k}})...(d_m-t_m)} f(t_1,...,c_{i_1},...,c_{i_{2k}},...,t_m) \geq 0 \quad (1)$$

$$\sum_{k=0}^{\left[\frac{m+1}{2}\right]} \sum_{I_{2k}\subset\overline{m}} \Delta_{(d_1-t_1)...(t_{i_1}-c_{i_1})...\left(t_{i_{2k-1}}-c_{i_{2k-1}}\right)...(d_m-t_m)} \times$$
$$\times f\left(t_1,...,c_{i_1},...,c_{i_{2k-1}},...,t_m\right) \geq 0 \quad (2)$$

here $[s]$ is an entire part of s, the following summand corresponds to the empty set I_0

$$\Delta_{(d_1-t_1)...(d_m-t_m)} f(t_1,...,t_m).$$

It is evident that in relations (1) and (2) involve only those points $t = (t_1,...,t_m)$ for which all points

$$(t\backslash t_{I_s}, c_{I_s}),\ (t\backslash t_{J_s},\ d_{J_s}),\quad s \in \overline{m}$$

taking part in considered differences belong to Q.

Consider the best approximation of the function f by the sums of functions of fewer numbers of the variables

$$N_K = \left\{\varphi/\varphi = \sum_{\nu=1}^{m} \varphi_\nu(t\backslash t_\nu)\right\}$$

on Q

$$E_f = E\left[f, N_K, Q\right] = \inf_{\varphi \in N_K} \sup_{x \in Q} \left|\left[f - \varphi\right](x)\right| = \inf_{\varphi \in N_K} \left\|f - \varphi\right\|_{M(Q)}$$

Denote by s the subset Q whose boundary s_0 represents the boundary of some n-dimensional parallelepiped $[x_1', x_1'', ..., x_n', x_n''] \subset Q$. Compare the set s with the quantity

$$L(f, s) = \Delta_{(t_1''-t_1')...(t_m''-t_m')} f\left(t'\right).$$

Consider the boundary of the parallelepiped $\Pi(a, h)$

$$\Pi^0(a, h) = \left\{ x \in R^n / \bigcup_{i=1}^{n} \left\{ \begin{array}{l} x_i = a_i,\ a_i + h;\ i \in \overline{n} \\ \\ a_j \le x_j \le a_j + h_j;\ j \in \overline{n} \backslash i \end{array} \right\} \right\}$$

and the subset $\Pi^0(a, h)$ corresponding to the division K:

$$\Pi_K^0(a, h) = \left\{ x \in R^n / \bigcup_{i=1}^{m} \left\{ \begin{array}{l} t_i = c_i, d_i;\ i \in \overline{m} \\ \\ c_j \le t_j \le d_j;\ j \in \overline{m} \backslash i \end{array} \right\} \right\}$$

where

$$t_i = c_i \iff x_{k_{i-1}^{+1}} = a_{k_{i-1}^{+1}}, ..., x_{k_i} = a_{k_i};$$

$$t_i = d_i \iff x_{k_{i-1}^{+1}} = a_{k_{i-1}^{+1}} + h_{k_{i-1}^{+1}}, ..., x_{k_i} = a_{k_i} + h_{k_i};$$

$$c_j \le t_j \le d_j \iff \left\{ \begin{array}{l} a_{k_{j-1}} \le x_{k_{j-1}^{+1}} \le a_{k_{j-1}^{+1}} + h_{k_{j-1}^{+1}}, \\ \\ , ...,\ \ a_{k_j} \le x_{k_j} \le a_{k_j} + h_{k_j} \end{array} \right.$$

Theorem 1. *The best approximation of the function* $f \in W_K\left(\Pi_K^0\right)$ *may be calculated by the formula*

$$E\left[f, N_K,\ \Pi_K^0\right] = 2^{-m} L\left(f, \Pi_K^0\right) = 2^{-m} \Delta_{(d_1-c_1)...(d_m-c_m)} f(c). \tag{3}$$

We'll need the auxiliary sentences for the proof of the theorem.

We'll conditionally denote by $T = [t_1',\ t_1'', ..., t_m', t_m'']$ the parallelepiped with the "sides" parallel to coordinate "planes" $t_\nu,\ \nu = \overline{1, m}$.

Lemma 1. *For arbitrary parallelepiped* $T \subset \Pi_K^0(a, h)$ *the functional* $L(f, T)$ *cancels each sum of functions depending on* $m - 1$ *groups of variables* $t_\nu,\ \nu = \overline{1, m}$.

Proof. Allowing for the linearity of the functional $L(f, T)$ we have:

$$L\left(\sum_{\nu=1}^{m} \varphi_\nu(t \backslash t_\nu), T\right) = \sum_{\nu=1}^{m} \left(L\left(\varphi_\nu(t \backslash t_\nu), T\right)\right) = \sum_{\nu=1}^{m} \Delta_{(t_1''-t_1'')...(t_m''-t_m')} \varphi_\nu(t \backslash t_\nu).$$

Using the known property of mixed finite difference we'll continue the process

$$= \sum_{\nu=1}^{m} \Delta_{t_\nu''-t_\nu'} \left[\Delta_{(t_1''-t_1')...(t_{\nu-1}''-t_{\nu-1}')(t_{\nu+1}''-t_{\nu+1}')...(t_m''-t_m')} \varphi_\nu(t \backslash t_\nu)\right] = 0,$$

since the expression in square bracket doesn't depend on a group of variables t_ν.

Lemma 1 is proved.

Lemma 2. *Let s^0 be a boundary of the parallelepiped s and be represented in the form $s^0 = \overset{k}{\underset{i=1}{U}} s_i^0$ where s_i^0 are boundaries of some parallelepipeds from s with sides parallel to coordinate "planes", in pairs not having common internal points.* Then

$$L\left(f, s^0\right) = \sum_{i=1}^{k} L\left(f, s_i^0\right).$$

This lemma follows from [1, lemma 3], since despite the fact that there the parallelepipeds were mentioned, and here we speak about their boundaries, in [1] and here the values of considered functionals are calculated only by the values of corresponding expressions on the boundaries of these parallelepipeds.

Determine the function

$$\Phi(t) = \sum_{p=0}^{m} (-1)^p \sum_{I_p \subset \overline{m}} \Delta_{(d_1-t_1)\ldots\left(t_{i_1}-c_{i_1}\right)\ldots\left(t_{i_p}-c_{i_p}\right)\ldots(d_m-t_m)} \times$$

$$\times f\left(t_1, ..., c_{i_1}, ..., c_{i_p}, ..., t_m\right). \tag{4}$$

Lemma 3.

$$f \in W_K\left(\Pi_K^0\right) \Longrightarrow \|\Phi(t)\|_{M(\Pi^0(a,h))} = \Delta_{(d_1-c_1)\ldots(d_m-c_m)} f(c).$$

Proof. From the determination of the function Φ we can write

$$\Phi(t) = \sum_{k=0}^{\left[\frac{m}{2}\right]} \sum_{I_{2k} \subset \overline{m}} \Delta_{(d_1-t_1)\ldots\left(t_{i_1}-c_{i_1}\right)\ldots\left(t_{i_{2k}}-c_{i_{2k}}\right)\ldots(d_m-t_m)} f\left(t_1, ..., c_{i_1}, ..., c_{i_{2k}}, ..., t_m\right) -$$

$$-\sum_{k=0}^{\left[\frac{m+1}{2}\right]} \sum_{I_{2k-1} \subset \overline{m}} \Delta_{(d_1-t_1)\ldots\left(t_{i_1}-c_{i_1}\right)\ldots\left(t_{i_{2k-1}}-c_{i_{2k-1}}\right)\ldots(d_m-t_m)} f\left(t_1, ..., c_{i_1}, ..., c_{i_{2k-1}}, ..., t_m\right)$$

Introduce the notation

$$\Delta_{(d_1-c_1)\ldots(d_m-c_m)} f(c) = \varepsilon, \tag{5}$$

$$\sum_{k=0}^{\left[\frac{m}{2}\right]} \Delta_{(d_1-t_1)\ldots\left(t_{i_1}-c_{i_1}\right)\ldots\left(t_{i_{2k}}-c_{i_{2k}}\right)\ldots(d_m-t_m)} \times$$

$$\times f\left(t_1, ..., c_{i_1}, ..., c_{i_{2k}}, ..., t_m\right) = \sum\nolimits_e^{(t)} \Delta f \tag{6}$$

$$\sum_{k=0}^{\left[\frac{m+1}{2}\right]} \Delta_{(d_1-t_1)\ldots\left(t_{i_1}-c_{i_1}\right)\ldots\left(t_{i_{2k-1}}-c_{i_{2k-1}}\right)\ldots(d_m-t_m)} \times$$

$$f\left(t_1, ..., c_{i_1}, ..., c_{i_{2k-1}}, ..., t_m\right) = \sum\nolimits_0^{(t)} \Delta f. \tag{7}$$

Using these notation we can write the function Φ in the following form

$$\Phi(t) = \sum\nolimits_e^{(t)} \Delta f - \sum\nolimits_0^{(t)} \Delta f \tag{8}$$

It is easy to note that for each fixed $t \in \Pi_K^0(a,h)$ the boundaries of parallelepipeds participating in (8), have no common internal points and their unification gives $\Pi_K^0(a,h)$. Therefore and according to lemma 2 we have

$$\varepsilon = \sum\nolimits_e^{(t)} \Delta f + \sum\nolimits_0^{(t)} \Delta f. \tag{9}$$

Using equality (9) and notation (6) and (7) in (8) we'll obtain

$$\Phi(t) = \varepsilon - 2\sum\nolimits_0^{(t)} \Delta f \ ,$$

whence by virtue of determination of the class

$$\Phi(t) - \varepsilon = -2H_0^{(t)} \Delta f \le 0$$

or

$$\Phi(t) \le \varepsilon. \tag{10}$$

Besides, from these relations we obtain

$$\Phi(t) + \varepsilon = 2\sum\nolimits_e^{(t)} \Delta f \ge 0$$

$$\Phi(t) \ge -\varepsilon. \tag{11}$$

Combining relations (10) and (11) in double inequality we'll obtain

$$-\varepsilon \le \Phi(t) \le \varepsilon.$$

Show that the function $\Phi(t)$ reaches the boundaries of these inequalities. We have

$$\Phi(d) = \sum_{p=0}^{m} (-1)^p \sum_{I_p \subset \overline{m}} \Delta_{(d_1-t_1)\ldots(t_{i_1}-c_{i_1})\ldots(t_{i_p}-c_{i_p})\ldots(d_m-t_m)} \times$$

$$\times f\left(d_1,\ldots,c_{i_1},\ldots,c_{i_p},\ldots,d_m\right) = (-1)^m \Delta_{(d_1-c_1)\ldots(d_m-c_m)} f(c).$$

Here the expression at the right-hand side of the equality corresponds to the case $I_p = \overline{m}$, and at $I_k \ne \overline{m}$ in each addend of finite difference in determination Φ there will be even one vector- increment equal to zero that will turn this finite difference in zero. Further, by this reason in case of $I_{\bar{p}} = (2,\ldots,m)$ we'll obtain

$$\Phi(c_1,d_2,\ldots,d_m) = (-1)^{m-1} \Delta_{(d_1-c_1)\ldots(d_m-c_m)} f(c).$$

Thus, regardless the evenness m the values $\Phi(d_1,\ldots,d_m)$ and $\Phi(c_1,d_2,\ldots,d_m)$ by virtue of notation (5) reach the numbers ε and $-\varepsilon$.

Whence it follows the assertion of lemma 3, i.e.,

$$\|\Phi(t)\|_{M\left(\Pi_K^0\right)} = \Delta_{(d_1-c_1)\ldots(d_m-c_m)} f(c).$$

Lemma 3 is proved.

Proof of theorem 1. Considering that according to lemma 1 the mixed difference $\Delta_{\delta_1...\delta_m}$ is an annihilator of the arbitrary sum $\sum\limits_{\nu=1}^{m}\varphi_\nu\left(t\backslash t_\nu\right)$ and in addition it is linear we have

$$\Delta_{\delta_1...\delta_m}f\left(t\right)=\Delta_{\delta_1...\delta_m}f\left(t\right)-\Delta_{\delta_1...\delta_m}\sum_{\nu=1}^{m}\varphi_\nu\left(t\backslash t_\nu\right)=$$

$$=\Delta_{\delta_1...\delta_m}\left[f\left(t\right)-\sum_{\nu=1}^{m}\varphi_\nu\left(t\backslash t_\nu\right)\right]\leq 2^m\left\|f\left(t\right)-\sum_{\nu=1}^{m}\varphi_\nu\left(t\backslash t_\nu\right)\right\|_{M\left(\Pi_K^0\right)}$$

Consequently,

$$\Delta_{\delta_1...\delta_m}f\left(t\right)\leq 2^mE\left[f,N_K,\Pi_K^0\right] \tag{12}$$

Consider the expression of the function $\Phi\left(t\right)$ from (1). At $p=0$ at the right-hand side we obtain the addend

$$\Delta_{(d_1-t_1)...(d_m-t_m)}f\left(t_1,...,t_m\right),$$

where the function $f\left(t\right)=f\left(t_1,...,t_m\right)$ takes part as $\left(-1\right)^mf\left(t\right)$. At $p=1$ we obtain m addends of the form

$$\Delta_{(d_1-t_2)...\left(t_{i_1}-c_{i_1}\right)...(d_m-t_m)}f\left(t_1,...,c_{i_1},...,t_m\right),\quad i=\overline{1,m}$$

(for example at $i_0=1$ these addend will be

$$\Delta_{(t_1-c_1)...(d_2-t_2)...(d_m-t_m)}f\left(c_1,t_2,...,t_m\right))$$

Each of these addends will contain the function $f\left(t\right)$ in the form

$$\left(-1\right)^{m-1}\left(-1\right)f\left(t_1,...,t_m\right)=\left(-1\right)^mf\left(t\right).$$

Thus at $p=1$ we'll have only $\left(-1\right)^mmf\left(t\right)$ expressions containing $f\left(t\right)$

At $p=2$ they will be

$$\left(-1\right)^{m-2}\left(-1\right)^2c_m^2f\left(t_1,...,t_m\right)=\left(-1\right)^mc_m^2f\left(t\right)$$

Further assuming $p=3,...,m$ we'll obtain that in ghe expression of the function $\Phi\left(t\right)$ they will be

$$\left(-1\right)^m\sum_{k=0}^{m}c_m^kf\left(t\right)=\left(-1\right)^m2^mf\left(t\right).$$

Since all other finite differences in the expression of the function $\Phi\left(t\right)$ from (4) contain the functions depending at most on $m-3$ groups of variables $t_\nu,\ \nu=\overline{1,m}$, then this allows to assert that the function $\Phi\left(t\right)$ has the form

$$\Phi\left(t\right)=\left(-1\right)^mf\left(t\right)-\sum_{\nu=1}^{m}\varphi_\nu^*\left(t\backslash t_\nu\right)$$

Then by virtue of lemma 3 and (12) we can write

$$\|\Phi(t)\|_{M\left(\Pi_K^0(a,h)\right)} = \left\|2^m f(t) - \sum_{\nu=1}^{m} \varphi_\nu^*(t \backslash t_\nu)\right\| \le 2^m E\left[f, N_K,\ \Pi_K^0(a,h)\right]$$

Or

$$\left\|f(t) - \sum_{\nu=1}^{m} \varphi_\nu^0(t \backslash t_\nu)\right\| \le 2^m E\left[f, N_K,\ \Pi_K^0\right].$$

But by virtue of definition of the best approximation here the sign of the strict inequality wouldn't be, consequently

$$\|\Phi\|_{M\left(\Pi_K^1\right)} = \Delta_{(d_1-c_4)\ldots(d_m-c_m)} f(c) = 2^m E\left[f, N_K, \Pi_K^0\right]$$

whence we finally obtain

$$E\left[f, N_K, \Pi_K^0\right] = 2^{-m} \Delta_{(d_1-c_1)\ldots(d_m-c_m)} f(c)$$

Theorem 1 is proved.

2. Denote by

$$f\left[(c)_{I_p}, (d)_{J_q}, t_{\underline{pq}}\right]$$

the values of the function f at the points whose coordinates consist of p groups $c_i,\ i \in I_p$, q groups $d_j,\ j \in J_q$ and $m-(p+q)$ groups $t_k,\ k \in pq$.

Theorem 2. *For the function $f \in W_K\left(\Pi_K^1\right)$ the sum*

$$\sum\nolimits_f^0 = \sum_{p+q=1}^{m} (-1)^{p+q+1} 2^{-(p+q)} \sum_{(I_p,I_q)\subset \overline{m}} f\left[(c)_{I_p}, (d)_{J_p}, t_{\underline{pq}}\right] \tag{13}$$

is the best approximating in the approximation

$$E\left[f, \sum_{\nu=1}^{m} \varphi_\nu(t \backslash t_\nu), \Pi_K^0\right].$$

The proof of theorem 2. At the proof of theorem 8 it was established that the function $f(t) = f(t_1, \ldots, t_m)$ in the expression of the function $\Phi(t)$ takes pari in the following form

$$(-1)^m 2^m f(t). \tag{14}$$

Let's now calculate the quantity

$$f_{pq} \overset{df}{=} f(c_1, \ldots, c_p, d_{p+1}, \ldots, d_{p+q}, t_{p+q+1}, \ldots, t_m) \tag{15}$$

taking part in the expression of the function $\Phi(t)$. To this end let's write the mixed finite difference in the form

$$\Delta_{(d_3-t_1)\ldots\left(t_{i_1}-c_{i_1}\right)\ldots\left(t_{i_p}-c_{i_p}\right)\ldots(d_m-t_m)} f\left(t_1, \ldots, c_{i_p}, \ldots, t_m\right) =$$

$$(-1)^m \sum_{s_1=5}^{1} ... \sum_{s_m=0}^{1} (-1)^{s_1+...+s_m} \cdot f\left(t_1 + (d_5 - t_1)\, s_1, ..., c_{i_1} + \right.$$

$$\left. + \left(t_{i_1} - c_{i_1}\right) s_{i_1}, ..., c_{i_p} + \left(t_{i_p} - c_{i_p}\right) s_{i_p}, ..., t_m + (d_m - t_m)\, s_m\right). \qquad (16)$$

Now using (4) and (16) we can express the function $\Phi(t)$ by the values of the function f in the following way

$$\Phi(t) = (-1)^m \sum_{r=7}^{m} (-1)^r \sum_{I_r \subset \bar{m}} \sum_{s_1=0}^{1} ... \sum_{s_m=1}^{1} (-1)^{s_1+...+s_m} \cdot$$

$$\cdot f\left(t_1 + (d_1 - t_1) = s_1, ..., c_{i_1} + \left(t_{i_1} - c_{i_1}\right) = \right.$$

$$\left. = s_{i_1}, ..., \left(t_{i_r} - c_{i_r}\right) = s_{i_r}, ..., t_m + (d_m - t_m) = s_m\right). \qquad (17)$$

In relation (17) at $r < p$ the expression

$$f\left(c_1, ..., c_p, d_{p+1}, ..., d_{p+q}, t_{p+q+1}, ..., t_m\right)$$

isn't contained, since the arguments c_i are obtained only from $c_i + (t_i - c_i)\, s_i$, moreover when $s_i = 0$. Assuming $s_i = 0$, $i = \overline{1,p}$; $s_j = 1$ at $r = p$ (then we obtain d_j) at $j = \overline{p+1, p+q}$ and $s_l = 0$ at $l = \overline{p+q+1, m}$ (then we obtain t_l) and we'll have the function (15) with the sign

$$(-1)^{m+p} (-1)^q = (-1)^{m+p+q}$$

Assume $s_i = 0$, $i = \overline{1,p}$ at $r = p+1$ and for getting (15) we must take $p+1$-th $s_i = 1$ beginning from the index $p+q+1$ till the m, and all other $s_\nu = 0$, $\nu = \overline{p+q+1, m}$ except this one, i.e., they will be c^1_{m-p-q} and they will be with the sign

$$(-1)^{m+p+1} (-1)^{q+1} = (-1)^{m+p+q}$$

So, at $r = p+1$ function (15) takes part in the expression of the function $\Phi(t)$ with the coefficient $(-1)^{m+p+q}\, c^1_{m-p-q}$.

Continuing analogously, we'll obtain that function (15) at $r = p+k$ takes part in the expression $\Phi(t)$ in the form

$$(-1)^{m+p+q}\, c^k_{m-p-q} f_{pq}.$$

From the aforesaid we obtain that function (15) takes part in the expression $\Phi(t)$ in the following form

$$(-1)^{m+p+q} \sum_{k=0}^{m-p-q} c^k_{m-p-q} f_{pq} = (-1)^{m+p+q}\, 2^{m-p-q} f_{pq}. \qquad (18)$$

Then as it is easy to observe using (14) and (18) we can write the function $\Phi(t)$ in the following form

$$\Phi(t) = (-1)^m\, 2^m f(t) + \sum_{p+q=1}^{m} (-1)^{m+p+q}\, 2^{m-(p+q)} \sum_{(I_p, J_q) \subset \bar{m}} f_{pq}$$

Therefore

$$\|\Phi(t)\| = 2^m \left\| f(t) - \sum_{p+q=1}^{m} (-1)^{p+q+1} 2^{-(p+q)} \sum_{(I_p,J_q)\subset \bar{m}} f\left[(c)_{I_p}, (d)_{J_q}, t_{\underline{pq}} \right] \right\|$$

and using lemma 3 we continue

$$= \Delta_{(d_1-c_1)\dots(d_m-c_m)} f(c)$$

Let's apply theorem 1 which leads to equalities

$$\left\| f - \sum\nolimits_f^0 \right\| = 2^{-m} \Delta_{(d_1-c_1)\dots(d_m-c_m)} f(c) = E\left[f, N_k, \Pi_k^0\right]$$

that mean that the sum

$$\sum\nolimits_f^0 = \sum_{p+q=1}^{m} (-1)^{p+q+1} 2^{-(p+q)} \sum_{(I_p,J_q)} f\left[(c)_{I_p}, (d)_{J_q}, t_{\underline{pq}} \right]$$

is the best approximating function in the approximation

$$E\left[f, N_K, \Pi_k^0\right].$$

Theorem 2 is proved.

3. Theorems 1 and 2 allow to find the extremals on the boundary of the domain, namely they give the accessible formula for calculation of the best approximation and allows to find the best approximating function in the approximation of the functions of many variables in the boundary $\Pi_K^0(a,h)$ of the $n-$dimensional parallelepiped $\Pi_K(a,h)$.

Let's consider now an arbitrary subset Q of the parallelepiped $\Pi_K(a,h)$ containing its boundary

$$\Pi_K^0(a,h) \subset Q \subset \Pi_K(a,h). \tag{19}$$

Each function $\varphi_\nu(t\backslash t_\nu)$, $\nu = \overline{1,m}$ is determined on the projection $\Pi_K(a,h)$ on the space $R_{n-|t_\nu|}(t\backslash t_\nu)$ ($|t_\nu|$ is the number of variables in the group).

Relation (19) allows to write

$$\left\| f - \sum_{\nu=1}^{m} \varphi_\nu \right\|_{M\left(\Pi_K^0\right)} \le \left\| f - \sum_{\nu=1}^{m} \varphi_\nu \right\|_{M(Q)} \le \left\| f - \sum_{\nu=1}^{m} \varphi_\nu \right\|_{M(\Pi_K)},$$

whence it follows that

$$\inf_{\Sigma\varphi_\nu} \left\| f - \sum_{\nu=1}^{m} \varphi_\nu \right\|_{M\left(\Pi_K^0\right)} \le \inf_{\Sigma\varphi_\nu} \left\| f - \sum_{\nu=1}^{m} \varphi_\nu \right\|_{M(Q)} \le \inf_{\Sigma\varphi_\nu} \left\| f - \sum_{\nu=1}^{m} \varphi_\nu \right\|_{M(\Pi_K)},$$

or

$$E\left[f, N_K, \Pi_K^0\right] \le E\left[f, N_K, Q\right] \le E\left[f, N_K, \Pi_K\right]. \tag{20}$$

In [1] it is established that value (3) and function (13) are extremals of the best approximation $E[f, N_K, \Pi_K]$, i.e., on whole parallelepiped $\Pi_K(a,h)$. Allowing for this and by virtue of theorems 1 and 2 from relations (20) we obtain that it is true the following theorem.

Theorem 3. *Let Q be an arbitrary subset $\Pi_K(a,h)$ containing its boundary. Then for each function $f \in W_K(\Pi)$ the basic extremals of the best approximation $E[f, N_K, Q]$ are determined by the following form: the best approximation is calculated by the following formula*

$$E[f, N_k, Q] = 2^{-m} \Delta_{(d_1-c_1)\ldots(d_m-c_m)} f(c),$$

and the sum

$$\sum\nolimits_f^0 = \sum_{p+q=1}^{m} (-1)^{p+q+1} 2^{-(p+q)} \sum_{(I_p,J_q)\subset\overline{m}} f\left[(c)_{I_p}, (d)_{J_q}, t_{\underline{pq}}\right]$$

is the best approximate function in this approximation.

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