MATHEMATICS

Adalat Sh. ABBASOV

ON CORRECT SOLVABILITY OF SOME BOUNDARY VALUE PROBLEMS FOR OPERATOR-DIFFERENTIAL EQUATIONS OF ODD ORDER

Abstract

In the paper the sufficient conditions providing correct solvability of some boundary value problems is obtained for differential equations of higher order whose main part contain a normal operator.

Let H be a separable Hilbert space, A be a normal inverse operator in H. Then the operator A has polar decomposition $A = UC$ where U is a unitary, and C is a positive definite operator in H, moreover $A = UC = CU$ and for any $x \in D(A)$: $\|Ax\| = \|A^*x\| = \|Cx\|$.

Denote by H_α the scale of Hilbert spaces generated by the operator C, i.e., $H_\alpha = D(C^\alpha), (x,y)_\alpha = (C_x^\alpha, C_y^\alpha)$, $x, y \in D(C^\alpha)$, $\alpha \geq 0$. Let $L_2(R_+; H)$ be Hilbert space of the vector-functions $f(t)$ determined in $R_+ = (0, \infty)$ almost everywhere with the values from H, G moreover

$$\|f\|_{L_2(R_+;H)} = \left(\int_0^\infty \|f(t)\|^2 dt \right)^{1/2}.$$

Further, determine the following Hilbert spaces [1].

$$W_2^n(R_+; H) = \left\{ u/u^{(n)} \in L_2(R_+; H),\ A^n u \in L_2(R_+; H) \right\},$$

$$\mathring{W}_2^n(R_+; H) = \left\{ u/u \in W_2^n(R_+; H), u(0) = ... = u^{m-1}(0) = 0 \right\},\ m < n.$$

Here and in future the derivatives are understood in terms of theory of distributions [1].

In future we'll assume that $n = 4k - 1$, and $m = 2k$ $(k = 1, 2, ...)$.

Consider the boundary value problem

$$\frac{d^n u(t)}{dt^n} - A^n u(t) + \sum_{j=1}^n A_j u^{(n-j)}(t) = f(t),\quad t \in R_+, \tag{1}$$

$$u(0) = u'(0) = ... = u^{(2k-1)}(0) = 0, \tag{2}$$

where relative to the coefficients of equation (1) the fulfilment of the following conditions are assumed:

1) A is the normal inverse operator with the spectrum in corner domain

$$S_\varepsilon = \{\lambda / |\arg \lambda| \leq \varepsilon\}, \quad 0 \leq \varepsilon < \pi/2n;$$

2) The operators $B_j = A_j A^{-j}$ $(j = \overline{1,n})$ are bounded in H.

Definition 1. *If vector-function $u(t) \in W_2^n(R_+; H)$ satisfies equation (1) almost everywhere in R_+, then we'll call it a regular solution of equation (1).*

Definition 2. *If at any $f(t) \in L_2(R_+; H)$ there exists the regular solution of equation (1) $u(t)$ which satisfies boundary conditions (2) in the sense*

$$\lim_{t\to+0} = \|u(t)\|_{n-j-1/2} = 0, \quad j = 0, 1, ..., 2k-1$$

and inequality

$$\|u\|_{W_2^n(R_+;H)} \leq const \|f\|_{L_2(R_+;H)},$$

then the problem (1), (2) is said to be regularly solvable.

In the present paper the conditions of regular solvability of boundary value problem (1), (2) are constructed. Note that when A is a self-adjoint operator then similar problems are considered in [2-5].

Denote by

$$P_0 u \equiv u^{(n)} - A^n u, \quad u \in \mathring{W}_2^n(R_+; H) \qquad (n = 4k-1)$$

$$P_1 u \equiv \sum_{j=1}^{n} A_j u^{(n-j)}(t), \quad u \in \mathring{W}_2^n(R_+; H)$$

It holds

Lemma 1. *Let condition (1) be fulfilled. Then the operator P_0 maps the space $\mathring{W}_2^n(R_+; H)$ onto $L_2(R_+; H)$ isomorphically.*

Proof. The general solution of the equation $P_0 u = 0$ from the space $W_2^n(R_+; H)$ is represented in the following form

$$u_0(t) = \sum_{\rho=1}^{2k} e^{\omega_\rho A t} \varphi_\rho,$$

where ω_ρ is a root from 1, moreover $\operatorname{Re}\omega_\rho < 0$ $(\rho = 1, 2, ..., 2k)$, $\varphi_\rho \in H_{n-1/2}$, and $e^{\omega_\rho A t}$ is a strongly continuous semigroup of bounded operators generated by the operator $\omega_\rho A$. From the condition $u \in \mathring{W}_2^n(R_+; H)$ $(u(0) = u'(0) = ... = u^{2k-1}(0) = 0)$ we obtain the system of equations relative to φ_ρ:

$$\sum_{\rho=1}^{2k} (\omega_\rho A)^j \varphi_\rho = 0, \quad j = 0, 1, ..., k-1.$$

Hence it follows that

$$\sum_{\rho=1}^{2k}\omega_\rho^j\varphi_\rho=0,\quad j=0,1,...,k-1.$$

Consequently $\varphi_\rho=0$, i.e., the operator P_0 vanishes only at zero.

Show that the equation $P_0u=f$ has solution $u\in\overset{\circ}{W}{}_2^n(R_+;H)$ at any $f\in L_2(R_+;H)$. It is easy to check that

$$u_1(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}((-i\xi)^nE+A^n)^{-1}\left(\int_0^\infty f(s)e^{-i\xi s}ds\right)e^{i\xi t}dt,\quad t\in R$$

satisfies the equation $P_0u=f$ almost everywhere in R_+. Show that $u_1(t)\in W_2^n(R;H)$. For this by the Plansherel theorem it is sufficient to show that $A^n\hat{u}_1(\xi)L_2(R;H)$, $\xi^n\hat{u}_1(\xi)\in L_2(R;H)$ where $\hat{u}_1(\xi)$ is a Fourier transformation of the vector-functions $u_1(t)$.

It is evident that

$$\|A^n\hat{u}_1(\xi)\|_{L_2}=\left\|(A^n((-i\xi)^nE+A^n)^{-1}\hat{f}(\xi)\right\|_{L_2}\le$$

$$\le\sup_{\xi\in R}\left\|A^n((-i\xi)^nE+A^n)^{-1}\right\|\cdot\left\|\hat{f}(\xi)\right\|_{L_2}$$

Since at any $\xi\in R$

$$\left\|A^n((-i\xi)^nE+A)^{-1}\right\|\le\sup_{\mu\in\sigma(A)}\left|\mu^n(-i\xi^n+\mu^n)^{-1}\right|=$$

$$=\sup_{\mu\in\sigma(A)}\left|\mu^n\left(\xi^{2n}+\mu^{2n}-2\xi^n\mu^n\sin n\varphi\right)^{-1/2}\right|\le$$

$$\le\sup_{\mu\in\sigma(A)}\left|\mu^n\left(\xi^{2n}+|\mu|^{2n}-2|\xi|^n|\mu|^n\sin n\varepsilon\right)^{-1/2}\right|\le$$

$$\le\sup_{\mu\in\sigma(A)}\left||\mu|^n\left(\xi^{2n}+|\mu|^{2n}-(\xi^{2n}+\mu^{2n}\sin^2 n\varepsilon)\right)^{-1/2}\right|\le\frac{1}{\cos n\varepsilon}.$$

Thus,

$$\|A^nu_1(\xi)\|_{L_2}=\frac{1}{\cos n\varepsilon}\|f\|_{L_2},$$

i.e. $A^n\hat{u}_1(\xi)\in L_2$. Analogously we obtain that $\xi^n\hat{u}_1(\xi)\in L_2$. Thus $u_1(t)\in W_2^n(R;H)$. Denote by $\bar{u}_1(t)$ the contraction of the vector-function $u_1(t)$ on $R_+=[0,\infty)$. Then $\bar{u}_1(t)\in W_2^n(R_+;H)$ and $\bar{u}_1^{(j)}(0)\in H_{n-j-1/2}$ We'll check the solution of the equation $P_0u=f$ in the following form

$$u(t)=u_0(t)+\bar{u}_1(t)\equiv\sum_{\rho=1}^{2k}e^{\omega_\rho tA}\varphi_\rho+\bar{u}_1(t),$$

where $\varphi_\rho \in H_{n-1/2}$ are the unknown vectors. From the condition $u \in \mathring{W}_2^n(R_+;H)$ for the determination φ_ρ we obtain the system of equations:

$$\sum_{\rho=1}^{2k} (\omega_\rho A)^j \varphi_\rho = \bar{u}_1^{(j)}(0), \qquad j=0,...,2k-1$$

or

$$\sum_{\rho=1}^{2k} (\omega_\rho A)^j \varphi_\rho = -A^{-j}\bar{u}_1^{(j)}(0).$$

Hence φ_ρ are determined uniquely, and from the condition $\bar{u}_1^{(j)}(0) \in H_{n-j-1/2}$ it follows that $\varphi_\rho \in H_{n-1/2}$. Consequently $u(t) \in \mathring{W}_2^n(R_+;H)$ and it is a solution of the equation $P_0 u = f$.

Since

$$\|P_0 u\| = \left\| \frac{d^m u}{dt^n} - A^n u \right\| \le \sqrt{2}\,\|u\|_{W_2^n},$$

then by Banach theorem on the inverse operator P_0 maps the space $W_2^n(R_+;H)$ onto $L_2(R_+;H)$ isomorphically.

Lemma 2. *Let condition 1) be fulfilled. Then for any $u \in \mathring{W}_2^n(R_+;H)$ it holds the inequality*

$$\|P_0 u\|_{L_2}^2 \ge \|u\|_{W_2^n}^2 - 2\sin n\varepsilon \left\|u^{(n)}\right\|_{L_2} \cdot \|A^n u\|_{L_2}.$$

Proof. Let $u \in \mathring{W}_2^n(R_+;H)$. Then

$$\|P_0 u\|_{L_2}^2 = \|u^n - A^n u\|_{L_2}^2 = \|u\|_{W_2^n}^2 - 2\operatorname{Re}\left(u^{(n)}, A^n u\right)_{L_2}. \tag{3}$$

Integrating by parts we have

$$\left(u^{(n)},\, A^n u\right)_{L_2} = -\left((A^*)^n u, u^{(n)}\right)_{L_2}.$$

Consequently

$$\left|2\operatorname{Re}\left(u^{(n)}, A^n u\right)_{L_2}\right| = \left|\left(\left(A^n - (A^*)\right)^n u, u^{(n)}\right)_{L_2}\right| =$$

$$= \left|\left(\left(E - (A^* A^{-1})^n\right) A^n u, u^{(n)}\right)_{L_2}\right| \le$$

$$\le \left\|E - (A^* A^{-1})^n\right\| \cdot \|A^n u\|_{L_2} \cdot \left\|u^{(n)}\right\|_{L_2}.$$

Since

$$\left\|E - (A^* A^{-1})^n\right\|_{L_2} \le \sup_{\mu \in \sigma(A)} \left|1 - \left(\bar{\mu}\mu^{-1}\right)^n\right| \le \sup_{|\varphi|<\varepsilon} \left|e^{in\varphi} - e^{-in\varphi}\right| \le 2\sin n\varepsilon,$$

then

$$2\operatorname{Re}\left(u^{(n)}, A^n u\right)_{L_2} \le 2\sin n\varepsilon\, \|A^n u\|_{L_2} \cdot \left\|u^{(n)} u\right\|_{L_2}.$$

Allowing for inequality in (3) we obtain the assertion of the lemma.

Lemma 3. *Let condition 1) be fulfilled. Then at any* $u \in \mathring{W}_2^n(R_+;H)$ *it holds the inequality*

$$\left\| A^{n-j}u^{(j)} \right\|_{L_2} \le c_j(\varepsilon) \left\| P_0 u \right\|_{L_2}, \qquad j=0,1,...,n, \tag{4}$$

where $c_0(\varepsilon) = c_n(\varepsilon) = (\cos n\varepsilon)^{-1}$,, and at $j=1,...,n-1$

$$c_j(\varepsilon) = (1-\sin n\varepsilon)^{1/2} c_j \left(\frac{n-j}{n}\right)^{\frac{n-j}{2n}} \left(\frac{j}{n}\right)^{\frac{j}{2n}}, \tag{5}$$

moreover

$$c_j = \begin{cases} 2^{\frac{j}{2n}(2k-1)(2k-2)}, & j=1,...,2k \\ 2^{\frac{n-j}{2n}(nj-2k(2k+1))}, & j=2k+1,...,n-1 \end{cases} \tag{6}$$

Proof. From lemma (2) it follows that

$$\|P_0u\|_{L_2}^2 \ge \left\|u^{(n)}\right\|_{L_2}^2 + \|A^n u\|_{L_2}^2 - \left(\left\|u^{(n)}\right\|_{L_2}^2 + \sin^2 n\varepsilon \|A^n u\|_{L_2}^2 \right) = \cos^2 n\varepsilon \|A^n u\|_{L_2}^2,$$

i.e.,

$$\|A^n u\|_{L_2} \le (\cos n\varepsilon)^{-1} \|P_0 u\|_{L_2}.$$

We analogously obtain that

$$\left\|u^{(n)}\right\|_{L_2} \le (\cos n\varepsilon)^{-1} \|P_0 u\|_{L_2}.$$

Prove the other inequalities. After the integrating by parts we obtain

$$\left\|C^{n-j}u^{(j)}\right\|_{L_2}^2 \le \left\|C^{n-(j-1)}u^{(j-1)}\right\|_{L_2} \cdot \left\|C^{n-(j+1)}u^{(j+1)}\right\|_{L_2}, \qquad j=1,...,2k,$$

and from equality

$$\left\|\lambda C^{n-p+1}u^{(p-1)} + C^{n-p}u^{(p)} + \frac{1}{\lambda}C^{n-p-1}u^{(p-1)}\right\| = \frac{1}{\lambda^2}\left\|C^{n-p-1}u^{p-1}\right\|_{L_2}^2 +$$

$$+\lambda^2 \left\|C^{n-p+1}u^{(p-1)}\right\|_{L_2}^2 - \left\|C^{n-p}u^{(p)}\right\|_{L_2}^2 -$$

$$-\left\|\sqrt{\lambda}C^{n-p+\frac{1}{2}}u^{(p-1)}(0) + \frac{1}{\sqrt{\lambda}}C^{n-p-\frac{1}{2}}u^{(p)}(0)\right\|_{L_2}^2$$

at $\lambda = \left\|C^{n-p+1}u^{(p-1)}\right\|_{L_2}^{-1/2} \cdot \left\|C^{n-p-1}u^{(p-1)}\right\|_{L_2}^{1/2}$ it follows that

$$\left\|C^{n-j}u^{(j)}\right\|_{L_2}^2 \le 2\left\|C^{n-(j-1)}u^{(j-1)}\right\|_{L_2} \cdot \left\|C^{n-(j+1)}u^{(j+1)}\right\|_{L_2}, \quad j=2k+1,...,n.$$

Thus we have the following system of equations

$$\left\|A^{n-1}u'\right\|_{L_2}^2 \le \left\|A^n u\right\|_{L_2} \cdot \left\|A^{n-2}u''\right\|_{L_2},$$

$$\left\|A^{n-2}u''\right\|_{L_2}^2 \le \left\|A^{n-1}u'\right\|_{L_2} \cdot \left\|A^{n-3}u'''\right\|_{L_2},$$

....

$$\left\|A^{2k-1}u^{(2k)}\right\|_{L_2}^2 \le \left\|A^{2k}u^{(2k-1)}\right\|_{L_2} \cdot \left\|A^{2k-2}u^{(2k+1)}\right\|_{L_2},$$

$$\left\|A^{2k-2}u^{(2k+1)}\right\|_{L_2}^2 \le 2\left\|A^{2k-1}u^{(2k)}\right\|_{L_2} \cdot \left\|A^{2k-3}u^{(2k+2)}\right\|_{L_2},$$

....

$$\left\|Au^{(n-1)}\right\|_{L_2}^2 \le \left\|A^2 u^{(n-2)}\right\|_{L_2} \cdot \left\|A^n u\right\|_{L_2}.$$

Hence assuming $p_1 = \frac{n-j}{n}$, $p_2 = 2\frac{n-j}{n}, ..., p_j = j\frac{n-j}{n}$, $p_{j+1} = (n-j-1)\frac{j}{n}$, $p_{j+2} = (n-j-2)\frac{j}{n}, ...p_{n-1} = \frac{j}{n}$ we obtain:

$$\left\|A^{n-1}u'\right\|_{L_2}^{2p_1} \cdot \left\|A^{n-2}u''\right\|_{L_2}^{2p_2} \cdot ... \cdot \left\|A^{2k-1}u^{(2k)}\right\|_{L_2}^{2p_{2k}} \times$$

$$\times \left\|A^{2k}u^{(2k+1)}\right\|_{L_2}^{2p_{2k+1}} \cdot ... \cdot \left\|Au^{(n-1)}\right\|_{L_2}^{2p_{n-1}} \le$$

$$\le 2^{p_{2k+1}+...+p_{n-1}} \left\|A^n u\right\|_{L_2}^{p_1} \cdot \left\|A^{n-1}u'\right\|_{L_2}^{p_2} \cdot ... \cdot \left\|A^{2k-1}u^{(2k)}\right\|_{L_2}^{p_{2k-1}} \cdot$$

$$\cdot \left\|A^{2k-2}u^{(2k+1)}\right\|_{L_2}^{p_{2k}} \cdot ... \cdot \left\|Au^{(n-1)}\right\|_{L_2}^{p_{n-2}} \cdot \left\|u^{(n)}\right\|_{L_2}^{p_{n-1}},$$

or

$$\left\|A^{n-j}u^{(j)}\right\|_{L_2}^{2p_j - p_{j-1} - p_{j+1}} \le 2^{p_{2k+1}+...+p_{n-1}} \left\|A^n u\right\|_{L_2}^{\frac{n-j}{n}} \cdot \left\|u^{(n)}\right\|_{L_2}^{\frac{j}{n}}.$$

Consequently

$$\left\|A^{n-j}u^{(j)}\right\|_{L_2} \le c_j^2 \left\|A^n u\right\|_{L_2}^{2\frac{n-j}{n}} \cdot \left\|u^{(n)}\right\|_{L_2}^{2\frac{j}{n}} \le$$

$$c_j^2 \left(\frac{n-j}{n}\right)^{\frac{n-j}{n}} \left(\frac{j}{n}\right)^{\frac{j}{n}} \left(\left\|A^n u\right\|_{L_2}^2 + \left\|u^{(n)}\right\|_{L_2}^2\right), \tag{7}$$

where the numbers c_j $(j = 0, ..., n)$ are determined from equality (6). Further, from lemma 2 by applying the Cauchy theorem we obtain

$$\left\|P_0 u\right\|_{L_2}^2 \ge (1 - \sin n\varepsilon) \left\|u\right\|_{W_2^n}^2. \tag{8}$$

Allowing for inequalities (8) in (7) we complete the proof of the lemma.

Theorem. *Let conditions 1) and 2) be fulfilled, moreover*

$$\alpha(\varepsilon)=\sum_{j=1}^{n} C_{n-j}(\varepsilon)\|B_j\|<1,$$

where the coefficients $C_j(\varepsilon)$ are determined from lemma 3. Then problem (1), (2) is regularly solvable.

Proof. Write problem (1), (2) in the form $Pu=P_0u+P_1u=f$, $u\in \overset{\circ}{W}{}_2^n(R_+;H)$, $f\in L_2(R_+;H)$. After substitution $\vartheta(t)=P_0u(t)$ we obtain the equation $\vartheta+P_1P_0^{-1}\vartheta=f$ in the space $L_2(R_+;H)$.

Then for any $\vartheta\in L_2(R_+;H)$ we have:

$$\left\|P_1P_0^{-1}\vartheta\right\|_{L_2}=\|P_1u\|_{L_2}\leq\sum_{j=1}^{n}\|B_{n-j}\|\left\|A^{n-j}u^{(j)}\right\|_{L_2}\leq$$

$$\leq\sum_{j=1}^{n}c_j(\varepsilon)\|B_{n-j}\|\left\|P_0^{n-j}u\right\|_{L_2}=\alpha(\varepsilon)\|\vartheta\|_{L_2}.$$

Since by the condition of the theorem $\alpha(\varepsilon)<1$, then $E+P_1P_0^{-1}$ turn into $L_2(R_+;H)$ and consequently

$$u=P_0^{-1}\left(E+P_1P_0^{-1}\right)^{-1}f.$$

The theorem is proved.

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Adalat Sh.Abbasov

Symgayit branch of Azerbaijan Teacher's Institute.

16, apt.38^a, AZ3200, Symgayit, Azerbaijan.

Tel.: (050) 201 70 99 (mob.)

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