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ON OPTIMAL CONTROL FOR PROCESSES DESCRIBED BY QUASILINEAR ELLIPTIC EQUATIONS

Abstract

In the present paper the optimal control problem for quasilinear elliptic equations with nonlinear optimality criterion of the general form is considered. The coefficients at higher derivatives and the function included to the nonlinear part of the equation are the controlling functions. The correctness problems of the statement of the considered problem are investigated, the sufficient conditions for differentiability of optimality criterion are found, the formula for its gradient is obtained, the necessary optimality condition in the form of variational inequality is established.

In the papers [1-5] and others the existence of optimal control and derivation of necessary optimality conditions for the optimal control problems by coefficients of linear elliptic equations are investigated.

In the present paper the optimal control problem for quasilinear elliptic equations with nonlinear optimality criterion of the general form is considered. The coefficients at higher derivatives and the function included to the nonlinear part of the equation are the controlling functions. The correctness problem of the statement of the considered problem are investigated, the sufficient conditions for differentiability of optimality criterion are found, the formula for its gradient is obtained, the necessary optimality condition in the form of variational inequality is established.

1. Statement of the problem. Let the domain Ω of the Euclidean space R^3 be a ball, ball layer, parallelepiped or can be transformed to one of these domains with the help of regular transformation from $C^2\left(\overline{\Omega}\right)$, Γ be a boundary of the domain Ω which is supposed to be continuous by Lipshitz, $x=(x_1,x_2,x_3)$ be an arbitrary point of the domain Ω. The functional spaces used in the present paper are defined for example, in [6, p.25-30]. Besides, everywhere below M denotes positive constants which are independent of feasible controls and estimated quantities.

Let's consider the controlled process described by the quasilinear equation

$$-\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left(k_i\left(x\right)\frac{\partial u}{\partial x_i}\right)+f\left(x,u,q\right)=f_0\left(x\right),\quad x\in\Omega \tag{1}$$

with the boundary condition

$$u\left(x;\vartheta\right)=0,\quad x\in\Gamma, \tag{2}$$

where $\vartheta=(k_1,k_2,k_3,q):\Omega\rightarrow R^4$ is controlling action, $u=u\left(x;\vartheta\right)$ is a state function corresponding to the control ϑ; $f\in L_2\left(\Omega\right)$ is a given function; $f\left(x,u,q\right)$ is a function satisfying the Karateodori condition in the domain $\Omega\times R\times G$, i.e. at all $u\in R,\ q\in G$, it is measurable by $x\in\Omega$ and is continuous almost at all $x\in\Omega$ by $u\in R,\ q\in G$, $G=\left\{\ q\in G: q_0\leq q\leq q_1\right\}$, $0\leq q_0<q_1$ are given numbers.

In the space $B=\prod\limits_{i=1}^{3}W_{m_i}^{1}(\Omega)\times L_2(\Omega)$ we define the following set of feasible controls

$$V=\prod_{i=1}^{3}V_i\times Q,$$

$$V_i=\Big\{\ k_i=k_i(x):k_i\in W_{m_i}^{1}(\Omega),\ \ 0<\nu_i\le k_i(x)\le\mu_i,$$
$$\overset{\circ}{\forall}x\in\Omega,\ \left\|\frac{\partial k_i}{\partial x_j}\right\|_{L_{m_i}(\Omega)}\le d_j^{(i)}\quad\left(j=\overline{1,3}\right)\ \Big\},\quad i=\overline{1,3},\tag{3}$$

$$Q=\left\{q=q(x):q\in L_2(\Omega),\ q(x)\in G,\ \overset{\circ}{\forall}x\in\Omega\right\}.$$

Here $\nu_i,\mu_i,d_j^{(i)}>0,\ m_i>3\ \left(i,j=\overline{1,3}\right)$ are given numbers, the symbol $\overset{\circ}{\forall}$ denotes "almost for all".

On the set of feasible controls V we give the functional

$$J(\vartheta)=\int_{\Omega}F\left(x,u(x;\vartheta),\nabla u(x;\vartheta),\vartheta(x)\right)dx,\tag{4}$$

where $F(x,u,p,\vartheta)$ is a given function satisfying the Karateodori condition in the domain $\Omega\times R\times R^3\times U,$ where $p=(p_1,p_2,p_3)\,,\ U=K\times G,\ K=\prod\limits_{i=1}^{3}K_i,$ $K_i=\{k_i\in R:\nu_i\le k_i\le\mu_i\}\ \left(i=\overline{1,3}\right).$

Assume that the functions $f(x,u,q)\,,\ F(x,u,p,\vartheta)$ satisfy the following conditions

1) there exists a constant $L>0$ such that $0\le[f(x,u,q)-f(x,w,q)](u-w)\le L(u-w)^2.$

2) For each $h\in(0,\infty)$ there exists a function $a_h\in L_1(\Omega)$ such that

$$|F(x,u,p,\vartheta)|\le a_h(x)+c\,|p|^r\,,$$

$$\overset{\circ}{\forall}x\in\Omega,\ \forall u\in R,\ |u|\le h,\ p\in R^3,\ \vartheta\in U,$$

where $c>0,\ r\in[2,6)$ are some numbers, $|p|=\left(\sum\limits_{i=1}^{3}p_i^2\right)^{1/2}.$

Let's put the following optimal control problem: to minimize the functional $J(\vartheta)$ on the set V under conditions (1),(2). This problem below is said to be problem (1)-(4).

Everywhere below the conditions accepted in p.1 are assumed to be fulfilled.

2. The correctness of the statement of the problem. Under a solution of boundary-value problem (1),(2) at any fixed $\vartheta\in V$ we'll understand the function

$u = u(x,\vartheta)$ from $\overset{\circ}{W}{}_2^1(\Omega)$ satisfying the identity

$$\int_\Omega \left[\sum_{i=1}^3 k_i(x) \frac{\partial u}{\partial x_i}\frac{\partial \eta}{\partial x_i} + f(x,u,q(x))\,\eta - f_0(x)\,\eta\right] dx = 0, \tag{5}$$

for $\eta \in \overset{\circ}{W}{}_2^1(\Omega)$.

Theorem 1. *Let condition 1) be satisfied. Then for any fixed $\vartheta \in V$ there exists a unique solution of boundary value problem (1),(2) from $\overset{\circ}{W}{}_2^1(\Omega)$. Besides, the solution of boundary value problem (1),(2) belongs to the space $W_{2,0}^2(\Omega) = W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)$, satisfies equation (1) almost everywhere in Ω and the estimate*

$$\|u\|_{W_2^2(\Omega)} \le M\,\|f_0\|_{L_2(\Omega)} \tag{6}$$

is valid.

Proof. By virtue of condition 1) for each given $\vartheta \in V$ the operator

$$A(u) = -\sum_{i=1}^3 \frac{\partial}{\partial x_i}\left(k_i(x)\frac{\partial u}{\partial x_i}\right) + f(x,u,q(x)), \quad A:\ \overset{\circ}{W}{}_2^1(\Omega) \to W_2^{-1}(\Omega)$$

is bounded, semi-continuous, monotone and coercive. Therefore, according to theorem 2.1 §2, ch.2 from [7] problem (1),(2) has a solution from $\overset{\circ}{W}{}_2^1(\Omega)$. We can be easily convinced that for each $\vartheta \in V$ the solution of problem (1),(2) is unique.

Besides, from condition 1) it follows that

$$\|f\|_{L_2(\Omega)}^2 \equiv \int_\Omega |f(x,u(x;\vartheta),q(x))|^2\,dx =$$
$$= \int_\Omega |f(x,u(x;\vartheta),q(x)) - f(x,0,q(x))|^2\,dx \le L^2\,\|u\|_{L_2(\Omega)}^2 . \tag{7}$$

It follows from (7) that $\Phi(x) \equiv f_0(x) - f(x,u(x;\vartheta),q(x)) \in L_2(\Omega)$. Then rewriting problem (1), (2) in the form

$$\mathcal{L}u \equiv -\sum_{i=1}^3 \frac{\partial}{\partial x_i}\left(k_i(x)\frac{\partial u}{\partial x_i}\right) = \Phi(x), \quad x \in \Omega$$

$$u(x;\vartheta) = 0, \ \ x \in \Gamma$$

using the second basic inequality for elliptic operators [6, p.213-216] and estimate (7) we obtain

$$\|u\|_{W_2^2(\Omega)} \le M\,\|\mathcal{L}u\|_{L_2(\Omega)} \le M\left[\|f_0\|_{L_2(\Omega)} + \|f\|_{L_2(\Omega)}\right] \le$$
$$\le M\left[\|f_0\|_{L_2(\Omega)} + \|u\|_{L_2(\Omega)}\right]. \tag{8}$$

Thus, $u \in W_{2,0}^2(\Omega)$ and consequently, the function also satisfies equation (1) almost everywhere in Ω.

We now show that estimate (6) is valid. Assuming $\eta = u$ in identity (5) we obtain

$$\int\limits_{\Omega}\left[\sum_{i=1}^{3}k_i(x)\left(\frac{\partial u}{\partial x_i}\right)^2 + f(x,u,q)\,u\right]dx = \int\limits_{\Omega} f_0 u dx.$$

Hence using condition (1) and the Cauchy-Bunyakovskii inequality we have

$$\nu\|\nabla u\|^2_{L_2(\Omega)} \le \|f_0\|_{L_2(\Omega)}\|u\|_{L_2(\Omega)},$$

where $\nu = \min\limits_{1\le i\le 3}\nu_i$. Here allowing for the Friedrichs inequality [8, p.62] we obtain

$$\|u\|_{L_2(\Omega)} \le \|f_0\|_{L_2(\Omega)}. \qquad (9)$$

Then allowing for (9) in (8) we derive (6). Theorem 1 is proved.

Remark 1. Note that the analogue of theorem 1 for two-dimensional quasilinear elliptic equation in the case of $k_1(x)\equiv 1,\ k_2(x)\equiv 1$ is proved in the paper [9].

Remark 2. According to theorem 1for each given $\vartheta \in V$ the solution of boundary-value problem (1)-(2) is an element of the space $W^2_{2,0}(\Omega)$. It is known from [6, p.75] that for any element $u \in W^2_{2,0}(\Omega)$ the inclusions $u \in C(\overline{\Omega})$, $\dfrac{\partial u}{\partial x_i} \in L_r(\Omega)$, where $r \in [2,6)$ is any number, are valid. From the boundedness of the embedding $W^2_{2,0}(\Omega) \to C(\overline{\Omega})$ [6, p.77] and from estimate (6) it follows that for any $\vartheta \in V$ the inequality

$$\max_{x\in\overline{\Omega}}|u(x;\vartheta)| \le M$$

holds.

It follows [10, p.375] from these reasons and from condition (2) that the superposition operator $F(u)$ generated by the function $F(x,u(x);\vartheta)\ \ \nabla u((x;\vartheta),\ \vartheta(x))$ for each fixed $\vartheta \in V$ acts from $W^2_{2,0}(\Omega)$ to $L_1(\Omega)$. Consequently, functional (4) is defined on V and gets finite values.

Assume that the following conditions are satisfied

3) for each fixed function $u \in W^2_{2,0}(\Omega)$ the mapping $f(q): Q \to L_2(\Omega)$ generated by the functions $f(x,u(x),q(x))$ is weakly continuous on Q;

4) The mapping $F(u,\vartheta): W^2_{2,0}(\Omega)\times V \to L_1(\Omega)$ generated by the function $F(x,u(x),\nabla u(x),\vartheta(x))$ is completely continuous.

We now study the correctness problems of the statement of problem (1)-(4).

Theorem 2. *Let conditions (1)-(4) be satisfied. Then the functional* $J(\vartheta)$ *is weakly continuous on* V*, the set of optimal controls of problem (1)-(4)* $V_* = \{\vartheta_* \in V : J(\vartheta_*) = \inf\{J(\vartheta) : \vartheta \in V\}\}$ *is nonempty, weakly compact in* B*, and any minimizing sequence* $\left\{\vartheta^{(n)}\right\}$ *converges weakly in* B *to the set* V_*.

Proof. Let $\vartheta = (k_1,k_2,k_3,q) \in V$ be some element, $\{\vartheta = (k_1,k_2,k_3,q)\} \in V$ be an arbitrary sequence such that

$$\vartheta^{(n)} \to \vartheta \text{ weakly in } B \text{ as } n\to\infty. \qquad (10)$$

Let $u^{(n)} = u\left(x;\vartheta^{(n)}\right)$ be a solution of problem (1),(2) at $\vartheta = \vartheta^{(n)}$. It follows from estimate (6) that

$$\left\|u^{(n)}\right\|_{W^2_2(\Omega)} \le M \quad (m = 1,2,...). \qquad (11)$$

It is known [6, p.77] that the embedding $W_{2,0}^2(\Omega) \to \overset{\circ}{W}{}_2^1(\Omega)$ is compact, and the inclusions $W_{m_i}^1(\Omega) \to C(\overline{\Omega})$ are compact at $m_i > 3$. Therefore, by virtue of (10) and (11) we can extract from the sequence $\left\{\left(\vartheta^{(n)}, u^{(n)}\right)\right\}$ such subsequence which again we denote by $\left\{\left(\vartheta^{(n)}, u^{(n)}\right)\right\}$, that

$$\begin{array}{c} k_i^{(n)} \to k_i \text{ weakly in } W_{m_i}^1(\Omega) \\ \text{strongly in } C(\overline{\Omega}) \ \left(i=\overline{1,3}\right) \end{array} \tag{12}$$

$$q^{(n)} \to q \text{ weakly in } L_2(\Omega), \tag{13}$$

$$\begin{array}{c} u^{(n)} \to u \text{ weakly in } W_{2,0}^2(\Omega) \\ \text{strongly in } \overset{\circ}{W}{}_2^1(\Omega) \end{array} \tag{14}$$

as $n \to \infty$, where $u = u(x)$ is some function from $W_{2,0}^2(\Omega)$.

We show that $u(x) = u(x;\vartheta)$, i.e. the function $u(x)$ is a solution of problem (1),(2) corresponding to the control $\vartheta \in V$. It is clear that the functions u^n $(n = 1, 2, ...)$ satisfy the identities

$$\begin{array}{c} \int\limits_\Omega \left[\sum\limits_{i=1}^3 k_i^n \dfrac{\partial u}{\partial x_i}^{(n)} \dfrac{\partial \eta}{\partial x_i} + f\left(x, u^{(n)}, q^{(n)}\right)\eta\right] dx = \\ = \int\limits_\Omega f_0(x)\,\eta dx, \quad \forall \eta \in \overset{\circ}{W}{}_2^1(\Omega) \ (n = 1, 2, ...)\ . \end{array} \tag{15}$$

Passing to the limit as $n \to \infty$ in the identity

$$\int\limits_\Omega \sum_{i=1}^3 k_i^n \frac{\partial u}{\partial x_i}^{(n)} \frac{\partial \eta}{\partial x_i} dx = \int\limits_\Omega \sum_{i=1}^3 k_i^n \left(\frac{\partial u}{\partial x_i} - \frac{\partial u}{\partial x_i}\right) \frac{\partial \eta}{\partial x_i} dx +$$

$$+ \int\limits_\Omega \sum_{i=1}^3 k_i^n \frac{\partial u}{\partial x_i} \frac{\partial \eta}{\partial x_i} dx\ .$$

and using (12),(14) we obtain that

$$\int\limits_\Omega \sum_{i=1}^3 k_i^n \frac{\partial u}{\partial x_i}^{(n)} \frac{\partial \eta}{\partial x_i} dx \to \int\limits_\Omega \sum_{i=1}^3 k_i \frac{\partial u}{\partial x_i} \frac{\partial \eta}{\partial x_i} dx\ . \tag{16}$$

We now show that

$$\int\limits_\Omega f\left(x, u^{(n)}, q^{(n)}\right)\eta dx \to \int\limits_\Omega f(x, u, q)\,\eta dx, \ \forall \eta \in L_2(\Omega) \tag{17}$$

as $n \to \infty$. It is clear that the identity

$$\int\limits_\Omega f\left(x, u^{(n)}, q^{(n)}\right)\eta dx =$$

$$= \int\limits_{\Omega} \left[f\left(x, u^{(n)}, q^{(n)}\right) - f\left(x, u, q^{(n)}\right) \right] \eta dx + \int\limits_{\Omega} f\left(x, u, q^{(n)}\right) \eta dx \qquad (18)$$

is valid.

Using condition 1), the Cauchy-Bunyakowsky inequality and (14) we have

$$\left| \int\limits_{\Omega} \left[f\left(x, u^{(n)}, q^{(n)}\right) - f\left(x, u, q^{(n)}\right) \right] \eta dx \right| \le$$
$$\le L \int\limits_{\Omega} \left| u^{(n)} - u \right| |\eta| \, dx \le L \left\| u^{(n)} - u \right\|_{L_2(\Omega)} \|\eta\|_{L_2(\Omega)} \to 0, \qquad (19)$$

as $n \to \infty$.

Besides, it follows from condition 3) and from (13) that

$$\int\limits_{\Omega} f\left(x, u, q^{(n)}\right) \eta dx \to \int\limits_{\Omega} f\left(x, u, q\right) \eta dx, \ \forall \eta \in L_2\left(\Omega\right),$$

as $n \to \infty$. Allowing for this relation and (19) in (18) we obtain (17).

Now, passing to the limit in (15) as $n \to \infty$ and using (16),(17) we obtain that the function $u = u\left(x\right)$ satisfies identity (5). It follows from here and from theorem 1 that $u\left(x\right) = u\left(x; \vartheta\right)$.

Thus, it is established that at fulfilling (10) from the sequence $\left\{ \left(\vartheta^{(n)}, u^{(n)} \right) \right\}$ we can select the subsequence which again we denote by $\left\{ \left(\vartheta^{(n)}, u^{(n)} \right) \right\}$, for which relations (12)-(14), where $u\left(x\right) = u\left(x; \vartheta\right)$, are valid. Using the uniqueness of solution of problem (1),(2) at any $\vartheta \in V$, it is easy to show that relations (12)-(14) are also valid for all sequence $\left\{ \vartheta^{(n)}, u^{(n)} \right\}$.

Now, using relations (10),(14) and condition 4) we obtain that $J\left(\vartheta^{n}\right) \to J\left(\vartheta\right)$ as $n \to \infty$, i.e. the functional $J\left(\vartheta\right)$ is weakly continuous on V. Besides, since the set defined by relation (3) is convex, close, bounded in reflexive Banach space B, it is weakly compact in B, [11, p.51]. Then by virtue of theorem 2 from [11, p.49] the set V_* is nonempty, weakly compact in B, and any minimizing sequence $\left\{ \vartheta^{(m)} \right\}$ converges weakly in B to V_*. Theorem 2 is proved.

We now consider the minimization problem of functional

$$J_\alpha\left(\vartheta\right) = J\left(\vartheta\right) + \alpha \left\| \vartheta - \omega \right\|_B^\beta \qquad (4')$$

on the set V determined by relation (3) under conditions (1),(2), where $\alpha \ge 0, \ \beta \ge 1$ are given numbers, $\omega \in B$ is a given element, the functional $J\left(\vartheta\right)$ is defined by formula (4). This problem is said to be problem (1)-(3), (4').

Theorem 3. *Let conditions 1)-4) be satisfied and* $\alpha \ge 0, \ \beta \ge 1$. *Then for any* $\omega \in B$ *problem (1)-(3), (4') has at least one solution. If* $\alpha > 0, \ \beta > 1$, *then there exists a dense subset* B_0 *of the space* B *such that for any* $\omega \in B_0$ *problem (1)-(3), (4') has a unique problem.*

Proof. The functional $J_\alpha\left(\vartheta\right)$ represents the sum of weakly continuous functional $J\left(\vartheta\right)$ on V and weakly lower semi-continuous functional $\alpha \left\| \vartheta - \omega \right\|_B^\beta$ $(\alpha \ge 0, \ \beta \ge 1)$.

Therefore, $J_\alpha(\vartheta)$ is weakly lower semi-continuous on V. Then by virtue theorem 2 from [11, p.49] problem (1)-(3), (4') at $\alpha \geq 0$, $\beta \geq 1$ has at least one solution.

Now let $\alpha > 0$, $\beta > 1$. By virtue of theorem 2 the functional $J(\vartheta)$ is continuous by norm of the space B and lower bounded on V. Besides, the space B is uniform convex, and the set V is closed and bounded in B. Then by virtue of the known theorem [12] there exists the dense subset B_0 of the space such that for any $\omega \in B_0$ at $\alpha > 0$, $\beta > 1$ problem (1)-(3), (4') has a unique solution. Theorem 3 is proved.

3. Differentiability of functional and necessary optimality condition. For the proof of differentiability of the functional $J(\vartheta)$ we assume that the following conditions are satisfied:

5) the function $f(x,u,q)$ almost at all $x \in \Omega$ and for all $u \in R$, $q \in G$ has the partial derivatives $\frac{\partial f}{\partial u}, \frac{\partial f}{\partial q}$ satisfying the Caratheodory conditions in the domain $\Omega \times R \times G$, the following inequalities are satisfied

$$0 \leq \frac{\partial f(x,u,q)}{\partial u} \leq b_h(x),$$

$$\left| \frac{\partial f(x,u,q)}{\partial q} \right| \leq C_h(x), \ \forall \overset{\circ}{x} \in \Omega, \ \forall u \in R, \ |u| \leq h, \ \ q \in G,$$

where $h \in (0,\infty)$ is any number, $b_h \in L_2(\Omega)$, $C_h \in L_\infty(\Omega)$ are some functions and besides, the mappings $\frac{\partial f}{\partial u}(u,q), \frac{\partial f}{\partial q}(u,q)$ generated by the functions $\frac{\partial f(x,u(x),q(x))}{\partial u}, \frac{\partial f(x,u(x),q(x))}{\partial q}$ continuously act from $L_\infty(\Omega) \times Q$ to $L_2(\Omega)$.

6) The function $F(x,u,p,\vartheta)$ doesn't depend on the variable p, the function $F(x,u,p,\vartheta)$ almost at all $x \in \Omega$ and for all $u \in R$, $\vartheta \in U$ has partial derivatives $\frac{\partial F}{\partial u}, \frac{\partial F}{\partial \vartheta}$ satisfying the Caratheodory condition in the domain $\Omega \times R \times U$ and besides, the mappings $\frac{\partial F}{\partial u}(u,\vartheta)$, $\frac{\partial F}{\partial \vartheta}(u,\vartheta)$ generated by the functions

$$\frac{\partial F(x,u(x),\vartheta(x))}{\partial u}, \ \frac{\partial F(x,u(x),\vartheta(x))}{\partial \vartheta}$$

continuously act from $L_\infty(\Omega) \times V$ to $L_2(\Omega)$, $L_2^{(4)}(\Omega)$, respectively.

We introduce the adjoint state $\psi = \psi(x;\vartheta)$ as a solution of the problem

$$-\sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(k_i(x)\frac{\partial \psi}{\partial x_i}\right) + \frac{\partial f(x,u,q)}{\partial u}\psi = -\frac{\partial F(x,u,\vartheta)}{\partial u}, \ \ x \in \Omega, \tag{20}$$

$$\psi(x;\vartheta) = 0, \ \ x \in \Gamma, \tag{21}$$

where $u = u(x;\vartheta)$ is a solution of problem (1),(2).

Under a solution of boundary-value problem (20),(21) at each fixed $\vartheta \in V$ we'll understand a generalized solution of this problem from the space $\overset{\circ}{W}{}_2^1(\Omega)$. It follows from the results of the paper [6, pp.191-195] and form suggestion with respect to the functions $\frac{\partial f}{\partial u}, \ \frac{\partial F}{\partial u}$ that at each fixed $\vartheta \in V$ problem (20), (21) has a unique solution from $\overset{\circ}{W}{}_2^1(\Omega)$. Besides, this solution also belongs to the space $W_{2,0}^2(\Omega)$, satisfies equation (20) almost at all $x \in \Omega$ and the estimate

$$\|\psi\|_{W_2^2(\Omega)} \le M \left\| \frac{\partial F}{\partial u} \right\|_{L_2(\Omega)} \tag{22}$$

holds.

Let's also introduce the following boundary value problems on defining the functions $\psi_i = \psi_i(x;\vartheta)$ $\left(i = \overline{1,3}\right)$ from the conditions

$$-\sum_{j=1}^{3} \frac{\partial^2 \psi_i}{\partial x_j^2} + \psi_i = \frac{\partial u}{\partial x_i}\frac{\partial \psi}{\partial x_i} + \frac{\partial F}{\partial k_i}, \quad x \in \Omega, \tag{23}$$

$$\left.\frac{\partial \psi_i}{\partial \nu}\right|_{\Gamma} = 0 \quad (i = 1,3), \tag{24}$$

where ν is exterior normal to Γ, $u = u(x;\vartheta)$, $\psi = \psi(x;\vartheta)$ are the solutions of problems (1),(2) and (20),(21), respectively.

Under a solution of boundary-value problem (23),(24) at fixed $i \in \{1,2,3\}$, $\vartheta \in V$ we'll understand the functions $\psi_i = \psi_i(x;\vartheta)$ from $W_2^1(\Omega)$ satisfying the identity

$$\int_{\Omega} \left(\sum_{j=1}^{3} \frac{\partial \psi_i}{\partial x_j}\frac{\partial \eta}{\partial x_j} + \psi_i \eta \right) dx = \int_{\Omega} \left(\frac{\partial u_i}{\partial x_i}\frac{\partial \psi}{\partial x_i} + \frac{\partial F}{\partial k_i} \right) dx, \forall \eta \in W_2^1(\Omega) \ \left(i = \overline{1,3}\right). \tag{25}$$

Since the functions u, ψ are elements of the space $W_{2,0}^2(\Omega)$, then it is known that $\frac{\partial u}{\partial x_i}\frac{\partial \psi}{\partial x_i} \in L_r(\Omega)$ $(i = 1,3)$, where $r \in [2,6)$ is any number. Hence and from condition 6) it follows that the right hand side of equation (23) belongs to the space $L_2(\Omega)$. Therefore, we obtain from the results of the paper [6, pp.200-203] that at fixed $i \in \{1,2,3\}$, $\vartheta \in V$ boundary value problem (23),(24) is uniquely solvable in $W_2^1(\Omega)$.

Theorem 4. *Let conditions 1)-6) be satisfied. Then the functional $J(\vartheta)$ is continuously differentiable by Frechet on V and its gradient is defined by the formula*

$$J'(\vartheta) = \left(\psi_1, \psi_2, \psi_3, \frac{\partial f}{\partial q}\psi + \frac{\partial F}{\partial q} \right). \tag{26}$$

Proof. Let $\delta\vartheta = (\delta k_1, \delta k_2, \delta k_3, \delta q) \in B$ be an increment of control on the element $\vartheta \in V$ such that $\vartheta + \delta\vartheta \in V$ and $\delta u = u(x;\vartheta + \delta\vartheta) - u(x;\vartheta)$ are increments of solution of problem (1),(2). Using conditions (1),(2) and the Lagrange finite increment formula, for the function δu we obtain the following boundary-value problem

$$\begin{gathered} -\sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left((k_i + \delta k_i) \frac{\partial \delta u}{\partial x_i} \right) + \frac{\partial f(x, u + \theta, \delta u, q + \delta q)}{\partial u}\delta u = \\ = \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(\delta_{k_i} \frac{\partial u}{\partial x_i} \right) - \frac{\partial f(x, u, q + \theta_2 \delta q)}{\partial q}\delta q, \quad x \in \Omega, \end{gathered} \tag{27}$$

$$\delta u(x) = 0, \quad x \in \Gamma, \tag{28}$$

where $\theta_1, \theta_2 \in (0,1)$ are some numbers.

It follows from [6, pp.221-226] and from condition 5) that for solution of problem (27),(28) the estimate

$$\|\delta u\|_{W_2^2(\Omega)} \leq M\left[\sum_{i=1}^{3}\left(\left\|\delta k_i \frac{\partial^2 u}{\partial x_i^2}\right\|_{L_2(\Omega)} + \left\|\frac{\partial k_i}{\partial x_i}\frac{\partial u}{\partial x_i}\right\|_{L_2(\Omega)}\right) + \|\delta q\|_{L_2(\Omega)}\right] \quad (29)$$

is valid.

Besides, using the boundedness of the embedding $W_{m_i}^1(\Omega) \to C\left(\overline{\Omega}\right)$ at $m_i > 3 \quad (i = \overline{1,3})$ from [6,c.75], we obtain

$$\left\|\delta k_i \frac{\partial^2 u}{\partial x_i^2}\right\|_{L_2(\Omega)} + \left\|\frac{\partial \delta k_i}{\partial x_i}\frac{\partial u}{\partial x_i}\right\|_{L_2(\Omega)} \leq \|\delta k_i\|_{C(\overline{\Omega})}\left\|\frac{\partial^2 u}{\partial x_i^2}\right\|_{L_2(\Omega)} +$$

$$+\left\|\frac{\partial \delta k_i}{\partial x_i}\right\|_{L_{m_i}(\Omega)}\left\|\frac{\partial u}{\partial x_i}\right\|_{L_{2m_i/(m_i-2)}(\Omega)} \leq M\|\delta k_i\|_{W_{m_i}^1(\Omega)}\|u\|_{W_2^2(\Omega)} \quad (i = \overline{1,3}).$$

Then allowing for this inequality and estimate (6) in (29) we obtain

$$\|\delta u\|_{W_2^2(\Omega)} \leq M\|\delta\vartheta\|_B. \quad (30) \qquad (30)$$

Now we consider an increment of the functional $J(\vartheta)$. Using equality (4) and the finite increment formula we obtain

$$\delta J(\vartheta) = J(\vartheta + \delta\vartheta) - J(\vartheta) = \int_\Omega \left[\frac{\partial F}{\partial u}u + \sum_{i=1}^{3}\frac{\partial F}{\partial k_i}\delta k_i + \frac{\partial F}{\partial q}\delta q\right]dx + R_1(\delta\vartheta), \quad (31)$$

where

$$R_1(\delta\vartheta) = \int_\Omega \left(\frac{\partial F(x, u+\theta_3\delta u,\ \vartheta+\theta_3\delta\vartheta)}{\partial u} - \frac{\partial F(x,u,\vartheta)}{\partial u}\right)\delta u +$$

$$+\sum_{i=1}^{3}\left(\frac{\partial F(x, u+\theta_3\delta u,\ \vartheta+\theta_3\delta\vartheta)}{\partial k_i} - \frac{\partial F(x,u,\vartheta)}{\partial k_i}\right)\delta k_i + \quad (32)$$

$$+\left(\frac{\partial F(x, u+\theta_3\delta u,\ \vartheta+\theta_3\delta\vartheta)}{\partial q} - \frac{\partial F(x,u,\vartheta)}{\partial q}\right)\delta q\ \Big]\ dx, \quad \theta_3 \in (0,1).$$

Using condition (27), (28) and (20), (21) we can easily represent the right hand side of equality (31) in the following form

$$\delta J(\vartheta) = \int_\Omega \left[\sum_{i=1}^{3}\left(\frac{\partial u}{\partial x_i}\frac{\partial \psi}{\partial x_i} + \frac{\partial F}{\partial k_i}\right)\delta k_i + \left(\frac{\partial f}{\partial q}\psi + \frac{\partial F}{\partial q}\right)\delta q\right]dx + R(\delta\vartheta), \quad (33)$$

where $R(\delta\vartheta) = R_1(\delta\vartheta) + R_2(\delta\vartheta)$

$$R_2(\delta\vartheta) =$$

$$= \left[\int_\Omega \sum_{i=1}^{3}\frac{\partial \delta u}{\partial x_i}\frac{\partial \psi}{\partial x_i}\delta k_i + \left(\frac{\partial f(x, u+\theta_1\delta u,\ q+\delta q)}{\partial u} - \frac{\partial f(x,u,q)}{\partial u}\right)\psi\delta u + \right. \quad (34)$$

$$\left. +\left(\frac{\partial f(x,u,q+\theta_2\delta q)}{\partial q} - \frac{\partial f(x,u,q)}{\partial q}\right)\times\psi\delta q\right]dx.$$

If we assume $\eta = \delta k_i \ \ (i = 1,3)$ in (25) and allow for the obtained equality in (34), then we have

$$\delta J(\vartheta) = \int_\Omega \left[\sum_{i=1}^{3} \left(\psi_i \delta k_i + \sum_{j=1}^{3} \frac{\partial \psi_i}{\partial x_j} \frac{\partial \delta k_i}{\partial x_j} \right)_i + \left(\frac{\partial f}{\partial q} \psi + \frac{\partial F}{\partial q} \right) \delta q \right] dx + R(\delta\vartheta) . \quad (35)$$

Besides, using equality (32),(34), boundedness of the embedding $W^1_{m_i}(\Omega) \to C(\overline{\Omega}) \ \ (i = 1,3)$, $W^2_{2,0}(\Omega) \to C(\overline{\Omega})$ and estimates (22),(30) we obtain the following estimates:

$$|R_1(\delta\vartheta)| \le M \left[\left\| \frac{\partial F(\cdot, u + \theta_3 \delta u, \ \vartheta + \theta_3 \delta\vartheta)}{\partial u} - \frac{\partial F(\cdot, u, \vartheta)}{\partial u} \right\|_{L_1(\Omega)} + \right.$$

$$+ \sum_{i=1}^{3} \left\| \frac{\partial F(\cdot, u + \theta_3 \delta u, \ \vartheta + \theta_3 \delta\vartheta)}{\partial k_i} - \frac{\partial F(\cdot, u, \vartheta)}{\partial k_i} \right\|_{L_1(\Omega)} + \quad (36)$$

$$\left. + \left\| \frac{\partial F(\cdot, u + \theta_3 \delta u, \ \vartheta + \theta_3 \delta\vartheta)}{\partial q} - \frac{\partial F(\cdot, u, \vartheta)}{\partial q} \right\|_{L_2(\Omega)} \right] \|\delta\vartheta\|_B ,$$

$$|R_2(\delta\vartheta)| \le M \left[\|\delta\vartheta\|_B + \left\| \frac{\partial f(\cdot, u + \theta_1 \delta u, \ q + \delta q)}{\partial u} - \frac{\partial f(\cdot, u, q)}{\partial u} \right\|_{L_1(\Omega)} + \right.$$

$$\left. + \left\| \frac{\partial f(\cdot, u, q + \theta_2 \delta q)}{\partial q} - \frac{\partial f(\cdot, u, q)}{\partial q} \right\|_{L_2(\Omega)} \right] \|\delta\vartheta\|_B . \quad (37)$$

Then using conditions 5),6) from (36) and (37) we obtain that

$$R(\delta\vartheta) = o\left(\|\delta\vartheta\|_B\right).$$

Hence and form equality (35) it follows that the functional $J(\vartheta)$ is differentiable by Frechet on V and for its gradient formula (26) is valid.

It remains to show that the mapping $\vartheta \to J'(\vartheta)$ continuously acts from V in B^*, where B^* is a space adjoint to B. Let $\delta\psi = \psi(x; \vartheta + \delta\vartheta) - \psi(x; \vartheta)$, $\delta\psi_i = \psi_i(x; \vartheta + \delta\vartheta) - \psi_i(x; \vartheta) \ \left(i = \overline{1,3}\right)$ be increments of solutions of problem (20),(21) and (23),(24), respectively. Reasoning quite analogously as was obtained estimation (30) for the function δu, we can easily show that for the functions $\delta\psi, \ \delta\psi_i \ \left(i = \overline{1,3}\right)$ the estimates

$$\|\delta\psi\|_{W_2^2(\Omega)} \le M \left[\sum_{i=1}^{3} \|\delta k_i\|_{W^1_{m_i}(\Omega)} + \left\| \frac{\partial f(\cdot, u + \delta u, q + \delta q)}{\partial u} - \frac{\partial f(\cdot, u, q)}{\partial u} \right\|_{L_2(\Omega)} + \right.$$

$$\left. + \left\| \frac{\partial F(\cdot, u + \delta u, q + \delta q)}{\partial u} - \frac{\partial F(\cdot, u, q)}{\partial u} \right\|_{L_2(\Omega)} \right],$$

$$\|\delta\psi_i\|_{W_2^2(\Omega)} \le M \left[\|\delta u\|_{W^1_{m_i}(\Omega)} + \|\delta\psi\|_{W_2^2(\Omega)} + \|\delta u\|_{W_2^2(\Omega)} \|\delta\psi\|_{W_2^2(\Omega)} + \right.$$

$$\left. + \left\| \frac{\partial F(\cdot, u + \delta u, q + \delta q)}{\partial k_i} - \frac{\partial F(\cdot, u, q)}{\partial k_i} \right\|_{L_2(\Omega)} \right] \quad \left(i = \overline{1,3}\right)$$

are valid.

From these estimates allowing for conditions 5),6) and estimate (30) we obtain that

$$\|\delta\psi\|_{W_2^2(\Omega)} \to 0, \quad \|\delta\psi_i\|_{W_2^1(\Omega)} \to 0 \quad \text{as} \quad \|\delta\vartheta\|_B \to 0. \tag{38}$$

Besides, using equality (26), estimate (22) and condition 5) we obtain

$$\|J'(\vartheta+\delta\vartheta) - J'(\vartheta)\|_B \le M\left[\sum_{i=1}^{3}\|\delta\psi_i\|_{W_2^1(\Omega)} + \|\delta\psi\|_{W_2^2(\Omega)} + \right.$$

$$+\left\|\frac{\partial f(\cdot,u+\delta u,q+\delta q)}{\partial q} - \frac{\partial f(\cdot,u,q)}{\partial q}\right\|_{L_2(\Omega)} +$$

$$\left. +\left\|\frac{\partial F(\cdot,u+\delta u,\vartheta+\delta\vartheta)}{\partial q} - \frac{\partial F(\cdot,u,q)}{\partial q}\right\|_{L_2(\Omega)}\right].$$

Hence, the continuity of $J'(\vartheta)$ on V follows from (38) and conditions 5),6). Theorem 4 is proved.

Now we formulate the necessary optimality condition for solution of problem (1)-(4).

Theorem 5. *Let conditions 1)-6) be satisfied, and* $\vartheta_* = (k_{1*}, k_{2*}, k_{3*}, q) \in V$ *be an optimal control for problem (1)-(4). Then for any control* $\vartheta = (k_1, k_2, k_3, q) \in V$ *the following inequality is satisfied*

$$\int_\Omega \left\{\sum_{i=1}^{3}\left[\psi_{i*}(x)(k_i(x) - k_{i*}(x)) + \sum_{j=1}^{3}\frac{\partial\psi_{i*}}{\partial x_j}\left(\frac{\partial k_i}{\partial x_j} - \frac{\partial k_{i*}}{\partial x_j}\right)\right] + \right.$$

$$\left. + \left[\frac{\partial f}{\partial q}(x,u_*,q_*)\psi_*(x) + \frac{\partial F}{\partial q}(x,u_*,\vartheta_*)\right](q(x) - q_*(x))\right\}dx \ge 0, \tag{39}$$

where $u_*(x), \psi_*(x)$ and $\psi_{i*}(x)$ $\left(i=\overline{1,3}\right)$ are solutions of problems (1),(2);(20),(21) and (23), (24) at $\vartheta = \vartheta_*$, respectively.

Proof. According to theorem 4 the functional $J(\vartheta)$ is continuously differentiable by Frechet on V and for its gradient formula (26) is valid. The set V defined by relation (3) is convex. Then by virtue of the known theorem [11, p.28] on the element $\vartheta_* \in V$ supplying a minimum of the functional $J(\vartheta)$, it is necessary the satisfaction of the inequality $< J'(\vartheta_*), \vartheta - \vartheta_* > \ge 0$ for any $\vartheta \in V$. Hence and from (26) the validity of inequality (39) follows. Theorem 5 is proved.

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